Lie algebra generated by logarithm of differentiation and logarithm

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Abstract. Let $\log\left(\frac{d}{dx}\right)$ be the generator of the1-parameter group $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ of fractional order differentiations acting on the space of operators of Mikusinski ([5]). The Lie algebra $\mathfrak{g}_{\log}$ generated by $\log\left(\frac{d}{dx}\right)$ and $\log x$ is a deformation and can be regarded as the logarithm of Heisenberg Lie algebra. We show $\mathfrak{g}_{\log}$ is isomorphic to the Lie algebra generated by $\frac{d}{ds}\log(\Gamma(1+s))$ and $\frac{d}{ds}$. Hence as a module, $\mathfrak{g}_{\log}$ is isomorphic to the module generated by $\frac{d}{ds}$ and polygamma functions. Structure of the group generated by 1-parameter groups $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ and $\{x^a|a \in \mathbb{R}\}$, is also determined.

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1 Introduction

Schrödinger representation of Heisenberg Lie algebra is generated by $\frac{d}{dx}$ and x. Replacing $\frac{d}{dx}$ by $F(\frac{d}{dx})$,

$$F(X) = \sum_n c_n X^n, \quad F\left(\frac{d}{dx}\right) = \sum_n c_n \frac{d^n}{dx^n},$$

we obtain a deformation of Heisenberg Lie algebra. This algebra is isomorphic to the Lie algebra generated by $\frac{d}{dx}$ and $F(x)$. It is nilpotent if $F(x)$ is a polynomial and generalized nilpotent if $F(x)$ is an infinite series..

An example of such deformation is the algebra generated by logarithm of differentiation $\log\left(\frac{d}{dx}\right)$ and $\log x$.

If we consider fractional order differentiation $\frac{d^a}{dx^a}$ acts on the space of Operators of Mikusinski ([5]), we can define $\log\left(\frac{d}{dx}\right)$ by $\lim_{a\to 0}\frac{d}{da}\frac{d^a}{dx^a}|_{a=0}$. Explicitly, we have

$$\log\left(\frac{d}{dx}\right) f(x) = -\left(\gamma f(x) + \int_0^x \log(x-t)\frac{df_+}{dt}dt\right),$$

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where γ is the Euler constant and $\frac{df_+}{dt} = \frac{df}{dt} + f(0)\delta$, δ is the Dirac function ([4, §2 Prop.1.], [6]). By using logarithm of differentiation and the formula

$$\frac{d^a}{dx^a}x^c = \frac{\Gamma(1+c)}{\Gamma(1+c-a)}x^{c-a}, \tag{1.1}$$

where c and $c-a$ are both not negative integers, we obtain the following arguments on fractional calculus:

Let $\mathcal{R}$ be an integral transformation from functions on $\mathbb{R}$ to functions on positive real axis defined by

$$\mathcal{R}[f(s)](x) = \int_{-\infty}^{\infty} x^s \frac{f(s)}{\Gamma(1+s)}ds. \tag{1.2}$$

Then we have

$$\frac{d^a}{dx^a}\mathcal{R}[f(s)](x) = \mathcal{R}[\tau_a f(s)](x), \quad \tau_a f(s) = f(s+a), \tag{1.3}$$

$$\log\left(\frac{d}{dx}\right)\mathcal{R}[f(s)](x) = \mathcal{R}\left[\frac{df(s)}{ds}\right](x) \tag{1.4}$$

(§3, Theorem 3.1 and its Corollary). Our study on the structures of $\mathfrak{g}_{\log}$ and the group generated by exponential image of $\mathfrak{g}_{\log}$ are based on these equalities.
By the variable change $x = e^t$, we have $x^s = e^{ts}$. Hence we have

$$\mathcal{R}[f(s)](x) = \mathcal{L}\left[\frac{f(s)}{\Gamma(1+s)}\right](t), \quad \mathcal{L}[g(s)](t) = \int_{-\infty}^{\infty} e^{st}g(s)ds.$$

Therefore we obtain

$$\left.\frac{d^a}{dx^a}\right|_{x=e^t}\mathcal{L}[f(s)](t) = \mathcal{L}\left[\tau_a\left(\frac{\Gamma(1+s)}{\Gamma(1+s-a)}f(s)\right)\right](t), \tag{1.5}$$

$$\left.\log\left(\frac{d}{dx}\right)\right|_{x=e^t}\mathcal{L}[f(s)](t) = \mathcal{L}\left[\left(\frac{d}{ds} + \frac{\Gamma'(1+s)}{\Gamma(1+s)}\right)f(s)\right](t). \tag{1.6}$$

