Tangent sphere bundles which are $\eta-$Einstein

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Abstract. We study some almost contact metric structures on the tangent sphere bundles, induced from some almost Hermitian structures of natural diagonal lift type on the tangent bundle of a Riemannian manifold (M, g). The above almost contact metric structures are not automatically contact metric structures. In order to get such properties we made some rescalings of the metric, of the fundamental vector field, and of the 1-form. Then we gave the characterization of the Sasakian structures on the tangent sphere bundles. In this case, the base manifold must be of constant sectional curvature. For the obtained Sasakian manifolds we got the condition under which they are η - Einstein.

1 Introduction

In the last years, many researchers have been exhibiting a great interest in investigating the geometry of tangent sphere bundles of constant radius (see e. g. the papers [1], [2], [4], [5], [7], [10], [13], [15]).

The tangent sphere bundles $T_r M$ of constant radius r are hypersurfaces of the tangent bundles, obtained by considering only the tangent vectors which have the norm equal to r. Every almost Hermitian structure from the tangent bundle induces an almost contact structure on the tangent sphere bundle of constant radius r. In papers such as [4], [5] and [15], the metric considered on the tangent bundle TM was the Sasaki metric, but then E. Boeckx remarked that the unit tangent bundle equipped with the induced Cheeger-Gromoll metric is isometric to the tangent sphere bundle of radius $\frac{1}{\sqrt{2}}$, endowed with the metric induced by the Sasaki metric. This suggested to O. Kowalski and M. Sekizawa the idea that the tangent sphere bundles with different constant radii and with the metrics induced from the Sasaki metric might possess different geometrical properties, and they showed how the geometry of the tangent sphere bundles depends on the radius. All the important results obtained by the two authors in the field of the Riemannian geometry of the tangent sphere

bundles with arbitrary constant radius, done since 2000, may be found in the survey [10] from 2008.

In the present paper we study some geometric properties of the tangent sphere bundle of constant radius $T_r M$, by considering on the tangent bundle TM of a Riemannian manifold (M, g), a natural almost complex structure J and a natural metric G, both of them obtained by the second author in [14] as diagonal lifts of the Riemannian metric g from the base manifold. We determine the almost contact structure on the tangent sphere bundles of constant radius r, induced by the almost Hermitian structure of natural diagonal lift type from the tangent bundle, we get the conditions under which these structures are Sasakian, then we find the conditions under which the determined Sasakian tangent sphere bundles are η - Einstein. Extensive literature concerning Einstein equations can be mentioned from different perspectives (e. g. see [3], [8], [11]).

The manifolds, tensor fields and other geometric objects considered in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). The Einstein summation convention is used throughout this paper, the range of the indices h, i, j, k, l, m, r, being always $\{1, \dots, n\}$.

2   Preliminary results

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \to M$. The total space TM has a structure of a $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from usual local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, where the local coordinates $x^i, y^j, i, j = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates $y^j, j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it (see [12]).

Denote by $\dot{\nabla}$ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

(2.1) $$TTM = VTM \oplus HTM$$

of the tangent bundle to TM into the vertical distribution $VTM = \text{Ker } \tau_*$ and the horizontal distribution HTM defined by $\dot{\nabla}$ (see [20]). The vertical and horizontal lifts of a vector field X on M will be denoted by X^V and X^H respectively. The set of vector fields $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$ on $\tau^{-1}(U)$ defines a local frame field for VTM, and for HTM we have the local frame field $\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\}$, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma_{0i}^h \frac{\partial}{\partial y^h}, \quad \Gamma_{0i}^h = y^k \Gamma_{ki}^h$, and $\Gamma_{ki}^h(x)$ are the Christoffel symbols of g.

The set $\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\}_{i,j=\overline{1,n}}$, denoted also by $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$, defines a local frame on TM, adapted to the direct sum decomposition (2.1).

Consider the energy density of the tangent vector y with respect to the Riemannian metric g

$$(2.2) \qquad t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y,y) = \frac{1}{2}g_{ik}(x)y^i y^k, \quad y \in \tau^{-1}(U).$$

Obviously, we have $t \in [0, \infty)$ for every $y \in TM$.

There are many types of lifts of a metric from the base manifold to the tangent, cotangent, or jet bundles (e.g. see [16]-[19]), the most general being the natural lifts (in the sense of [9]), studied in a few recent papers, such as [1], [2], [6], [13].

The second author considered in [14] an (1,1)-tensor field J on the tangent bundle TM, obtained as natural 1-st order lift of the metric g from the base manifold to the tangent bundle TM:

$$(2.3) \qquad \begin{aligned} JX_y^H &= a_1(t)X_y^V + b_1(t)g_{\tau(y)}(X,y)y_y^V, \\ JX_y^V &= -a_2(t)X_y^H - b_2(t)g_{\tau(y)}(X,y)y_y^H, \end{aligned}$$

$\forall X \in \mathcal{T}_0^1(M)$, $\forall y \in TM$, a_1, a_2, b_1, b_2 being smooth functions of the energy density.

The above (1,1)-tensor field J defines an almost complex structure on the tangent bundle if and only if

$$(2.4) \qquad a_2 = \frac{1}{a_1}, \ b_2 = -\frac{b_1}{a_1(a_1 + 2tb_1)}.$$

Then it was considered on TM a Riemannian metric $\widetilde{G}$ of natural diagonal lift type:

$$(2.5) \qquad \begin{aligned} \widetilde{G}(X_y^H, Y_y^H) &= c_1(t)g_{\tau(y)}(X,Y) + d_1(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ \widetilde{G}(X_y^V, Y_y^V) &= c_2(t)g_{\tau(y)}(X,Y) + d_2(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ \widetilde{G}(X_y^V, Y_y^H) &= \widetilde{G}(X_y^H, X_y^V) = 0, \end{aligned}$$

$\forall X, Y \in \mathcal{T}_0^1(TM)$, $\forall y \in TM$, where c_1, c_2, d_1, d_2 are smooth functions of the energy density on TM. The conditions for $\widetilde{G}$ to be a Riemannian metric on TM (i.e. to be positive definite) are $c_1 > 0$, $c_2 > 0$, $c_1 + 2td_1 > 0$, $c_2 + 2td_2 > 0$ for every $t \geq 0$.

