Tietze-type theorem in 2-dimensional Riemannian manifolds without conjugate points

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Abstract. Let A be an open connected subset of a C^∞ complete simply connected 2-dimensional Riemannian manifold without conjugate points W^2. The main result of this short article states that: a point x of A has a local maximal visibility if and only if x is a point of the convex kernel of A. Thus we obtain a Tietze-type theorem in W^2.

M.S.C. 2010: 52A10, 52A30, 53B20.
Key words: maximal visibility, starshaped set, kernel, conjugate points, Tietze-type theorem.

1 Introduction

The first local versus global result involving usual convexity is due to Tietze [10]. Tietze and Nakajima proved that a closed connected locally convex set in Euclidean space is convex, thus they established a global property from a local one [9, 12, 11, 16, 17]. In [16], the authors obtained similar results in which local convexity was replaced by weaker conditions called $C-$convexity and strong local $C-$convexity. In [12], the authors proved that under certain conditions a starshaped set is characterized by the existence of points enjoying a local condition, maximal visibility. Using maximal visibility, J. Cell presented a similar result for open connected set and for its closure and he obtained a Tietze-type theorem for partially convex planar set [10]. M. Breen studied the union of starshaped sets and the union of orthogonally starshaped sets using the concept of local maximal visibility in the plane [9]. In the present work we get a Tietze-type theorem for open connected subsets of a C^∞ complete simply connected 2-dimensional Riemannian manifold without conjugate points W^2.

Now, we introduce some properties of C^∞ complete simply connected n-dimensional Riemannian manifold without conjugate points W^n. At first recall that, by the well-known Hopf-Rinow theorem, if a Riemannian manifold is complete, then it is geodesically connected. Moreover, any two points p and q can be joined by a minimal geodesic. So it is worth pointing out that in order to obtain convexity in Riemannian manifolds, the assumption of completeness can not be removed [2]. The behavior of

Balkan Journal of Geometry and Its Applications, Vol.16, No.2, 2011, pp. 133-137.

geodesics in manifolds without conjugate or focal points has been discussed by many geometers as Morse, Hedlund, Green, Eberlein and others [13]. Now let W^n be a C^∞ complete simply connected n-dimensional Riemannian manifold without conjugate points. The Euclidean space as well as the Hyperbolic space are examples of complete Riemannian manifolds without conjugate points. In such Riemannian manifolds W^n, no two geodesics intersect twice due to the absence of conjugate points and hence for any two different points p and q there is a unique and hence minimal closed geodesic segment, denoted by $[pq]$, joining them. This fact implies that the three types of convexities in complete Riemannian manifolds that were introduced in [1] are identical in $W^n$and each of them is equivalent to the classical concept of convexity in the Euclidean space E^n. In the following we introduce this classical concept of convexity in W^n. For more properties of W^n and convex sets in it see [4, 3, 6, 7, 8, 18, 14, 15].

2 Notations and definitions

Let A be a subset of W^n. We say that A is starshaped if there exists a point p in A such that for any point x in A the closed geodesic segment$[px]$, joining p and x, is in A. In this case we say that p sees x via A. The set of all such points p is called the kernel of A and is denoted by $\ker A$. M. Beltagy proved that $\ker A$ is convex for $n = 2$[6]. A is convex if $\ker A = A$ i.e. for each $x, y \in A$, the closed geodesic segment $[xy]$ joining them is contained in A and hence x sees y via A. The open and closed geodesic discs are both convex sets in W^n.

A set A is said to be locally convex at a point p in A if there exists a neighborhood N of p such that $N \cap A$ is convex. It is clear that the open set is a locally convex set. The convex hull of a set A is the smallest convex set that contains A and is denoted by $C(A)$. It is clear that $C(A) = A$ when A is convex. Let A_p be the set of all points x of A that p sees via A. We say that p has higher visibility via A than q if $A_q \subset A_p$. A point of (local) maximal visibility of A is a point $p \in A$ such that there exists a neighborhood N of p satisfying that p has higher visibility than any other point of $N \cap A$ [10].

A geodesic path between two points p and q is the union of n closed geodesic segments $[x_i x_{i+1}], 0 \leq i \leq n-1$ where x_i's are distinct points of W^n with $x_0 = p$ and $x_n = q$. Every geodesic segment $[x_i x_{i+1}]$ is called a side of the geodesic path. A set A is called geodesically connected if for each two points x and y in A there exists a geodesic path in A joining x and y [5]. $\overrightarrow{xy}$ denotes the geodesic ray starting from x and passing through y, where (xy) denotes the open geodesic segment joining x and y.

3 Maximal visibility in W^2

In this section we present the main theorem of this paper that introduces the assumptions of a subset A of W^2 to get a characterization of the kernel of A using the concept of local maximal visibility. We begin with the following two lemmas.

Lemma 3.1. *If A is a nonempty open connected subset of W^2, then A is geodesically connected.*

Proof. Let $p \in A$ and let A_p denote the set of all points in A which can be joined to p by a geodesic path in A. We claim that A_p is both open and closed as a subset of A. To see that A_p is open, let $q \in A_p$. Since A is locally convex, there exists a neighborhood N of q such that $N \cap A$ is convex. It follows that each point of $N \cap A$ can be joined to q and hence to p by a geodesic path. Thus $N \cap A$ (as an open set in the relative topology) is a subset of A_p and A_p is open in A. To see that A_p is closed, let $z \in \bar{A}_p$. Since A is locally convex, there exist a neighborhood N of z such that $N \cap A$ is convex and is a neighborhood of z in the relative topology on A, and hence $N \cap A$ must also intersect A_p since $z \in \bar{A}_p$. Thus there exists a point w in $(N \cap A) \cap A_p$. Since $N \cap A$ is convex, $[wz] \subset A$. But w can be joined to p by a geodesic path in A, hence z can also. Thus $z \in A_p$, and A_p is closed. Since A_p is closed and open in A and A is connected, it follows that A_p must equal to A and hence A is geodesically connected. □

