Basic evolution PDEs in Riemannian geometry

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Abstract. The purpose of this paper is threefold: (1) to analyze the Riemann flow and the Riemann wave; (2) to introduce and study the generalized Riemann flow; (3) to define the Einstein flow, the Einstein wave and Einstein inequalities. This means to control the geometric quantities associated to a Riemannian metric as it evolves with respect to a parameter via a geometric PDE. The evolution PDEs lead to some families of the Riemannian metrics: expanding, collapsing, solitonic, geodesically, conformally or concircular related. This approach and our open problems on Riemann waves and sectional curvature, Einstein flows, Einstein waves, multitime flows, multitime waves, and multitime solitons inaugurate new understandings of certain phenomena in differential geometry and physics.

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1 Introduction

Hamilton published some groundbreaking papers [4], [5], introducing and studying the concept of the Ricci flow. This was the first means to study the geometric quantities associated to a metric $g(x,t)$, $(x,t) \in M \times R$ as the metric evolves via a PDE, where M is a differentiable manifold. For a Riemannian manifold $(M, g_0(x))$ the *Ricci flow PDE* is

$$\frac{\partial g}{\partial t}(x,t) = -2\,S(g(x,t)),\; g(x,0) = g_0(x),$$

where $S(g(x,t))$ denotes the Ricci curvature tensor associated to the metric $g(x,t)$. Hamilton [5] proved that closed 3-manifolds, which admit metrics of strictly positive Ricci curvature, are diffeomorphic to quotients of the round 3-sphere by finite groups of isometries acting freely.

The papers of Hamilton give a new perspective of understanding to differential geometers and other mathematicians to introduce and study geometrical evolution PDEs. Usually, the idea is to evolve the metric in some way that will make the

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manifold "what we want". The main hope is to underline topological properties of the manifolds from the existence of such round metrics.

The Hamilton Ricci flow was used by Perelman [12], [13] to prove the geometrization and Poincaré conjectures, i.e., every simply connected closed 3-dimensional manifold is homeomorphic to the 3-sphere. Also, other important mathematicians contributed to the subject from different perspectives [2], [14], [3], [30].

Kong and Liu [9] studied the *wave character of metrics* (ultra-hyperbolic PDEs).

Recently, the second author [19], [23] introduced and studied two touchstone notions in Riemannian geometry: *Riemann flow* and *Riemann wave* via the bialternate product Riemannian metric. The aims of this work are: (1) to continue with new properties of the Riemann flow or wave, (2) to find special classes of metrics determined by some Riemann type flows or waves; (3) to introduce the Einstein flow and the Einstein wave.

Section 2 classifies the ultra-hyperbolic-parabolic geometric evolutions and recall some properties of Riemann flows and Riemann waves. The relation between a Riemann wave and the sectional curvature is formulated as an open problem. Section 3 introduces and studies T-Riemann type flows, T-Riemann solitons and classes of metrics determined by generalized Riemann type flows. Section 4 proposes the open problems of Einstein flows, Einstein waves, Einstein inequalities and multitime flows, multitime waves, multitime solitons.

2 Geometric evolutions

2.1 Ultra-hyperbolic-parabolic Ricci evolutions

The Ricci flow is a powerful tool to understand the geometry and topology of some Riemann manifolds. Any solution of Ricci flow equation will help us to understand its behavior for general cases and the singularity formation, further the basic topological and geometrical properties as well as analytic properties of the underlying manifolds.

An ultra-hyperbolic Ricci evolution is the Ricci wave. The Ricci flow and the Ricci wave PDEs are prolongations of the Einstein equation, which plays significant role in general relativity and modern theoretical physics. Any solution of them can help us to find new solutions of the Einstein equation.

Let $S(g(x,t))$ be the Ricci tensor associated to the metric $g(x,t)$. Both, the Ricci flow and the Ricci wave PDEs are particular cases of the following PDEs system

$$\alpha(x,t)\,\frac{\partial^2 g}{\partial t^2}(x,t)+\beta(x,t)\,\frac{\partial g}{\partial t}(x,t)+\gamma(x,t)\,g(x,t)+2S(g(x,t))=0,$$

where α,β,γ are certain smooth functions. Particularly,

a) if $\alpha=1$; $\beta=\gamma=0$, then the formula goes back to the wave metric (ultra-hyperbolic equations) [9];

b) if $\alpha=0$; $\beta=1$, $\gamma=0$, then one gets the famous Ricci flow (ultra-parabolic equations) [4], [5], [12];

c) if $\alpha=0$; $\beta=0$, $\gamma=$ const, we obtain the case of Einstein PDEs (ultra-hyperbolic equations).