The Riemannian metric $\widetilde{G}$ is almost Hermitian with respect to the almost complex structure J if and only if

$$(2.6) \qquad \frac{c_1}{a_1} = \frac{c_2}{a_2} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu,$$

where $\lambda > 0$, $\mu > 0$ are functions of t.

The symmetric matrix of type $2n \times 2n$

$$\begin{pmatrix} \widetilde{G}_{ij}^{(1)} & 0 \\ 0 & \widetilde{G}_{ij}^{(2)} \end{pmatrix} = \begin{pmatrix} c_1(t)g_{ij} + d_1(t)g_{0i}g_{0j} & 0 \\ 0 & c_2(t)g_{ij} + d_2(t)g_{0i}g_{0j} \end{pmatrix},$$

associated to the metric $\widetilde{G}$ in the adapted frame $\{\delta_j, \partial_i\}_{i,j=\overline{1,n}}$, has the inverse

$$\begin{pmatrix} \widetilde{H}_{(1)}^{kl} & 0 \\ 0 & \widetilde{H}_{(2)}^{kl} \end{pmatrix} = \begin{pmatrix} p_1(t)g^{kl} + q_1(t)y^k y^l & 0 \\ 0 & p_2(t)g^{kl} + q_2(t)y^k y^l \end{pmatrix},$$

where g^{kl} are the entries of the inverse matrix of $(g_{ij})_{i,j=\overline{1,n}}$, and p_1, q_1, p_2, q_2, are some real smooth functions of the energy density. More precisely, they may be expressed as rational functions of c_1, d_1, c_2, d_2 :

$$(2.7) \qquad p_1 = \frac{1}{c_1}, \; p_2 = \frac{1}{c_2}, \; q_1 = -\frac{d_1}{c_1(c_1 + 2td_1)}, \; q_2 = -\frac{d_2}{c_2(c_2 + 2td_2)}.$$

Proposition 2.1. *The Levi-Civita connection $\widetilde{\nabla}$ associated to the Riemannian metric $\widetilde{G}$ from the tangent bundle TM has the form*

$$\begin{cases} \widetilde{\nabla}_{X^V}Y^V = Q(X^V, Y^V), \; \widetilde{\nabla}_{X^H}Y^V = (\dot{\nabla}_X Y)^V + P(Y^V, X^H), \\[2mm] \widetilde{\nabla}_{X^V}Y^H = P(X^V, Y^H), \; \widetilde{\nabla}_{X^H}Y^H = (\dot{\nabla}_X Y)^H + S(X^H, Y^H), \end{cases} \quad \forall X, Y \in \mathcal{T}_0^1(M),$$

where the M-tensor fields Q, P, S, have the following components with respect to the adapted frame $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$:

$$Q_{ij}^h = \tfrac{1}{2}(\partial_i \widetilde{G}_{jk}^{(2)} + \partial_j \widetilde{G}_{ik}^{(2)} - \partial_k \widetilde{G}_{ij}^{(2)})\widetilde{H}_{(2)}^{kh},$$

$$(2.8)$$

$$P_{ij}^h = \tfrac{1}{2}(\partial_i \widetilde{G}_{jk}^{(1)} + R_{0jk}^l \widetilde{G}_{li}^{(2)})\widetilde{H}_{(1)}^{kh}, \quad S_{ij}^h = -\tfrac{1}{2}(\partial_k \widetilde{G}_{ij}^{(2)} + R_{0ij}^l \widetilde{G}_{lk}^{(2)})\widetilde{H}_{(2)}^{kh},$$

R_{kij}^h being the components of the curvature tensor field of the Levi Civita connection $\dot{\nabla}$ from the base manifold (M, g) and $R_{0ij}^h = y^k R_{kij}^h$. The vector fields $Q(X^V, Y^V)$ and $S(X^H, Y^H)$ are vertically valued, while the vector field $P(Y^V, X^H)$ is horizontally valued.

Using the relations (2.8), we may easily prove that the M-tensor fields Q, P, S, have invariant expressions of the forms

$$Q(X^V, Y^V) = \frac{c_2'}{2c_2}[g(y, X)Y^V + g(y, Y)X^V]$$
$$-\frac{c_2' - 2d_2}{2(c_2 + 2td_2)}g(X, Y)y^V + \frac{c_2 d_2' - 2c_2' d_2}{2c_2(c_2 + 2td_2)}g(y, X)g(y, Y)y^V,$$

$$P(X^V, Y^H) = \frac{c_1'}{2c_1}g(y, X)Y^H + \frac{d_1}{2c_1}g(y, Y)X^H + \frac{d_1}{2(c_1 + 2td_1)}g(X, Y)y^H$$
$$+\frac{c_1 d_1' - c_1' d_1 - d_1^2}{2c_1(c_1 + 2td_1)}g(y, X)g(y, Y)y^H - \frac{c_2}{2c_1}(R(X, y)Y)^H - \frac{c_2 d_1}{2c_1(c_1 + 2td_1)}g(X, R(Y, y)y)y^H,$$

$$S(X^H, Y^H) = -\frac{d_1}{2c_2}[g(y, X)\, Y^V + g(y, Y)\, X^V]$$
$$-\frac{c_1'}{2(c_2 + 2td_2)}g(X, Y)\, y^V - \frac{c_2 d_1' - 2d_1 d_2}{2c_2(c_2 + 2td_2)}g(y, X)g(y, Y)y^V - \tfrac{1}{2}(R(X, Y)y)^V,$$

for every vector fields $X, Y \in \mathcal{T}_0^1(M)$ and every tangent vector $y \in TM$.