By (1.6), $\mathfrak{g}_{\log}$ is isomorphic to the Lie algebra generated by $\frac{d}{dt}$ and $\frac{\Gamma'(1+s)}{\Gamma(1+s)}$ (§4.Theorem 4.2). As a module, this algebra is generated by $\frac{d}{dt}$ and $\psi^{(m)}(1+s)$, $m = 0, 1, 2, \ldots$. Here $\psi^{(m)}(s)$ is the m-th polygamma function $\frac{d^m}{dt^m}\psi(s)$, $\psi(s) = \psi^{(0)}(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ ([1, §6.4]).

Since $e^{a\log\left(\frac{d}{dx}\right)} = \frac{d^a}{dx^a}$ and $e^{at} = x^a$, and e^{at} acts as the translation operator τ_a in the images of Laplace transformation, the group $G_{\log}$ generated by 1-parameter groups $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ and $\{x^a|a \in \mathbb{R}\}$ is the crossed product $G_{\log} \cong \mathbb{R} \ltimes G_\Gamma$, where G_Γ is the group generated by $\{\frac{\Gamma(1+s+a)}{\Gamma(1+s)}|a \in \mathbb{R}\}$ by multiplication ([3, §5. Prop.2.]). Definition of G_Γ in [3] is different. But it gives same group).

$G_{\log}$ is the essential part of the group G_Ψ generated by exponential images of the elements of $\mathfrak{g}_{\log}$. Precise structures of G_Ψ and generalization of this construction to the Heisenberg Lie algebra generated by $x_1, \ldots, x_n$ and $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are also studied (§5.Theorem 5.1 and §6).

In Appendix, we give an alternative proof of Theorem 3.1, which derives the integral transformation $\mathcal{R}$ naturally.

Note. Results in §4 and §5 are improvements of our previous results given in [4] and [3], while results in §3 and §6 are new.

2 Review on fractional calculus and logarithm of differentiation

Let $f(x)$ be a function on positive real axis, and let $a > 0$. Then a-th order indefinite integral of f from the origin is given by the Riemann-Liouville integral

$$I^a f(x) = \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t) dt. \tag{2.1}$$

Hence we may define $(n-a)$-th order differentiation $\frac{d^{n-a}}{dx^{n-a}}$ of f by $\frac{d^n}{dx^n} I^a f$ (Riemann-Liouville) or $I^a(\frac{d^n f}{dx^n})$ (Caputo). They are different if we consider in the category of functions. But if we use the space of operators of Mikusinski ([5]), they coincide and $\{\frac{d^a}{dx^a} | a \in \mathbb{R}\}$ $\{\frac{d^a}{dx^a} | a \in \mathbb{R}\}$ becomes a 1-parameter group. As a price, we can not investigate fractional order functions. The constant function 1 is replaced by the Heaviside function Y. Its derivative is the Dirac function δ.

Proposition 2.1. *The generating operator* $\log\left(\frac{d}{dx}\right) = \frac{d}{da}\frac{d^a}{dx^a}|_{a=0}$ *of the 1-parameter group* $\{\frac{d^a}{dx^a} | a \in \mathbb{R}\}$ *is given by*

$$\log\left(\frac{d}{dx}\right) = -\left(\gamma f(x) + \int_0^x \log(x-t)\frac{df_+}{dt}dt\right), \tag{2.2}$$

$$= -(\log x + \gamma) f(x) - \int_0^x \log\left(1 - \frac{t}{x}\right)\frac{df_+}{dt}dt. \tag{2.3}$$

Here γ *is the Euler constant and* $\frac{df_+}{dx}$ *means* $\frac{df}{dx} + f(0)\delta$ *([4, 6]).*

Note. If we assume $f(0) = 0$, or replace $f(x)$ be $f(x) - f(0)$, then we can avoid the use of distribution (cf.[2]).

By definition, we have

$$e^{a\log\left(\frac{d}{dx}\right)} = \frac{d^a}{dx^a}. \tag{2.4}$$

We also have

$$\log\left(\frac{d}{dx} + g(x)\right) = G(x)^{-1}\log\left(\frac{d}{dx}\right)\cdot G(x), \quad G(x) = e^{\int_0^x g(t)dt}.$$

Because we have $\frac{d}{dx} + g(x) = G(x)^{-1}\frac{d}{dx}\cdot G(x)$, where $G(x)$ is regarded as a linear operator acting by multiplication.