In this sense, the foregoing evolution equation represent a ultra-hyperbolic-parabolic evolution.

Theorem 2.1 [9]. *Let $g(x,t)$ be the Ricci wave and R_{ijkl}, $i,j,k,l = \overline{1,n}$ be the associated Riemann curvature tensor, S_{ij} be the Ricci tensor and ρ be the scalar curvature. Then the Ricci wave $g(x,t)$ is solution of the following PDEs*

$$\frac{\partial^2 R_{ijkl}}{\partial t^2} = \Delta R_{ijkl} + \text{(lower order terms)}$$

$$\frac{\partial^2 S_{ij}}{\partial t^2} = \Delta S_{ij} + \text{(lower order terms)}$$

$$\frac{\partial^2 \rho}{\partial t^2} = \Delta \rho + \text{(lower order terms)},$$

where Δ is the Laplacian with respect to the evolving metric, the lower order terms only contain lower order derivatives of $g(x,t)$.

Example. Let us consider the initial Einstein metric

$$ds^2 = \frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2,$$

where k is a constant taking the values $-1, 0, 1$. The metric

$$ds^2 = (-2kt^2 + c_1 t + c_2)\left\{\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right\}$$

is a solution of the hyperbolic geometric flow, where c_1 and c_2 are two constants related by some conditions. It plays an important role in cosmology.

2.2 Ultra-hyperbolic-parabolic Riemann evolutions

For $(0,2)$-tensors a and b, their *Kulkarni-Nomizu product* $a \wedge b$ is given by

$$(a \wedge b)(X_1, X_2; X, Y) = a(X_1, X)b(X_2, Y) + a(X_2, Y)b(X_1, X)$$
$$-a(X_1, Y)b(X_2, X) - a(X_2, X)b(X_1, Y).$$

The Kulkarni-Nomizu product $G = \frac{1}{2}\, g \wedge g$ is the Riemann metric induced by g on 2-forms. It coincides to the *bialternate product Riemannian metric*

$$G = g \odot g, \;\; G_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}, \;\; i,j,k,l = \overline{1,n}.$$

If $n \geq 3$, then the bialternate product Riemannian metric G determines the Riemannian metric g (see [19], [23]).

The basic changes of the Riemannian metric work as follows: (1) a change of the form $\bar{g}_{ij} = g_{ij} + h_{ij}$ leads to $\bar{G} = G + H + g \wedge h$, where $H_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}$; (2) a conformal change $\bar{g} = e^{2\varphi}\, g$ gives $\bar{G} = e^{4\varphi}\, G$; (3) a change $g(x,t) = \varphi_t^* g(x,0)$, by a time-dependent family of diffeomorphisms $\varphi_t : M \mapsto M$, $\varphi_0 = id$, produces $G = \varphi_t^*\, G_0$.

Let $R(g(x,t))$ be the Riemann curvature tensor associated to the metric $g(x,t)$. The study of the Riemann flow PDE

$$\frac{\partial G}{\partial t}(x,t) = -2\, R(g(x,t))$$

and of the Riemann wave PDE

$$\frac{\partial^2 G}{\partial t^2}(x,t) = -2\,R(g(x,t))$$

started in the papers [19], [23]. These PDEs systems are particular forms of the following PDEs system

$$A(x,t)\,\frac{\partial^2 G}{\partial t^2}(x,t) + B(x,t)\,\frac{\partial G}{\partial t}(x,t) + C(x,t)\,G(x,t) + 2R(g(x,t)) = 0,$$

where A, B, C are certain smooth functions. The most familiar cases are:

a) if $A = 1$; $B = C = 0$, then the formula goes back to the wave metric (ultra-hyperbolic equations);

b) if $A = 0$; $B = 1$, $C = 0$, then one gets the Riemann flow (ultra-parabolic equations);

c) if $A = 0$; $B = 0$, $C = \text{const}$, we obtain the case of flat manifold PDEs (ultra-hyperbolic equations).

In this sense, the foregoing evolution PDE represent an ultra-hyperbolic-parabolic evolution.