Lemma 3.2. *Let A be an open connected subset of W^2. If $[xy] \subset A$ and $[yz] \subset A$, then there exists a point q in $[xy]$ with $q \neq y$, such that the convex hull of $\{q, y, z\}$ is contained in A.*

Proof. Let K be the set of all points p in $[yz]$ such that $C\{q, y, p\} \subset A$ for some q in $[xy]$ with $q \neq y$. Since $y \in K$, K is not empty. We claim that K is both open and closed in $[yz]$. Since $[yz]$ is connected, this claim implies that $K = [yz]$ and the proof is complete. To see that K is open in $[yz]$, let $p \in K$. Then there exists a point $q \in [xy)$ such that $C\{q, y, p\} \subset A$. Since A is open and hence locally convex then there is a neighborhood N of p such that $N \cap A$ is convex. Let $a \in N \cap [pz]$ and $b \in N \cap [pq]$, then the ray $\overrightarrow{ab}$ meets $[yq]$ at f, and hence $C\{a, y, f\} \subset A$, since the convex hull of $C\{a, b, p\}$ is contained in $N \cap A \subset A$ see Figure 1.

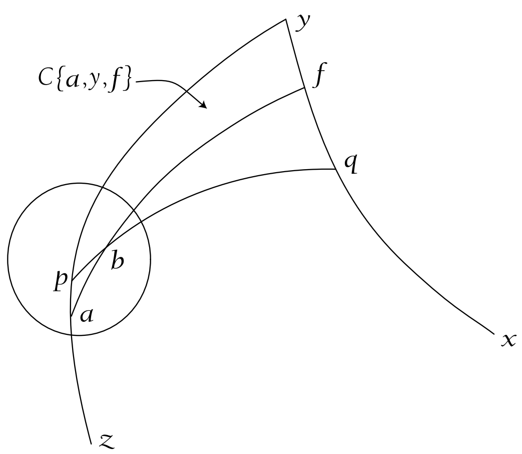

Figure 1: The set K is open in $[yz]$

Therefore, $a \in K$ and consequently p is an interior point in K. To see that K is closed in $[yz]$, let $p \in \bar{K}$. Since A is locally convex, then there is a neighborhood N of p such that $N \cap A$ is convex. Moreover, N contains a point a of K such that

$a \in N \cap [yp]$. Then there exists a point $q \in [xy)$ such that $C\{q, y, a\} \subset A$. Choose $b \in N \cap [aq]$, then the ray $\overrightarrow{pb}$ meets $[qy)$ at f see Figure 2.

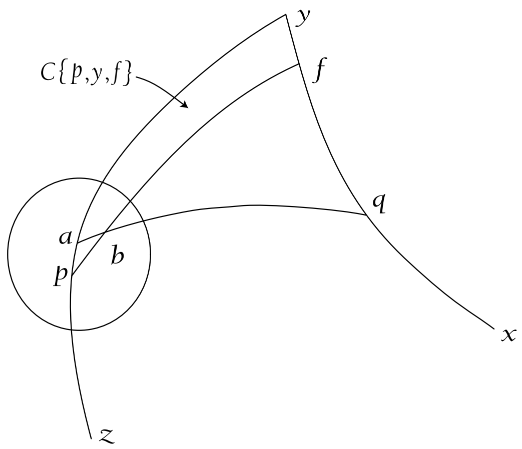

Figure 2: The set K is closed in $[yz]$

Since $C\{a, q, y\} \subset A$ by definition of K and $C\{a, p, b\} \subset A$ by local convexity, then $C\{p, f, y\} \subset C\{a, q, y\} \cup C\{a, p, b\} \subset A$ and hence $p \in K$. Then K is closed and the proof is complete. □

Theorem 3.3. *Let A be an open connected subset of W^2. Then the kernel of A is the set of all points of maximal visibility.*

Proof. Let V be the set of all points of maximal visibility in A. We want to prove that $V = \ker A$. It is clear that $\ker A \subset V$, so we will show that $V \subset \ker A$. Let $x \notin \ker A$ i.e. there is a point y in A such that $[xy] \not\subset A$. By Lemma 3.1, A is geodesically connected since A is an open connected subset of W^2. Therefore, there is a geodesic path with n sides such that $x = x_0, x_1, ..., x_n = y$ and

$$U_{i=0}^{n-1}[x_i x_{i+1}] \subset A$$

Choose a geodesic path P with minimal n and so P must be simple (dose not intersect itself). Now, the points x_0, x_1, x_2 are non-geodesic triple. By Lemma 3.2, there exists a point t in $[x_1x_2]$ such that $C([x_0x_1] \cup [x_1t]) \subset A$. Let M be the set of all such points t in $[x_1x_2]$. It is clear that M is convex, so we get a point z such that $M = [xz]$ or $M = [xz)$. Since A is open, $M = [xz)$. Now, for any neighborhood N of x, all points of $N \cap [x_0x_1]$ see z via A where x does not i.e. x is not a point of maximal visibility in A and therefore x is not in V. Hence $V \subset \ker A$ and the proof is complete. □

Theorem 3.3 is valid in the Euclidean space E^n as a manifold without conjugate points[12]. Also it is valid in the hyperbolic space H^n since the Beltrami (or central projection) map defined in [5] takes H^n to E^n and preserves geodesics. But the generalization of Theorem 3.3 to any W^n is more difficult and is left as an open question.

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