Since in the following sections we shall work on the subset $T_r M$ of TM consisting of spheres of constant radius r, we shall consider only the tangent vectors y for which the energy density t is equal to $\frac{r^2}{2}$, and the coefficients from the definition (2.5) of the metric $\widetilde{G}$ become constant. So we may consider them constant from the beginning. Then the M-tensor fields involved in the expression of the Levi-Civita connection

become simpler:
(2.9)
$$Q(X^V, Y^V) = \frac{d_2}{c_2 + r^2 d_2} g(X, Y) y^V,$$

$$P(X^V, Y^H) = \frac{d_1}{2c_1} g(Y, y) X^H + \frac{d_1}{2c_1(c_1 + r^2 d_1)} [c_1 g(X, Y) - d_1 g(X, y) g(Y, y)] y^H$$
$$- \frac{c_2}{2c_1} (R(X, y)Y)^H - \frac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} g(X, R(Y, y)y) y^H,$$

$$S(X^H, Y^H) = -\frac{d_1}{2c_2} [g(X, y) Y^V + g(Y, y) X^V]$$
$$+ \frac{d_1 d_2}{c_2(c_2 + 2t d_2)} g(y, X) g(y, Y) y^V - \frac{1}{2}(R(X, Y)y)^V.$$

3 Sasakian tangent sphere bundles of natural diagonal lifted type

Let $T_r M = \{y \in TM : g_{\tau(y)}(y, y) = r^2\}$, with $r \in [0, \infty)$, and the projection $\bar{\tau}$: $T_r M \to M$, $\bar{\tau} = \tau \circ i$, where i is the inclusion map.

The horizontal lift of any vector field on M is tangent to $T_r M$, but the vertical lift is not always tangent to $T_r M$. The tangential lift of a vector X to $(p, y) \in T_r M$ is tangent to $T_r M$ and is defined by

$$X_y^T = X_y^V - \frac{1}{r^2} g_{\tau(y)}(X, y) y_y^V.$$

Remark that the tangential lift of the tangent vector $y \in T_r M$, vanishes, i.e. $y_y^T = 0$.

The tangent bundle to $T_r M$ is spanned by δ_i and $\partial_j^T = \partial_j - \frac{1}{r^2} g_{0j} y^k \partial_k$, $i, j, k = \overline{1, n}$, although $\{\partial_j^T\}_{j=\overline{1,n}}$ are not independent. They fulfill the relation

(3.1)
$$y^j \partial_j^T = 0,$$

and in any point $y \in T_r M$ they span an $(n-1)$-dimensional subspace of $T_y T_r M$.

Denote by G' the metric on $T_r M$ induced by the metric $\widetilde{G}$ from TM. Remark that the functions c_1, c_2, d_1, d_2 become constant, since in the case of the tangent sphere bundle of constant radius r, the energy density t becomes a constat equal to $\frac{r^2}{2}$. It follows

$$\begin{cases} G'(X_y^H, Y_y^H) = c_1 g_{\tau(y)}(X, Y) + d_1 g_{\tau(y)}(X, y) g_{\tau(y)}(Y, y), \\ G'(X_y^T, Y_y^T) = c_2 [g_{\tau(y)}(X, Y) - \frac{1}{r^2} g_{\tau(y)}(X, y) g_{\tau(y)}(Y, y)], \\ G'(X_y^H, Y_y^T) = G'(Y_y^T, X_y^H) = 0, \end{cases}$$

$\forall X, Y \in \mathcal{T}_0^1(M)$, $\forall y \in T_r M$, where c_1, d_1, c_2 are constants. The conditions for G' to be positive are $c_1 > 0$, $c_2 > 0$, $c_1 + r^2 d_1 > 0$.

When the tangent bundle TM endowed with the metric $\widetilde{G}$ and the almost complex structure J defined respectively by the relations (2.3) and (2.5) is almost Hermitian, we may construct an almost contact metric structure on the tangent sphere bundle $T_r M$.

To this aim, we choose an appropriate normal (non unitary) vector field N to $T_r M$, given by

(3.2) $$N = (a_1 + r^2 b_1)y^V.$$

Using the almost complex structure J on TM, we define a vector field ξ', and a 1-form η', on $T_r M$ as follows:

$$\xi' = -JN, \ \eta'(\overline{X}) = G'(\overline{X}, \xi'), \ \forall \overline{X} \in \mathcal{T}_0^1(T_r M).$$

We may easily prove that

(3.3) $\ \xi' = y^H, \ \eta'(X^T) = 0, \ \eta'(X^H) = (c_1 + r^2 d_1)g(X, y), \forall X \in \mathcal{T}_0^1(M), \ y \in T_r M.$

The local coordinate expressions of ξ' and η' are

$$\xi' = y^i \delta_i, \ \eta' = (c_1 + r^2 d_1)g_{0i}dx^i, \ \forall i = \overline{1, n}.$$

Since $\eta'(\xi') = r^2(c_1 + r^2 d_1) \neq 1$, we shall define another 1-form, η and an appropriate vector field ξ, given by

(3.4) $$\eta = \beta \eta', \ \xi = \frac{1}{\beta r^2(c_1 + r^2 d_1)}\xi',$$

where β is an appropriate constant, whose value will be determined later. Obviously, we have $\eta(\xi) = 1$.