2.3 Properties of the Riemann flow

Let $R(g(x,t))$ be the Riemann curvature tensor associated to the metric $g(x,t)$. The Riemann flow $g(x,t)$, i.e., a solution of the PDE

$$\frac{\partial G}{\partial t}(x,t) = -2\,R(g(x,t))$$

has the following properties [19], [23]:

Short time existence and uniqueness. Let (M, g_0) be a compact Riemann manifold. Then there exists $\epsilon > 0$ such that the initial value problem

$$\frac{\partial G}{\partial t}(x,t) = -2\,R(g(x,t)),\;\; g(x,0) = g_0$$

has a unique solution $g(x,t)$ on $M \times [0, \epsilon]$.

Expanding hyperbolic space. *If (M, g_0) is a Riemann manifold $(n \geq 2)$ of constant sectional curvature -1, then the evolution metric of the Riemann flow is $g(t) = (1 + (n-1)t)\,g_0$. The manifold expands homothetically for all time.*

Collapsing the sphere. *For the round unit sphere (S^n, g_0), $n \geq 2$, the evolution metric of the Riemann flow is $g(t) = (1 - (n-1)t)\,g_0$ and the sphere collapses to a point in finite time.*

Equilibrium points. *A Riemannian manifold is flat, i.e., local isometric to the Euclidean space, if and only if the Riemannian curvature tensor vanishes. The corresponding metric is an equilibrium point of the Riemann flow.*

Now let us comment the connection between a Riemann flow and the sectional curvature.

Riemann flow and sectional curvature. Given a Riemannian manifold (M, g) and two local linearly independent vector fields X and Y, the *sectional curvature* is defined by

$$K(X,Y) = \frac{R(X,Y,X,Y)}{G(X,Y,X,Y)}\,.$$

The sectional curvature is a further, equivalent but more geometrical, description of the curvature of Riemannian manifolds.

Supposing $g(x,t)$ is a Riemann flow and denoting

$$\sigma(x,t) = G(x,t)(X(x), Y(x), X(x), Y(x)),$$

we obtain the first order partial differential equation $\sigma_t(x,t) = -2K(x,t)\sigma(x,t)$. Imposing the initial condition $\sigma(x,0) = c(x)$, we find the solution

$$\sigma(x,t) = c(x) \exp\left(-2 \int_0^t K(x,s) ds\right).$$

2.4 Properties of the Riemann wave

Let $R(g(x,t))$ be the Riemann curvature tensor associated to the metric $g(x,t)$. The Riemann wave $g(x,t)$, i.e., a solution of the PDE

$$\frac{\partial^2 G}{\partial t^2}(x,t) = -2\, R(g(x,t))$$

has the following properties [19], [23]:

Short time existence and uniqueness. *Let $(M, g_0(x))$ be a compact Riemannian manifold and $k_1(x)$ be a $(0,2)$ symmetric tensor field on M. Then there exists a constant $\epsilon > 0$ such that the initial value problem*

$$\frac{\partial^2 G}{\partial t^2}(t,x) = -2\ R(g(t,x)),\ g(0,x) = g_0(x),\ \frac{\partial g}{\partial t}(0,x) = k_1(x)$$

has a unique smooth solution $g(t,x)$ on $M \times [0,\epsilon]$.

Expanding hyperbolic space. *If (M, g_0) is a Riemann manifold $(n \geq 2)$ of constant sectional curvature -1, then an evolution metric of the Riemann wave is $g(t) = (1 + vt - \frac{\lambda}{6}\, t^2)\, g_0$ and the manifold expands homothetically for all time.*

Collapsing the sphere. *For the round unit sphere (S^n, g_0), $n \geq 2$, the evolution metric of the Riemann wave is of the form $g(t) = f(t)\, g_0$, where $f : [0,T) \to R$ is a concave function, with $\lim_{t\to T} f(t) = 0$, and the sphere collapses to a point when $t \to T$.*

Steady state points. *A Riemannian manifold is flat, i.e., local isometric to the Euclidean space, if and only if the Riemannian curvature tensor vanishes. The corresponding metric is a steady state point of the Riemann wave.*

Now let us introduce the connection between a Riemann wave and the sectional curvature.