Example. By (2.2), we have

$$\log\left(\frac{d}{dx}\right)x^c = -\left(\log x + \left(\gamma - \sum_{n=1}^{\infty}\frac{c}{n(n+c)}\right)\right)x^c,$$
$$\log\left(\frac{d}{dx}\right)x^n = -\left(\log x + \gamma - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right)x^n.$$

We also have

$$\log\left(\frac{d}{dx}\right)(\log x)^n = -(\log x + \gamma)(\log x)^n + \\ + \sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}n!\zeta(n-k+1)}{k}!\,(\log x)^k.$$

Here $\zeta(k)$ is the value of Riemann's ζ-function at k. Introducing an infinite order differential operator $\mathfrak{d}_{\log}$ by

$$(2.5)\qquad \mathfrak{d}_{\log,X} = \frac{d}{dt}\log(\Gamma(1+t))\Big|_{t=\frac{d}{dX}} = \left(-\gamma + \sum_{n=1}^{\infty}(-1)^{n-1}\zeta(n+1)\frac{d^n}{dX^n}\right),$$

we have $\log(d/dx)(\log x)^n = (-X + \mathfrak{d}_{\log,X})X^n|_{X=\log x}$.

3 Hidden hierarchy of calculus involved in fractional calculus

We introduce an integral transformation $\mathcal{R}$ by

$$(3.1)\qquad \mathcal{R}[f(s)](x) = \int_{-\infty}^{\infty} x^s\frac{f(s)}{\Gamma(1+s)}ds,\ x > 0.$$

To define $\mathcal{R}[f]$, f needs to satisfy some estimate. For example, if $f(s)$ satisfies

$$(3.2)\qquad |f(s)| = O(e^{Ms}), s \to \infty, \quad |f(s)| = O(e^{-|s|^{\alpha}}), s \to -\infty,\ \alpha > 1,$$

then $\mathcal{R}[f]$ is defined. But appropriate domain and range of $\mathcal{R}$ are not known.

Note. In this paper, we consider $\mathcal{R}[f](x)$ to be a function on positive real axis.But it is better to consider $\mathcal{R}[f](x)$ to be a (many valued) function on $\mathbb{C}^{\times} = \mathbb{C}\setminus\{0\}$. Then $\mathcal{R}[f](e^{it})$ is defined as a function on $\mathbb{R}$. Here we need to consider $\mathcal{R}[f](e^{it})$ and $\mathcal{R}[f](e^{i(t+2\pi)})$ take different values.. Then by Fourier inversion formula, we have the following inversion formula

$$(3.3)\qquad f(s) = \frac{\Gamma(1+s)}{2\pi}\int_{-\infty}^{\infty} e^{-its}\mathcal{R}[f](e^{it})dt.$$

This shows if $\mathcal{R}[f](x)$ is a periodic function on the unit circle of $\mathbb{C}^{\times}$, then f is not a function, but a distribution. Studies in this direction will be a future problem.

Theorem 3.1. *If f is sufficiently mild, $e, g,$ if f satisfies* (3.1)*, then*

$$\frac{d^a}{dx^a}\mathcal{R}[f(s)](x) = \mathcal{R}[\tau_a f(s)](x). \tag{3.4}$$

Proof. If f is sufficiently mild, then

$$\frac{d^a}{dx^a}\mathcal{R}[f(s)](x) = \int_{-\infty}^{\infty} \frac{d^a}{dx^a}x^s \frac{f(s)}{\Gamma(1+s)}ds = \int_{-\infty}^{\infty} x^{s-a}\frac{\Gamma(1+s)}{\Gamma(1+s-a)}\frac{f(s)}{\Gamma(1+s)}ds,$$

whence the variable change $t = s - a$ yields

$$\frac{d^a}{dx^a}\mathcal{R}[f](x) = \int_{-\infty}^{\infty} x^t \frac{f(t+a)}{\Gamma(1+t)}dt, = \mathcal{R}[\tau_a f](x).$$

□

Hence we have

Corollary 3.2. *Under same assumption on f, we have*

$$\log\left(\frac{d}{dx}\right)\mathcal{R}[f(s)](x) = \mathcal{R}\left[\frac{df(s)}{ds}\right](x). \tag{3.5}$$

Proof. By (3.2), we infer

$$\frac{d}{da}\frac{d^a}{dx^a}\mathcal{R}[f(s)](x) = \mathcal{R}\left[\frac{d}{da}\tau_a f(s)\right](x).$$

Since $\frac{d}{da}\tau_a f(s)|_{a=0} = \frac{df(s)}{ds}$, we obtain the claimed result. □

Note. $\dfrac{df(s)}{ds}$ in (3.4) is taken in the sense of distribution.. For example, if $f(s)$ is continuous on $s \geq c$, differentiable on $s > c$, and $f(s) = 0$, $s < c$, then

$$\log\left(\frac{d}{dx}\right)\mathcal{R}[f(s)](x) = \mathcal{R}[f'(s)](x) + \frac{x^c}{\Gamma(1+c)}f(c),$$

where $f'(s)$ means $\dfrac{df(s)}{ds}$, $s > c$.