Taking into account the relations (3.3) and (3.4), we obtain the invariant expressions of η and ξ:

$$\eta(X^T) = 0, \ \eta(X^H) = \beta(c_1 + r^2 d_1)g(X, y), \ \xi = \frac{1}{\beta r^2(c_1 + r^2 d_1)}y^H,$$

for every vector field X tangent to M, and every tangent vector y from $T_r M$.

The (1,1)-tensor field φ on $T_r M$, obtained by eliminating the normal component of the almost complex structure J from TM, will be expressed by

(3.5) $\ \varphi X^H = a_1 X^T, \varphi X^T = -a_2 X^H + \frac{a_2}{r^2}g(X, y)y^H, \ \forall X, Y \in \mathcal{T}_0^1(M), y \in T_r M.$

Since the (1,1)-tensor field φ given by (3.5) and ξ, η from (3.4) verify the relations

$$\varphi^2 = -I + \eta \otimes \xi, \ \varphi \xi = 0, \ \eta \circ \varphi = 0, \eta(\xi) = 1,$$

we have that (φ, ξ, η) is an almost contact structure on $T_r M$.

The aim of this section is to find the Sasakian structures (φ, ξ, η, G) of natural diagonal lifted type on $T_r M$, i.e. the conditions under which the almost contact structure (φ, ξ, η) is normal contact metric with respect to a certain metric G on $T_r M$, which we shall determine.

The obtained almost contact structure is not almost contact metric with respect to the metric G', since the relation $G'(\varphi \overline{X}, \varphi \overline{Y}) = G'(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}), \ \forall \overline{X}, \overline{Y} \in \mathcal{T}_0^1(T_r M)$ is not satisfied, i.e. the metric G' is not compatible with the almost contact structure (φ, ξ, η).

Using (3.3) and (3.4) we obtain the exterior differential of η:

$$d\eta(X^H, Y^H) = d\eta(X^T, Y^T) = 0,$$
$$d\eta(X^T, Y^H) = -d\eta(X^H, Y^T) = \beta \frac{c_1 + r^2 d_1}{2}[g(X,Y) - \frac{1}{r^2}g(X,y)g(Y,y)],$$

for every tangent vector fields X, Y on M and every tangent vector $y \in T_r M$.

We have too that the condition $G'(\overline{X}, \varphi\overline{Y}) = d\eta(\overline{X}, \overline{Y}), \ \forall \overline{X}, \overline{Y} \in \mathcal{T}_0^1(T_r M)$, is not satisfied, so the almost contact metric structure (φ, ξ, η) is not a contact metric structure with respect to G'.

We shall change the metric G', multiplying it by a scalar α. Thus, the new metric G on $T_r M$ will have the form

$$G = \alpha G'.$$

The scalars $\alpha, \ \beta$ may be determined from the conditions

$$G(\xi, \xi) = 1, \ \ G(X^T, \varphi Y^H) = d\eta(X^T, Y^H), \ \ G(X^H, \varphi Y^T) = d\eta(X^H, Y^T),$$

which, due to the proportionality relation (2.6), are equivalent to a simple system in α and β:

$$\begin{cases} \alpha = \beta^2 r^2 (c_1 + r^2 d_1), \\ \alpha\lambda = \beta \frac{c_1 + r^2 d_1}{2}. \end{cases}$$

The solution of this system is

$$\alpha = \frac{c_1 + r^2 d_1}{4r^2\lambda^2}, \ \beta = \frac{1}{2r^2\lambda}.$$

Thus, the final expressions for φ, ξ, η and G are:

$$(3.6) \qquad \varphi X^H = a_1 X^T, \varphi X^T = -a_2 X^H + \frac{a_2}{r^2}g(X,y)y^H,$$

$$(3.7) \qquad \xi = \frac{2\lambda}{c_1 + r^2 d_1}y^H, \ \ \eta(X^T) = 0, \ \eta(X^H) = \frac{c_1 + r^2 d_1}{2r^2\lambda}g(X,y),$$

$$(3.8) \qquad \begin{cases} G(X_y^H, Y_y^H) = \frac{c_1 + r^2 d_1}{4r^2\lambda^2}[c_1 g_{\tau(y)}(X,Y) + d_1 g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y)], \\[2mm] G(X_y^T, Y_y^T) = \frac{c_2(c_1 + r^2 d_1)}{4r^2\lambda^2}[g_{\tau(y)}(X,Y) - \frac{1}{r^2}g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y)], \\[2mm] G(X_y^H, Y_y^T) = G(Y_y^T, X_y^H) = 0, \end{cases}$$

for every tangent vector fields $X, Y \in \mathcal{T}_0^1(M)$, and every tangent vector $y \in T_r M$.

The nonzero components of the metric G in the generator system $\{\delta_i, \partial_j^T\}_{i,j=\overline{1,n}}$ are

$$(3.9) \qquad \begin{cases} G_{ij}^{(1)} = G(\delta_i, \delta_j) = \frac{c_1 + r^2 d_1}{4r^2\lambda^2}(c_1 g_{ij} + d_1 g_{0i}g_{0j}), \\[2mm] G_{ij}^{(2)} = G(\partial_i^T, \partial_j^T) = \frac{c_1 + r^2 d_1}{4r^2\lambda^2}c_2(g_{ij} - \frac{1}{r^2}g_{0i}g_{0j}). \end{cases}$$

In the computations we shall use the following shorter equivalent expressions for $\xi,\ \eta,\ G$:

(3.10) $\qquad \xi = \dfrac{1}{2\lambda r^2 \alpha} y^H,\ \eta(X^T) = 0,\ \eta(X^H) = 2\alpha\lambda g(X, y),\ G = \alpha G',$

where $\alpha = \frac{c_1 + r^2 d_1}{4r^2\lambda^2}$.

Now we may prove the following result.