Riemann wave and sectional curvature. Supposing $g(x,t)$ is a Riemann wave and denoting

$$\sigma(x,t) = G(x,t)(X(x), Y(x), X(x), Y(x)),$$

we obtain the second order partial differential equation

$$\sigma_{t^2}(x,t) + 2K(x,t)\sigma(x,t) = 0.$$

Imposing the initial conditions $\sigma(x,0) = c(x)$, $\sigma_t(x,0) = v(x)$, we can adapt the theory in [8] to this geometric PDE (**open problem**).

3 Generalized Riemann flows

3.1 T-Riemann type flow

Let M be a smooth closed manifold endowed with a Riemann metric $g(x,t)$. Let T be a $(0,4)$-*generalized curvature tensor field* depending on $g(x,t)$, i.e., a tensor field which has the same symmetries as the Riemann curvature tensor and verifies the first Bianchi identity. A generalized Riemann flow or a T-Riemann type flow is a means of processing the Riemann metric $g(x,t)$ by allowing it to evolve under the PDEs system

$$\frac{\partial G}{\partial t}(x,t) = -2\,T(g(x,t)),\; g(x,0) = g_0(x).$$

Usually, the Riemann curvature tensor field R_{ijkl} splits into three pieces

$$R_{ijkl} = S_{ijkl} + E_{ijkl} + W_{ijkl},$$

which are called respectively the *scalar part*, the *semi-traceless part* and the *fully traceless part* [10].

The fully traceless part, i.e., the *Weyl curvature tensor field*

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}S_{jl} - g_{il}S_{jk} - g_{jk}S_{il} + g_{jl}S_{ik}) + \frac{\rho}{(n-1)(n-2)}G_{ijkl}$$

measures the deviation of the Riemann manifold from conformal flatness. If it vanishes, the manifold is (locally) conformally equivalent to a flat manifold. From physical point of view, the $(0,4)$-conformal Weyl curvature tensor field represents the part of the gravitational field which can propagate as a gravitational wave through a region containing no matter or nongravitational fields.

Theorem 3.1[19], [23]. *On a Riemannian manifold $(M, g(x,t))$ the Ricci type flow*

$$\frac{\partial g_{ij}}{\partial t}(x,t) = \alpha\, g_{ij}(x,t) + \beta\, S_{ij}(g(x,t))$$

determines a Riemann type flow

$$\frac{\partial G_{ijkl}}{\partial t}(x,t) = 2\alpha\, G_{ijkl}(x,t) + \beta\, E_{ijkl}(g(x,t)),$$

where α and β are functions on M, and $E_{ijkl}(g(x,t))$ is the semi-traceless part of the Riemann curvature tensor.

3.2 T-Riemann solitons

If the Riemann space is $T(g(t))$-flat, then the corresponding metric $g(x,t)$ is a fixed point of the T-Riemann type flow. We can regard as generalized fixed points of the T-Riemann type flow, those manifolds which change by a diffeomorphism and a rescaling.

Let $g(x,t)$ be a solution of a T- Riemann type flow on the manifold M. One considers

$$\varphi_t : M \mapsto M,\; \varphi_0 = id$$

to be a time-dependent family of diffeomorphisms and $\sigma(t)$ be a time dependent scale factor. If we have

$$g(x,t) = \sigma(t)\varphi_t^* g(x,0),\ \sigma(0) = 1,$$

then the solution $(M, g(x,t))$ or $(\Lambda^2(M), G(x,t) = g(x,t) \odot g(x,t))$ is called a *generalized Riemann soliton.*

Since

$$G_{ijkl} = \sigma^2(t)\varphi_t^* g_{ik}(0)\varphi_t^* g_{jl}(0) - \sigma^2(t)\varphi_t^* g_{il}(0)\varphi_t^* g_{jk}(0),$$

it follows

$$\frac{\partial G}{\partial t}(t)\ |_{t=0} = 2\sigma'(0)g(0) \odot g(0) + \mathcal{L}_V g(0) \odot g(0) + g(0) \odot \mathcal{L}_V g(0),$$

where $V = \frac{\partial \varphi_t}{\partial t}$ and $\mathcal{L}_V$ is the Lie derivative.