Theorem 3.1 and its Corollary show the simplest 1-parameter group (or dynamical system) $\{\tau_a | a \in \mathbb{R}\}$ and its generating operator $\frac{d}{ds}$ are changed to the 1-parameter group of fractional order differentiations $\{\frac{d^a}{dx^a} | a \in \mathbb{R}\}$ and its generating operator $\log\left(\frac{d}{dx}\right)$ via the transformation $\mathcal{R}$. Hence they suggest there may exist hierarchy of calculus involved in fractional calculus.

For the convenience, we use $\mathcal{L}[f(s)](t) = \int_{-\infty}^{\infty} e^{st}f(s)ds$ as the Laplace transformation in this paper. Since $\mathcal{L}$ is the bilateral Laplace transformation, we have

$$e^{at}\mathcal{L}[fx(s)](t) = \mathcal{L}[\tau_{-a}f(s)](t).$$

By definitions, we have

$$\mathcal{R}[f(s)](e^t) = \mathcal{L}\left[\frac{f(s)}{\Gamma(1+s)}\right](t).$$

Since $\tau_a(fg) = (\tau_a f)\tau_a g$, we have

$$\left.\frac{d^a}{dx^a}\right|_{x=e^t} \mathcal{L}[f(s)](t) = \mathcal{L}\left[\tau_a\left(\frac{\Gamma(1+s)}{\Gamma(1+s-a)}f(s)\right)\right](t). \tag{3.6}$$

Similarly, we infer

$$\log\left(\frac{d}{dx}\right)\Big|_{x=e^t} \mathcal{L}[f(s)](t) = \mathcal{L}\left[\left(\frac{d}{ds} + \frac{\Gamma'(1+s)}{\Gamma(1+s)}\right) f(s)\right](t). \tag{3.7}$$

Since $\frac{d}{dt}\mathcal{L}[f(s)](t) = \mathcal{L}[sf(s)](t)$, we obtain

$$\left.\frac{d^a}{dx^a}\right|_{x=e^t} = e^{-at}\mathfrak{d}_a, \quad \mathfrak{d}_a = \left(\frac{\Gamma(1+X)}{\Gamma(1+X-a)}\right)|_{X=\frac{d}{dt}},$$

We also have

$$\log\left(\frac{d}{dx}\right)\Big|_{x=e^t} = -t + \mathfrak{d}_{\log}, \quad \mathfrak{d}_{\log} = \left(\frac{\Gamma'(1+X)}{\Gamma(1+X)}\right)|_{X=\frac{d}{dt}},$$

which was already shown as (2.4).

4 Structure of $\mathfrak{g}_{\log}$

Let $\mathfrak{g}_{\log}$ be the Lie algebra generated by $\log\left(\frac{d}{dx}\right)$ and $\log x$. We take

$$\mathrm{H}_{\log} = \left\{\sum_{n=0}^{\infty} c_n(\log x)^n \;\middle|\; \sum_{n=0}^{\infty}|c_n|^2 < \infty\right\},$$

or similar space with the Sobolev type metric as the Hilbert space on which $\mathfrak{g}_{\log}$ acts.

By the variable change $x = e^t$ and Laplace transformation, multiplication by $\log x$ is changed to $\frac{d}{ds}$ and $\log\left(\frac{d}{dx}\right)$ is changed to $\frac{d}{ds} + \frac{\Gamma'(1+s)}{\Gamma(1+s)}$. Hence we have

Lemma 4.1. *Let $\mathfrak{g}_\Psi$ be the Lie algebra generated by $\frac{d}{ds}$ and $\Psi^{(0)}(1+s)$; let $\Psi^{(0)}(s) = \dfrac{d}{ds}\log(\Gamma(s)) = \dfrac{\Gamma'(s)}{\Gamma(s)}$. Then $\mathfrak{g}_\Psi$ is isomorphic to $\mathfrak{g}_{\log}$.*

Note. Since we use variable change $\log x = t$, we must regard $\mathfrak{g}_\Psi$ acts on Hilbert space spanned by polynomials. By the variable change $\log x = t$, $\mathrm{H}_{\log}$ is unitary equivalent to $W^{1/2}[0,1]$, the Sobolev $\frac{1}{2}$-space on [0,1]. Hence it is natural to consider $\mathfrak{g}_\Psi$ acts on $W^{1/2}[0,1]$. But we do not use such argument in this paper.