Theorem 3.1. *The almost contact metric structure* (φ, ξ, η, G) *on* $T_r M$, *with* $\varphi,\ \xi, \eta$, *and* G *given by* (3.6) *and* (3.10), *is a contact metric structure, and it is Sasakian if and only if the base manifold has constant sectional curvature* $c = \frac{a_1^2}{r^2}$.

Proof: Since the metricity and the contact conditions

$$G(\varphi\overline{X}, \varphi\overline{Y}) = G(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}),\quad G(\overline{X}, \varphi\overline{Y}) = d\eta(\overline{X}, \overline{Y}),\ \forall \overline{X}, \overline{Y} \in \mathcal{T}_0^1(T_r M),$$

are fulfilled, it follows that the almost contact structure (φ, ξ, η, G) on $T_r M$ is a contact metric structure.

The normality condition for the contact metric structure found on $T_r M$ is

(3.11) $\qquad N_\varphi(\overline{X}, \overline{Y}) + 2d\eta(\overline{X}, \overline{Y})\xi = 0,\ \forall \overline{X}, \overline{Y} \in \mathcal{T}_0^1(T_r M),$

where N_φ is the Nijenhuis tensor field of the $(1,1)-$tensor field φ, and it is given by

(3.12) $\qquad N_\varphi(\overline{X}, \overline{Y}) = \varphi^2[\overline{X}, \overline{Y}] + [\varphi\overline{X}, \varphi\overline{Y}] - \varphi[\varphi\overline{X}, \overline{Y}] - \varphi[\overline{X}, \varphi\overline{Y}].$

When both $\overline{X}$ and $\overline{Y}$ are horizontal vector fields, $\delta_i,\ \delta_j$, the mentioned relation becomes

(3.13) $\qquad R_{0ij}^h + \dfrac{a_1^2}{r^2}(g_{i0}\delta_j^h - g_{j0}\delta_i^h) = 0,$

where $R_{0ij}^h = R_{kij}^h y^k,\ g_{i0} = g_{ik}y^k$.

By differentiating (3.13) with respect to the tangential coordinates y^k, we obtain that the base manifold M has constant sectional curvature:

(3.14) $\qquad R_{kij}^h = \dfrac{a_1^2}{r^2}(\delta_i^h g_{jk} - \delta_j^h g_{ik}).$

Next the relation (3.11) is identically fulfilled when we replace at least one of the vector fields $\overline{X}, \overline{Y}$ by ∂_i^T. Thus (3.14) is the only condition for the contact metric structure (φ, ξ, η, G) on $T_r M$ to be Sasakian (normal contact metric). Thus the theorem is proved.

In the sequel we shall see that the Levi-Civita connection associated to the metric G satisfies the necessary Sasaki condition

(3.15) $\qquad (\nabla_{\overline{X}}\varphi)\overline{Y} = G(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X},\ \forall \overline{X}, \overline{Y} \in \mathcal{T}_0^1(T_r M),$

for the almost contact metric manifold $(T_r M, \varphi, \xi, \eta, G)$.

The explicit expression of ∇ is given in the following proposition.

Proposition 3.2. *The Levi-Civita connection ∇, associated to the Riemannian metric G on the tangent sphere bundle $T_r M$ of constant radius r has the expression*

$$
\begin{cases}
\nabla_{\partial_i^T} \partial_j^T = A_{ij}^h \partial_h^T, \ \nabla_{\delta_i} \partial_j^T = \Gamma_{ij}^h \partial_h^T + B_{ji}^h \delta_h, \\[2mm]
\nabla_{\partial_i^T} \delta_j = B_{ij}^h \delta_h, \ \nabla_{\delta_i} \delta_j = \Gamma_{ij}^h \delta_h + C_{ij}^h \partial_h^T,
\end{cases}
$$

where the M-tensor fields involved as coefficients have the expressions:

$$
\begin{aligned}
& A_{ij}^h = -\tfrac{1}{r^2} g_{0j} \delta_i^h, \quad C_{ij}^h = -\tfrac{d_1}{2c_2}(\delta_j^h g_{0i} - \delta_i^h g_{0j}) - \tfrac{1}{2} R_{0ij}^h, \\
(3.16) \ & B_{ij}^h = P_{ij}^h - \tfrac{1}{r^2} g_{i0} P_{0j}^h = \tfrac{d_1}{2c_1} \delta_i^h g_{0j} + \tfrac{d_1}{2(c_1 + r^2 d_1)}(g_{ij} - \tfrac{2c_1 + r^2 d_1}{r^2 c_1} g_{0i} g_{0j}) y^h \\
& \qquad - \tfrac{c_2}{2c_1} R_{jik}^h y^k - \tfrac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} R_{ikjl} y^h y^k y^l,
\end{aligned}
$$

where $g_{0j} = g_{j0} = y^i g_{ij}$ and $P_{0j}^h = y^i P_{ij}^h$.

We mention that A and C have these quite simple expressions, since they are the coefficients of the tangential part of the Levi-Civita connection. All the terms containing $y^h \partial_h^T = 0$ have been cancelled.