Theorem 3.2. *If $\sigma'(0) = \lambda$ and $V = \nabla f$, then the function f is a solution of the PDEs system*

$$T_{ijkl} + \lambda G_{ijkl}(0) + g_{jl}(0)\nabla_i\nabla_k f - g_{jk}(0)\nabla_i\nabla_l f + g_{ik}(0)\nabla_j\nabla_l f - g_{il}(0)\nabla_j\nabla_k f = 0.$$

The solutions $g(x,t)$ of a T- Riemann type flow on the manifold M are called *generalized gradient Riemann solitons.* A generalized gradient Riemann soliton is called *shrinking* if $\lambda < 0$, *static* if $\lambda = 0$, and *expanding* if $\lambda > 0$.

3.3 Classes of metrics determined by Riemann type flows

a) Geodesically related metrics

A diffeomorphism $f : V_n = (M, g) \mapsto \overline{V}_n = (M, \bar{g})$ is called *geodesic mapping* if it maps geodesics of the Riemannian metric g into geodesics of the Riemannian metric $\bar{g}$. There exists a geodesic mapping f if and only if the Weyl formulae are satisfied

$$\overline{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X,$$

where $\psi \in \wedge^1(M)$.

The $(0,4)$-*projective Weyl curvature tensor*

$$P(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-1}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)]$$

is invariant under the geodesic mappings, i.e., $\overline{P} = P$.

The pseudo-symmetric Riemannian spaces, for which the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent at every point of the manifold, constitute a generalization of spaces of constant sectional curvature, along the line of locally symmetric ($\nabla R = 0$) and semi-symmetric spaces ($R \cdot R = 0$) [6]. We have

$$(R \cdot R)(X_1, \ldots, X_4; X, Y) = (R(X,Y) \cdot R)(X_1, \ldots, X_4) =$$

$$= -R(R(X,Y)X_1, \ldots, X_4) - \cdots - R(X_1, \ldots, R(X,Y)X_4),$$

$$Q(g,R)(X_1, \ldots, X_4; X, Y) = -((X \wedge Y) \cdot R)(X_1, \ldots, X_4) =$$

$$= R((X \wedge Y)X_1, \ldots, X_4) + \cdots + R(X_1, \ldots, (X \wedge Y)X_4),$$

where $(X \wedge_g Y)U = g(U,Y)X - g(U,X)Y$.

This notion arose during the study of totally umbilical submanifolds of semi-symmetric spaces, as well as during the consideration of geodesic mappings.

Theorem 3.4. *Let $(M, g_0(x))$ be a Riemann manifold. The class $g(x,t)$ of geodesically related metrics with g_0, given by the P-Riemann type flow, satisfies*

$$G(x,t) = -2P(g_0(x))\, t + G_0(x).$$

Moreover, if $(M, g_0(x))$ is a pseudo-symmetric space, then $(M, g(x,t))$ is also pseudo-symmetric.

Proof. Implicit solution of a Cauchy problem associated to a P-Riemann type flow. □

b) Concircular transformations of metrics

Let $g \mapsto e^{2u} g$ be a conformal transformation of the metric g on the Riemann space (M,g) [7]. The *tensor field of the conformal change* $B \in \mathcal{T}^{0,2}(M)$ has the components $B_{ij} = u_{i,j} - u_i u_j$, $u_i = \frac{\partial u}{\partial x^i}$, $i,j = \overline{1,n}$. If $B = \frac{1}{n}\, Tr(B) g$, then the conformal change is called *concircular transformation.*

A concircular transformation carries all the circles of the manifold into circles (a curve in a Riemannian manifold is called *circle* when the first curvature is constant and all the other curvatures are identically zero).

The $(0,4)$-*concircular curvature tensor*

$$Z(X,Y,Z,W) = R(X,Y,Z,W) - \frac{\rho}{n(n-1)} G(X,Y,Z,W)$$

is invariant under concircular transformations, where ρ is the scalar curvature.

Theorem 3.5. *Let $(M, g_0(x))$ be a Riemann manifold. The class $g(x,t)$ of concircular related metrics with $g_0(x)$, given by the Z-Riemann type flow, satisfies*

$$G(x,t) = -2Z(g_0(x))\, t + G_0(x).$$

Proof. Implicit solution of a Cauchy problem associated to a Z-Riemann type flow. □

c) Conharmonic transformations of metrics

It is known that a *harmonic function* is defined as a function whose Laplacian vanishes. In general, a harmonic function does not transform into a harmonic function, by a conformal change of the Riemannian metric.