Since $[\frac{d}{ds}, F(s)] = F'(s)$, $\mathfrak{g}_\Psi$ is generated by $\frac{d}{ds}$ and $\frac{d^m}{ds^m}\Psi^{(0)}(1+s)$, $m = 0, 1, \dots$ as a module. $\frac{d^m}{ds^m}\Psi^{(0)}(s)$ is known as m-th polygamma function and denoted by $\Psi^{(m)}(s)$.

Therefore we can say $\mathfrak{g}_\Psi$ is generated by $\frac{d}{ds}$ and polygamma functions $\Psi^{(m)}(1+s)$, $m = 0, 1, \ldots$. Since

$$\left[\frac{d}{ds}, \Psi^{(m)}(s)\right] = \Psi^{(m+1)}(s), \quad [\Psi^{(m)}(s), \Psi^{(n)}(s)] = 0,$$

denoting $\mathrm{I}_\Psi^{(m)}$ the subspace of $\mathfrak{g}_\Psi$ spanned by $\Psi^{(m)}(s), \Psi^{(m+1)}(s), \ldots$, $\mathrm{I}_\Psi^{(m)}$ is an abelian ideal of $\mathfrak{g}_\Psi$, $m = 0, 1, \ldots$. By definition, we have

$$\mathrm{I}_\Psi^{(m)} \supset \mathrm{I}_\Psi^{(m+1)}, \quad \bigcap_{m=0}^{\infty} \mathrm{I}_\Psi^{(m)} = \{0\}.$$

We also have

$$\left[\frac{d}{ds}, \mathrm{I}_\Psi^{(m)}\right] = \mathrm{I}_\Psi^{(m+1)}.$$

Hence we have

$$\dim(\mathrm{I}_\Psi^{(m)}/\mathrm{I}_\Psi^{(m+1)}) = 1.$$

Therefore we obtain

$$\dim(\mathfrak{g}_\Psi/\mathrm{I}_\Psi^{(m)}) = m + 1.$$

$\mathfrak{g}_\Psi/\mathrm{I}_\Psi^{(1)}$ is an abelian Lie algebra and the class of $\Psi^{(1)}$ in $\mathfrak{g}_\Psi/\mathrm{I}_\Psi^{(2)}$ is the basis of the center. Hence $\mathfrak{g}_\Psi/\mathrm{I}_\Psi^{(2)}$ is isomorphic to Heisenberg Lie algebra.

Let $\iota_{\log}^{\Psi} : \mathfrak{g}_{|log} \cong \mathfrak{g}_\Psi$ be the isomorphism defined by

$$\iota_{\log}^{\Psi}(\log x) = \frac{d}{ds}, \quad \iota_{\log}^{\Psi}(\log\left(\frac{d}{dx}\right)) = \frac{d}{ds} + \Psi^{(0)}(1+s),$$

and let $\iota_{\Psi}^{\log} = (\iota_{\log}^{\Psi})^{-1}$. We set $\mathrm{I}_{\log}^{(m)} = \iota_{\Psi}^{\log}(\mathrm{I}_\Psi^{(m)}$. Then we obtain

Theorem 4.2. $\mathfrak{g}_{\log}$ *has a descending chain of abelian ideals* $\mathrm{I}_{\log}^{(0)} \supset \mathrm{I}_{\log}^{(1)} \supset \cdots$ *such that*

$$\left[\log x, \mathrm{I}_{\log}^{(m)}\right] = \mathrm{I}_{\log}^{(m+1)}, \quad \bigcap_{m-0}^{\infty} \mathrm{I}_{\log}^{(m)} = \{0\}. \tag{4.1}$$

We have $\dim(\mathfrak{g}_{\log}/\mathrm{I}_{\log}^{(m)}) = m + 1$, $m = 0, 1, \ldots$. $\mathfrak{g}_{\log}/\mathrm{I}_{\log}^{(m)}$ *is abelian Lie algebras if* $m = 0$ *and 1. If* $m = 2$, *it is isomorphic to Heisenberg Lie algebra.*