The invariant form of the Levi-Civita connection from $T_r M$ is

$$
\begin{cases}
\nabla_{X^T} Y^T = -\tfrac{1}{r^2} g(Y, y) X^T, \\[3mm]
\nabla_{X^T} Y^H = \tfrac{d_1}{2c_1} g(Y, y) \ X^H + \tfrac{d_1}{2(c_1 + d_1 r^2)} g(X, Y) y^H \\
\quad - \tfrac{d_1(2c_1 + r^2 d_1)}{2r^2 c_1(c_1 + r^2 d_1)} g(X, y) g(Y, y) y^H - \tfrac{c_2}{2c_1} (R(X, y)Y)^H - \tfrac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} g(X, R(Y, y)y) y^H, \\[3mm]
\nabla_{X^H} Y^T = (\dot{\nabla}_X Y)^T + \tfrac{d_1}{2c_1} g(X, y) \ Y^H + \tfrac{d_1}{2(c_1 + r^2 d_1)} g(X, Y) y^H \\
\quad - \tfrac{d_1(2c_1 + r^2 d_1)}{2r^2 c_1(c_1 + r^2 d_1)} g(X, y) g(Y, y) y^H - \tfrac{c_2}{2c_1}(R(Y, y)X)^H - \tfrac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} g(Y, R(X, y)y) y^H, \\[3mm]
\nabla_{X^H} Y^H = (\dot{\nabla}_X Y)^H - \tfrac{d_1}{2c_2} [g(X, y) \ Y^T - g(Y, y) \ X^T] - \tfrac{1}{2}(R(X, Y)y)^T.
\end{cases}
$$

The condition for $(T_r M, \varphi, \eta, \xi, G)$ to be a Sasakian manifold is given by (3.15). When $\overline{X}$ and $\overline{Y}$ are both of them tangential vector fields, or horizontal vector fields, respectively, (3.15) becomes

$$
(\nabla_{\partial_i^T} \varphi) \partial_j^T - G_{ij}^{(2)} \xi = 0, \quad (\nabla_{\delta_i} \varphi) \delta_j - G_{ij}^{(1)} \xi + \eta(\delta_j) \delta_i = 0.
$$

After some quite long computations we get

$$
\tfrac{1}{r^2}[g_{0j} \delta_i^h + \tfrac{1}{2}(g_{ij} - \tfrac{3}{r^2} g_{0i} g_{0j}) y^h] + (\tfrac{1}{r^2} g_{0j} y^l - \delta_j^l) B_{il}^h - (\tfrac{1}{r^2} g_{0l} y^h - \delta_l^h) A_{ij}^l = 0,
$$

$$
a_1 B_{ji}^h - a_2 (\tfrac{1}{r^2} g_{0l} C_{ij}^l y^h - C_{ij}^h) - \tfrac{1}{2\lambda r^2 \alpha} G_{ij}^{(1)} y^h + 2\lambda \alpha g_{0j} \delta_i^h = 0,
$$

and it follows easily that these relations are identically fulfilled.

When $\overline{X}$ and $\overline{Y}$ are horizontal and tangential vector field, respectively, (3.15) takes the forms

$$
(\nabla_{\partial_i^T} \varphi) \delta_j + \eta(\delta_j) \partial_i^T = 0, \quad (\nabla_{\delta_i} \varphi) \partial_j^T = 0,
$$

which, after the computations, become

$$[a_1(A_{ij}^h - B_{ij}^h) + 2\alpha\lambda g_{0j}\delta_i^h]\partial_h^T = 0, \quad [a_2(\frac{1}{r^2}g_{0j}C_{i0}^h - C_{ij}^h) - a_1 B_{ji}^h]\partial_h^T = 0,$$

and they are satisfied, since the final values,

$$-\frac{1}{2r^2}\big[a_1 g_{ij} - \frac{b_1\lambda + (a_1 + r^2 b_1)\mu}{\lambda} g_{0i}g_{0j}\big]y^h\partial_h^T, \quad -\frac{1}{2r^2}(a_1 g_{ij} - \frac{a_1}{r^2}g_{0i}g_{0j})y^h\partial_h^T,$$

are zero due to the relation $y^h\partial_h^T = 0$, fulfilled by the generators ∂_h^T.

4 $\eta-$Einstein Sasakian tangent sphere bundles of natural diagonal lifted type

In this section we shall find the condition under which the normal contact metric manifold $(T_r M, \varphi, \xi, \eta, G)$, with φ, ξ, η, and G given respectively by (3.6), (3.7), and (3.8), is $\eta-$Einstein, i.e. the corresponding Ricci tensor may be written as

(4.1) $$Ric = \rho G + \sigma\eta \otimes \eta,$$

where ρ and σ are smooth real functions.

In the case where A, B, C are given by (3.16), it can be shown (see e.g. [7]) that their covariant derivatives are:

(4.2) $$\begin{cases} \dot{\nabla}_k A_{ij}^h = 0, \quad \dot{\nabla}_k C_{ij}^h = -\frac{1}{2}\dot{\nabla}_k R_{lij}^h y^l, \\ \dot{\nabla}_k B_{ij}^h = -\frac{c_2}{2c_1}\dot{\nabla}_k R_{jil}^h y^l - \frac{c_2 d_1}{2c_1(c_1 + r^2 d_1)}\dot{\nabla}_k R_{iljr}y^l y^r y^h, \end{cases}$$

where $\dot{\nabla}_k R_{kij}^h, \; \dot{\nabla}_k R_{hkij}$ are the usual local coordinate expressions of the covariant derivatives of the curvature tensor field and the Riemann-Christoffel tensor field of the Levi Civita connection $\dot{\nabla}$ from the base manifold M. If the base manifold (M, g) is locally symmetric then, obviously, $\dot{\nabla}_k A_{ij}^h = 0, \; \dot{\nabla}_k B_{ij}^h = 0, \; \dot{\nabla}_k C_{ij}^h = 0$.

In the paper [7] we obtained by a standard straightforward computation the horizontal and tangential components of the curvature tensor field K, denoted by sequences of H and T, to indicate horizontal, or tangential argument on a certain position. For example, we have

$$K(\delta_i, \delta_j)\partial_k^T = HHTH_{kij}^h \delta_h + HHTT_{kij}^h \partial_h^T.$$

The expressions of the non-zero M-tensor fields which appear as coefficients may be found in [7].