The condition under which the harmonic functions remain invariant have been studied by Ishii, who introduced the *conharmonic transformation* as a subgroup of the conformal transformations satisfying the condition

$$u_{ij} = u_{i,j} - u_i u_j + \frac{1}{2} u_k u^k g_{ij} = 0, \;\; i,j,k = \overline{1,n}.$$

The $(0,4)$-*conharmonic curvature tensor*

$$C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}[g(Y,Z)S(X,W)$$

$$-g(X,Z)S(Y,W)+g(X,W)S(Y,Z)-g(Y,W)S(X,Z)]$$

is invariant under conharmonic transformations.

Theorem 3.6. *Let $(M, g_0(x))$ be a Riemann manifold. The class $g(x,t)$ of conharmonically related metrics with $g_0(x)$, given by the C-Riemann type flow, satisfies*

$$G(x,t) = -2C(g_0(x))\,t + G_0(x).$$

Proof. Implicit solution of a Cauchy problem associated to a C-Rieman type flow.□

4 Open Problems

4.1 Einstein flow and Einstein wave

Let $(M, g_{\alpha\beta})$, $\alpha, \beta = \overline{1,4}$, be a spacetime. The metric $g_{\alpha\beta}$ determines the *Ricci curvature tensor* $S_{\alpha\beta}$ and the *scalar curvature* ρ. If $T_{\alpha\beta}$ is the *stress-energy tensor*, Λ is the *cosmological constant*, G is the *Newton gravitational constant*, c is the *speed of light* in vacuum, then the *Einstein field equations* (EFE) or *Einstein PDEs*

$$(EFE) \qquad S_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\rho + g_{\alpha\beta}\Lambda - \frac{8\pi G}{c^4}\, T_{\alpha\beta} = 0$$

(set of 10 PDEs) describe the fundamental interaction of gravitation as a result of spacetime being curved by matter and energy. Solution techniques for the EFE include simplifying assumptions such as symmetry. Special classes of exact solutions in general relativity model many gravitational phenomena, such as rotating black holes and the expanding universe. Further simplification is achieved taking $g_{\alpha\beta}(\epsilon)$ as a differentiable variation of a solution $g_{\alpha\beta}(0)$. We denote $\frac{\partial g_{\alpha\beta}}{\partial \epsilon}|_{\epsilon=0} = h_{\alpha\beta}$. The relations $g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha^\gamma$ implies

$$\frac{\partial g^{\lambda\gamma}}{\partial \epsilon}|_{\epsilon=0} = -h_{\alpha\beta}g^{\beta\gamma}g^{\alpha\lambda} = -h^{\lambda\gamma}.$$

In EFE, we take the partial derivative with respect to ϵ and set $\epsilon = 0$, we find the *infinitesimal deformation of EFE* or the *linearized EFE* around a solution $g_{\alpha\beta}(0)$. These equations are used to study phenomena such as *gravitational waves.*

Let us define the *Einstein flow* $g(\tau)$ by

$$\frac{\partial g_{\alpha\beta}}{\partial \tau} = S_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\rho + g_{\alpha\beta}\Lambda - \frac{8\pi G}{c^4}\, T_{\alpha\beta},$$

where τ is an appropriate evolution parameter (for example, time, mass etc). Similarly, we introduce the *Einstein wave* $g(\tau)$ defined by

$$\frac{\partial^2 g_{\alpha\beta}}{\partial \tau^2} = S_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\rho + g_{\alpha\beta}\Lambda - \frac{8\pi G}{c^4}\, T_{\alpha\beta}.$$

The solutions $g(0)$ of Einstein PDEs are *steady states* for the Einstein flow PDE or, respectively, for the Einstein wave PDE.

Study the properties of Einstein flow and Einstein wave. Extend the ideas to black holes.

4.2 Einstein partial differential inequalities

Study the spacetime characterized by the partial differential inequations [25]

$$S_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\rho + g_{\alpha\beta}\Lambda \geq \frac{8\pi G}{c^4}\, T_{\alpha\beta},$$

the inequality being in the sense of positive semidefinite matrix. Generally, study the geometric entities related by partial differential inequalities, as defined in [25].

4.3 Multitime flows, multitime waves, multitime solitons

Study the multitime flows, waves, solitons having in mind the model of multitime sine-Gordon solitons via geometric characteristics [11]. For related subjects, see also [1], [6] - [10], [29] - [26], [27]-[22].

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