$\mathfrak{g}_{\log}$ can be regarded as a kind of logarithm of Heisenberg Lie algebra. $\log x$ is a (deformed) creation operator if we consider $\mathfrak{g}_{\log}$ acts on $\mathrm{H}_{\log}$. But $\log\left(\frac{d}{dx}\right)$ is not a (deformed) annihilation operator. To get (deformed) annihilation operator, we need to replace $\log\left(\frac{d}{dx}\right)$ by $d_{\log} = \log\left(\frac{d}{dx}\right) + \log x + \gamma$. The Lie algebra $\mathfrak{g}_{d_{\log}}$ generated by $\log x$ and $d_{\log}$ is isomorphic to $\mathfrak{g}_{\log}$. $\mathfrak{g}_{\log}$ and $\mathfrak{g}_{d_{\log}}$ are different. But we have

$$\mathfrak{g}_{\log} \oplus \mathbb{R}\mathrm{Id} = \mathfrak{g}_{d_{\log}} \oplus \mathbb{R}\mathrm{Id}. \quad \mathbb{R}\mathrm{Id} = \{x\mathrm{Id} | x \in \mathbb{R}\}.$$

If we do not demand generators of Heisenberg Lie algebra to be creation and annihilation operators, Heisenberg Lie algebra has generators such as $\frac{d}{dx}+x$, $\frac{d}{dx}-x$. Since

$$\log\left(\frac{d}{dx}\pm x\right)=e^{\mp\frac{x^2}{2}}\cdot\frac{d}{dx}\cdot e^{\pm\frac{x^2}{2}},$$

the Lie algebra generated by $e^{-x^2/2}\cdot\frac{d}{dx}\cdot e^{x^2/2}$ and $e^{x^2/2}\cdot\frac{d}{dx}\cdot e^{-x^2/2}$ is another candidate of logarithm of Heisenberg Lie algebra. It is not yet known whether this algebra is isomorphic to $\mathfrak{g}_{\log}$ or not.

If $\xi\in\mathfrak{g}_\Psi$, then ξ is uniquely written as $c_0\frac{d}{ds}+\sum_{m\geq 0}c_n\Psi^{(m)}(1+s)$, and we have

$$\left[a_0\frac{d}{ds}+\sum_m a_m\Psi^{(m)}\ ,\ b_0\frac{d}{ds}+\sum_m b_m\Psi\ (m)\right]=\sum_m(a_0b_m-a_mb_0)\Psi^{(m+1)}.$$

Hence (semi) norm completions of $\mathfrak{g}_\Psi$ (and $\mathfrak{g}_{\log}$) become Lie algebras. For example, ℓ^2-completion $\mathfrak{g}_{\Psi,\ell^2}$ of $\mathfrak{g}_\Psi$ defined by

$$\mathfrak{g}_{\Psi,\ell^2}=\left\{c_0\frac{d}{ds}+\sum_{m=0}^{\infty}c_m\Psi^{(m)}\ \middle|\ |c_0|^2+\sum_{m=0}^{\infty}|c_m|^2<\infty\right\},$$

is a Lie algebra having the structure of Hilbert space. Study in this direction is a future problem.

5 Structure of the group generated by exponential image of $\mathfrak{g}_{\log}$

Since $e^{a\frac{d}{x}}=\frac{d^a}{dx^a}$ and $e^{a\log x}=x^a$, first we study the group $G_{\log}$ generated by 1-parameter groups $\{\frac{d^a}{dx^a}|a\in\mathbb{R}\}$ and $\{x^a|a\in\mathbb{R}\}$.

By the variable change $x=e^t$ and Laplace transformation, the operators x^a acting by multiplication, and $\frac{d^a}{dx^a}$ are changed to $\tau_a:\tau_af(s)=f(s+a)$ and $\tau_a\left(\frac{\Gamma(1+s)}{\Gamma(1+s-a)}\right)$. We set G_Γ the group generated by $\{\frac{\Gamma(1+s)}{\Gamma(1+s-a)}|a\in\mathbb{R}\}$ by multiplication. $a\in\mathbb{R}$ acts on G_Γ by $a\cdot f=\tau_a(f)$.