Let us remark some facts concerning the obtaining of the Ricci tensor field for the tangent bundle TM. We have the well known formula

$$Ric(Y, Z) = trace(X \to K(X, Y)Z),$$

where X, Y, Z are vector fields on TM. Then we get easily the components of the Ricci tensor field on TM

$$\widetilde{RicHH}_{jk} = \widetilde{Ric}(\delta_j, \delta_k) = HHHH_{khj}^h + VHHV_{khj}^h,$$

$$\widetilde{RicVV}_{jk} = \widetilde{Ric}(\partial_j, \partial_k) = VVVV_{khj}^h - VHVH_{kjh}^h,$$

where the components $VVVV_{kij}^h, VHVH_{kij}^h, VHHV_{kij}^h$, are obtained from the curvature tensor field on TM in a similar way as the components $TTTT_{kij}^h, THTH_{kij}^h$, $THHT_{kij}^h$ are obtained from the curvature tensor field on T_rM. In the expression of $\widetilde{RicHV}_{jk} = \widetilde{Ric}(\delta_j, \partial_k) = \widetilde{RicVH}_{jk} = \widetilde{Ric}(\partial_j, \delta_k)$ there are involved the covariant derivatives of the curvature tensor field R. If (M, g) is locally symmetric then $RicHV_{jk} = RicVH_{jk} = 0$. In particular we have $RicHV_{jk} = RicVH_{jk} = 0$ in the case where (M, g) has constant sectional curvature.

Equivalently, we may use an orthonormal frame $(E_1, \ldots, E_{2n})$ on TM and we may use the formula

$$\widetilde{Ric}(Y, Z) = \sum_{i=1}^{2n} G(E_i, K(E_i, Y)Z).$$

We may choose the orthonormal frame $(E_1, \ldots, E_{2n})$ such that the first n vectors $E_1, \ldots, E_n$ are the vectors of a (orthonormal) frame in HTM and the last n vectors $E_{n+1}, \ldots E_{2n}$ are the vectors of a (orthonormal) frame in VTM. Moreover, we may assume that the last vector E_{2n} is the unitary vector of the normal vector $N = y^i \partial_i$ to T_rM.

The components of the Ricci tensor field of T_rM can be obtained in a similar way by using the above traces. However the vector fields $\partial_1^T, \ldots, \partial_n^T$ are not independent. On the open set from T_rM, where $y^n \neq 0$ we can consider the basis $\{\delta_1, \ldots, \delta_n, \partial_1^T, \ldots, \partial_{n-1}^T\}$ for TT_rM. The last vector ∂_n^T is expressed as

$$(4.3) \qquad\qquad \partial_n^T = -\frac{1}{y^n} \sum_{i=1}^{n-1} y^i \partial_i^T.$$

Remark that the basis $\{\delta_1, \ldots, \delta_n, \partial_1^T, \ldots, \partial_{n-1}^T\}$ can be completed with the normal vector $N = y^V = y^h \partial_h$.

Using the relation (4.3), we obtained in [7] the components of the Ricci tensor on T_rM:

$$RicHH_{jk} = Ric(\delta_j, \delta_k) = HHHH_{khj}^h + THHT_{khj}^h - \frac{d_1(4c_1 + r^2d_1)}{4c_1c_2r^2} g_{0j}g_{0k}.$$

$$RicTT_{jk} = Ric(\partial_j^T, \partial_k^T) = HTTH_{khj}^h + TTTT_{khj}^h - \frac{1}{r^2} g_{jk} + \frac{1}{r^4} g_{0j}g_{0k}.$$

Taking into account the above relations and the $\eta-$Einstein condition (4.1), we obtain that

$$(4.4) \qquad \begin{aligned} &RicTT_{jk} - \rho G_{jk}^{(2)} = -\frac{c_2^2}{4c_1^2} R^i{}_{hj0} R^h{}_{ik0} - \frac{c_2^2 d_1}{2c_1^2(c_1 + r^2 d_1)} R_{h0j0} R^h{}_{0k0} \\ &+ \frac{2c_1^2(n-2) - 2c_1[\rho c_1 c_2 - d_1(n-2)]r^2 - d_1(2\rho c_1 c_2 + d_1)r^4}{2c_1 r^2(c_1 + r^2 d_1)} \left(g_{jk} - \frac{1}{r^2} g_{0j}g_{0k}\right), \end{aligned}$$

$$\begin{aligned}
&RicHH_{jk} - \rho G^{(1)}_{jk} - \sigma\eta(\delta_j)\eta(\delta_k) = \frac{2c_1(\rho c_1 c_2 + d_1) + d_1(2\rho c_1 c_2 + d_1)r^2}{2c_2(c_1 + r^2 d_1)} g_{jk} \\
&(4.5) \quad + Ric_{jk} + \left\{ \frac{d_1\{2c_1^2 n + c_1[3d_1 n - 2(\rho c_1 c_2 + d_1)]r^2 + d_1[d_1(n-1) - 2\rho c_1 c_2]r^4\}}{2c_1 c_2 r^2(c_1 + r^2 d_1)} \right. \\
&\left. - \frac{\sigma(c_1 + d_1 r^2)^2}{\lambda^2 r^4} \right\} g_{0j} g_{0k} + \frac{c_2}{2c_1} R^l_{kh0} R^h_{0jl} + \frac{c_2 d_1}{2c_1(c_1 + d_1 r^2)} R_{h0k0} R^h_{0j0}.
\end{aligned}$$

In the sequel we shall prove the main theorem of this paper.

Theorem 4.1. *The Sasakian manifold $(T_r M, \varphi, \xi, \eta, G)$, with φ, ξ, η, and G given respectively by (3.6), (3.7), and (3.8), is $\eta-$Einstein if and only if*

$$Case\ I)\ c_1 = cc_2 r^2,\ \rho = cc_2(2n-3) - \frac{d_1}{2cc_2^2 r^2},\ \sigma = \lambda^2 \frac{d_1 n - cc_2(n-2)}{2cc_2^2(cc_2 + d_1)r^2};$$

$$CaseII)\ d_1 = \frac{cc_2 r^2 - c_1(n-2)}{(n-1)r^2}, \rho = \frac{n(n-2)(c_1 + cc_2 r^2)}{2c_1 c_2(n-1)r^2},$$

$$\sigma = \lambda^2 \frac{2r^2 cc_1 c_2\{n[n(n-4)+6] - 2\} - n^2(n-2)(c_1^2 + c^2 c_2^2 r^4)}{2c_1 c_2(c_1 + cc_2 r^2)^2},$$

where $c = \frac{a_1^2}{r^2}$ is the constant sectional curvature of the base manifold M.
Replacing c by the the mentioned value, the cases become

$$Case\ I)\ c_1 = a_1^2 c_2, \rho = -\frac{\frac{d_1}{a_1^2} + \frac{c_2(3-2n)}{r^2}}{2c_2^2}, \sigma = \lambda^2 \frac{d_1 n r^2 - a_1^2 c_2(n-2)}{2a_1^2 c_2^2(a_1^2 c_2 + d_1 r^2)};$$

$$Case\ II)\ d_1 = \frac{a_1^2 c_2 - c_1(n-2)}{(n-1)r^2}, \rho = \frac{(c_1 + a_1^2 c_2)(n-2)n}{2c_1 c_2(n-1)r^2},$$

$$\sigma = -\lambda^2 \frac{n^2(n-2)(c_1^2 + a_1^4 c_2^2) - 2a_1^2 c_1 c_2\{n[n(n-4)+6] - 2\}}{2c_1 c_2(c_1 + a_1^2 c_2)^2}.$$

Proof: If $(T_r M, \varphi, \xi, \eta, G)$, is Sasakian, the base manifold has constant sectional curvature c, so the relation (4.4) becomes

$$RicTT_{jk} - \rho G^{(2)}_{jk} = \frac{2c_1(n - 2 - \rho r^2 c_2)(c_1 + r^2 d_1) + r^4(c^2 c_2^2 - d_1^2)}{2c_1 c_2 r^2(c_1 + r^2 d_1)} \left(g_{jk} - \frac{1}{r^2} g_{0j} g_{0k}\right).$$

The above quantity vanishes if and only if the function ρ has the expression

$$\rho = \frac{2c_1(n-2)(c_1 + r^2 d_1) + r^4(c^2 c_2^2 - d_1^2)}{2c_1 c_2 r^2(c_1 + r^2 d_1)}.$$

Replacing this value into (4.5), and taking into account that the base manifold has constant sectional curvature c, we obtain

$$\begin{aligned}
&RicHH_{jk} - \rho G^{(1)}_{jk} - \sigma\eta(\delta_j)\eta(\delta_k) = -\frac{(c_1 - cc_2 r^2)[c_1(n-2) + d_1 r^2(n-1) - cc_2 r^2]}{r^2 c_2(c_1 + r^2 d_1)} g_{jk} \\
&(4.6) \\
&+ \frac{\lambda^2 r^2\{4c_1^2 d_1 - c_1[c^2 c_2^2(n-2) - d_1^2(n+2)]r^2 - r^4 d_1(c^2 c_2^2 - d_1^2)n\} - 2\sigma c_1 c_2(c_1 + r^2 d_1)^3}{2r^4 \lambda^2 c_1 c_2(c_1 + r^2 d_1)} g_{0j} g_{0k}.
\end{aligned}$$

From the relation (4.6), we have that

$$RicHH_{jk} = \rho G_{jk}^{(1)} + \sigma\eta(\delta_j)\eta(\delta_k)$$

if and only if the coefficients of g_{jk} and $g_{0j}g_{0k}$ vanish.

From the vanishing condition of the first coefficient in (4.6), we obtain two cases:
I) $c_1 = cc_2 r^2$, and *II)* $d_1 = \frac{cc_2 r^2 - c_1(n-2)}{(n-1)r^2}$.

In the first case the numerator of the second coefficient in (4.6) becomes

$$-(cc_2 + d_1)^2 r^6 \{\lambda^2[cc_2(n-2) - d_1 n] + 2\sigma cc_2^2(cc_2 + d_1)r^2\},$$

and it vanishes if and only if *I.1)* $d_1 = -cc_2$, or *I.2)* $\sigma = \lambda^2 \frac{d_1 n - cc_2(n-2)}{2cc_2^2(cc_2+d_1)r^2}$. The subcase $d_1 = -cc_2$ reduces to $c_1 + r^2 d_1 = 0$, which should be excluded, since the constant ρ, some components of K, as well as the Levi Civita connection are not defined. Thus, case I reduces to the subcase I.2, and the values of c_1, ρ, σ are those presented in the theorem.

In the second case the coefficient of $g_{0j}g_{0k}$ in (4.6) has the numerator equal to

$$-(c_1 + cc_2 r^2)\left\{2\sigma c_1 c_2(c_1 + cc_2 r^2)^2 + \lambda^2\left[n^2(n-2)(c_1^2 + r^4 c^2 c_2^2) - 2cc_1 c_2\big[n[6 + (n-4)n] - 2\big]r^2\right]\right\}.$$

Thus the subcases are

$$II.1) \quad c_1 = -cc_2 r^2,$$

$$II.2) \quad \sigma = \lambda^2 \frac{2r^2 cc_1 c_2\{n[n(n-4)+6] - 2\} - n^2(n-2)(c_1^2 + c^2 c_2^2 r^4)}{2c_1 c_2(c_1 + cc_2 r^2)^2}.$$

The first subcase reduces again to $c_1 + r^2 d_1 = 0$, which should be excluded. Hence the case *II* presented in the theorem is practically the subcase *II.2*.

Thus the theorem is proved.

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