Proposition 1. *We have*

$$G_{\log}\cong\mathbb{R}\ltimes G_\Gamma. \tag{5.1}$$

Since G_Γ is an abelian group $G_{\log}$ is a solvable group of derived length 1. By the map

$$\iota_\Sigma(f(s))=\tau_0f(s),$$

G_Σ is embedded isomorphically in $G_{\log}$. $\iota_\Sigma(G_\Sigma)$ is a normal subgroup of $G_{\log}$ and we have

$$G_{\log}/\iota_\Sigma(G_\Sigma)\cong\mathbb{R}.$$

While there are no canonical isomorphic embedding of $\mathbb{R}$ in $G_{\log}$.

G_Γ is an abelian group. But its structure seems complicated. For example, since

$$\frac{\Gamma(1+s)}{\Gamma(1+s-b)}\left(\frac{\Gamma(1+s)}{\Gamma(1+s-a)}\right)^{-1}=\frac{\Gamma(1+s-a)}{\Gamma(1+s-b)},$$

real coefficients rational function having only real roots and poles belong to G_Γ. This equality also shows definition of G_Γ in this paper coincides our previous definition of G_Γ in [2], where G_Γ is defined as the group generated by $\{\frac{\Gamma(1+s-a)}{\Gamma(1+s-b)}|a,b\in\mathbb{R}\}$ by multiplication.

Note. In this paper, we work in real category. If we work in complex category, then G_Γ should be the group generated by $\frac{\Gamma(1+s)}{\Gamma(1+s-a)}|a\in\mathbb{C}$ by multiplication. In this case, G_Γ contains all non zero rational functions.

To study the group generated by exponential image of $\mathfrak{g}_{\log}$, it is convenient to use $\mathfrak{g}_\Psi$ instead of $\mathfrak{g}_{\log}$. We set $G_{\Psi^{(m)}}$ the group generated by $e^{a\Psi^{(m)}}; a\in\mathbb{R}$ by multiplication and actions of $\tau_a, a\in\mathbb{R}$. By using $G_{\Psi^{(m)}}$, we define an abelian group G_{Ψ^∞} by

$$G_{\Psi^\infty}=\prod_{m\geq 0}G_{\Psi^{(m)}}.$$

Then G_{Ψ^∞} is an abelian group. $a\in\mathbb{R}$ acts as the translation operator τ_a on G_{Ψ^∞}. The group G_Ψ generated by exponential image of $\mathfrak{g}_\Psi$ is written as follows:

$$G_\Psi\cong\mathbb{R}\ltimes G_{\Psi^\infty}. \tag{5.2}$$

By (5.1), we have

Theorem 5.1. *The group generated by exponential image of $\mathfrak{g}_{\log}$ is isomorphic to G_Ψ. Hence it is a solvable group of derived length 1.*

Similar to G_Γ, we can regard G_{Ψ^∞} to be a normal subgroup of G_Ψ. It is an abelian group, but seems to have complicated structure. Since it is an infinite product of abelian groups, we must consider its topology. Then (together with the topology of $\mathfrak{g}_\Psi$ (or $\mathfrak{g}_{\log}$)), it may be possible to investigate $\mathfrak{g}_\Psi$ as the Lie algebra of G_Ψ. This will be a next problem.

6 Remarks on higher dimensional case

If we take $x_1,\ldots,x_n$ and $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$ as generator of Heisenberg Lie algebra $\mathfrak{h}_n$, the Lie algebra generated by $\log x_1,\ldots,\log x_n$ and $\log(\frac{\partial}{\partial x_1}),\ldots,\log(\frac{\partial}{\partial x_n})$ is isomorphic to $\overbrace{\mathfrak{g}_{\log}\otimes\cdots\otimes\mathfrak{g}_{\log}}^{n}$. But if the matrix (a_{ij}) is regular, $\sum_j a_{1j}x_j,\ldots,\sum_j a_{n,j}x_j$ and $\sum_j a_{1,j}\frac{\partial}{\partial x_j},\ldots,\sum_j a_{n,j}\frac{\partial}{\partial x_j}$ are alternative generators of $\mathfrak{h}_n$. They are also alternative creation and annihilation operators. In this case, we need to compute $\log(\sum_j a_{ij}\frac{\partial}{\partial x_j})$. Computation of this kind of operators are done as follows. We take $\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ as the example. We rewrite

$$\frac{\partial}{\partial x}+\frac{\partial}{\partial y}=e^{-x\frac{\partial}{\partial y}}\left(\frac{\partial}{\partial x}\right)e^{x\frac{\partial}{\partial y}}.$$

Since

$$e^{x\frac{\partial}{\partial y}}f(x,y)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{\partial^n}{\partial y^n}f(x,y)=f(x,y+x),$$

if f is sufficiently regular, we infer $e^{x\frac{\partial}{\partial y}}=\tau_{y;x}$ and $\tau_{y;a}f(x,y)=f(x,y+a)$. Hence we have

$$\log\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)=\tau_{y;-x}\log\left(\frac{\partial}{\partial y}\right)\cdot\tau_{y;x}. \tag{6.1}$$

Similarly, we obtain

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^a=\tau_{y;-x}\frac{\partial^a}{\partial x^a}\cdot\tau_{y;x}. \tag{6.2}$$

If $a=1$, we have $\tau_{y;-x}\frac{\partial}{\partial x}\cdot\tau_{y;x}=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$. Because we have

$$\frac{\partial}{\partial x}f(x,y+x)=\left(\frac{\partial}{\partial x}f(x,Y)+\frac{\partial}{\partial Y}f(x,Y)\right)\Big|_{Y=x+y}.$$

By the repeating use of this equality, we obtain

$$\tau_{y;-x}\frac{\partial^n}{\partial x^n}\cdot\tau_{y;x}=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{\partial^n}{\partial x^k\partial y^{n-k}}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^n. \tag{6.3}$$

Otherwise, it seems $\tau_{y;-x}\frac{\partial^a}{\partial x^a}\cdot\tau_{y;x}$ and $\tau_{y;-x}\log(\frac{\partial}{\partial x})\cdot\tau_{y;x}$ have no simpler expressions.

Since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ commute, rewriting

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^a=\frac{\partial^a}{\partial x^a}\left(1+\left(\frac{\partial}{\partial x}\right)^{-1}\frac{\partial}{\partial y}\right)^a,$$

and use Taylor expansion, we obtain another expression of $(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})^a$. Similar expression of $\log(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})$ is also possible. But these expressions seem much more complicated than (5.2) and (6.1).

Appendix. Alternative proof of Theorem 3.1

In this Appendix, we sketch an alternative proof of Theorem 3.1. In this proof, the integral transformation $\mathcal{R}$ appears naturally.

First we note that, since

$$\frac{\Gamma(1+X)}{\Gamma(1+X-a)}\Big|_{X=\frac{d}{dt}}e^{ct}=\frac{\Gamma(1+c)}{\Gamma(1+c-a)}e^{ct},$$

we have $(x^a\frac{d^a}{dx^a}|_{x=e^t})e^{ct}=\frac{\Gamma(1+X)}{\Gamma(1+X-a)}|_{X=\frac{d}{dt}}e^{ct}$ by (1.1). Therefore, if $f(x)$ is a power series converges rapidly, or $f(x)=\int x^s g(s)ds$, $x=e^t$, then

$$\left(x^a\frac{d^a}{dx^a}f(x)\right)\Big|_{x=e^t}=\frac{\Gamma(1+X)}{\Gamma(1+X-a)}\Big|_{X=\frac{d}{dt}}f(e^t). \tag{6.4}$$

Since $\frac{d}{da}(x^a \frac{d^a}{dx^a})|_{a=0} = \log x + \log\left(\frac{d}{dx}\right)$ and

$$\frac{d}{da}\left(\frac{\Gamma(1+X)}{\Gamma(1+X-a)}\right)\bigg|_{a=0} = \frac{\Gamma'(1+X)}{\Gamma(1+X)},$$

we obtain

$$\left(\log x + \log\left(\frac{d}{dx}\right)\right) f(x)\bigg|_{x=e^t} = \frac{\Gamma'(1+X)}{\Gamma(1+X)}\bigg|_{X=\frac{d}{dt}} f(e^t),$$

which recovers (2.4). Hence we obtain alternative proof of formulae of $\mathfrak{d}_a$ and $\mathfrak{d}_{\log}$ given in §3. Therefore by using Laplace transformation $\mathcal{L}[f(s)](t) = \int_{-\infty}^{\infty} e^{st} f(s)ds$, we obtain (3.5) and (3.6).

Since $\tau_a\left(\frac{\Gamma(1+s)}{\Gamma(1+s-a)} f(s)\right) = \frac{1}{\Gamma(1+s)}\tau_a(\Gamma(1+s)f(s))$, we obtain

$$\frac{d^a}{dx^a}\bigg|_{x=e^t} \mathcal{L}\left[\frac{f(s)}{\Gamma(1+s)}\right](t) = \mathcal{L}\left[\frac{\tau_a f(s)}{\Gamma(1+s)}\right](t).$$

Hence we have Theorem 3.1 by the variable change $x = e^t$.

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