The curvatures of lightlike hypersurfaces of an indefinite Kenmotsu manifold

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Abstract. We study the forms of curvatures of lightlike hypersurfaces M of an indefinite Kenmotsu manifold $\bar{M}$ subject to the conditions: (1) M is locally symmetric, i.e., the curvature tensor R of M be parallel on TM, or (2) M is a semi-symmetric manifold, i.e., $R(X,Y)R = 0$ on TM.

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1 Introduction

In the classical theory of Sasakian manifolds, the following result is well-known: If a Sasakian manifold is locally symmetric, then it is of constant positive curvature 1 [9]. Recently we studied the forms of curvatures of locally symmetric lightlike hypersurfaces M of an indefinite Sasakian manifold [7]. We obtained the following result: If M is totally geodesic, then it is of constant positive curvature 1.

Further in 1971, K. Kenmotsu proved the following result [8]: If a Kenmotsu manifold is locally symmetric, then it is of constant negative curvature -1.

The objective of this paper is the study of curvatures of lightlike hypersurfaces of an indefinite Kenmotsu manifold subject to the conditions: (1) M is locally symmetric, i.e., the curvature tensor R of M be parallel on TM, or (2) M is a semi-symmetric manifold, i.e., $R(X,Y)R = 0$ on TM. We prove the following results:

Theorem 1.1. *Let M be a locally symmetric lightlike hypersurface of an indefinite Kenmotsu manifold $\bar{M}$ equipped with an almost contact metric structure $(J, \zeta, \theta, \bar{g})$.*

(1) *If the structure vector field ζ is tangent to M, then M is a totally geodesic space of constant negative curvature -1. In this case, the induced connection on M is a unique torsion-free metric connection, the transversal connection of M is flat and the Ricci type tensor of M is an induced symmetric Ricci tensor on M.*

(2) *The screen distribution $S(TM)$ of M is not totally geodesic in M.*

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Theorem 1.2. *Let M be a semi-symmetric lightlike hypersurface of an indefinite Kenmotsu manifold $\bar{M}$.*

(1) *If ζ is tangent to M, then M is a totally geodesic space of constant negative curvature -1. In this case, the induced connection on M is a unique torsion-free metric connection on M, the transversal connection of M is flat and the Ricci type tensor of M is an induced symmetric Ricci tensor on M.*

(2) *If $S(TM)$ is totally geodesic in M, the projection $Proj\zeta$ of ζ on M is a null vector field on M. Moreover if the transversal connection of M is flat, then M is totally umbilical and the curvature tensor R of M is given by*

$$R(X,Y)Z = 2\theta(Z)\{\theta(X)Y - \theta(Y)X\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

2 Lightlike hypersurfaces

An odd dimensional semi-Riemannian manifold $\bar{M}$ is said to be an *indefinite almost contact metric manifold* [8, 10] if there exist a structure set $(J,\, \zeta,\, \theta,\, \bar{g})$, where J is a $(1,1)$-type tensor field, ζ is a vector field which called the characteristic vector field, θ is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ such that

$$\begin{aligned} &J^2X = -X + \theta(X)\zeta, \quad J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\zeta) = 1, \\ &\theta(X) = \bar{g}(\zeta, X), \quad \bar{g}(JX, JY) = \bar{g}(X,Y) - \theta(X)\theta(Y), \end{aligned} \tag{2.1}$$

for any vector fields X, Y on $\bar{M}$. An indefinite almost contact metric manifold $\bar{M}$ is called an *indefinite Kenmotsu manifold* [8, 10] if

$$\bar{\nabla}_X\zeta = -X + \theta(X)\zeta, \tag{2.2}$$

$$(\bar{\nabla}_X J)Y = -\bar{g}(JX, Y)\zeta + \theta(Y)JX, \tag{2.3}$$

for any vector fields X, Y on $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$.

A hypersurface M of an indefinite Kenmotsu manifold $\bar{M}$ is called a *lightlike hypersurface* if the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM of M, of rank 1. Then there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^\perp$ in TM, called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{orth} S(TM), \tag{2.4}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(\bar{M})$ the algebra of smooth functions on $\bar{M}$ and by $\Gamma(E)$ the $F(\bar{M})$ module of smooth sections of a vector bundle E over $\bar{M}$. It is well-known [2] that, for any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ of rank 1 in the orthogonal complement $S(TM)^\perp$ of $S(TM)$ in $\bar{M}$ satisfying

$$\bar{g}\,(\xi, N) = 1, \quad \bar{g}(N,N) = \bar{g}(N,X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.5}$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen $S(TM)$ respectively.

Let P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.4). Then the local Gauss and Weingartan formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.6}$$

$$\bar{\nabla}_X N = -A_{_N} X + \tau(X)N\,; \tag{2.7}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.8}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.9}$$

for all $X,\,Y \in \Gamma(TM)$, where ∇ and ∇^* are the liner connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, $A_{_N}$ and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric on TM. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y,\,\xi)$ for all $X,\,Y \in \Gamma(TM)$, we show that B is independent of the choice of a screen distribution and satisfies

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.10}$$

Two local second fundamental forms B and C are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.11}$$

$$C(X,PY) = g(A_N X, PY), \quad \bar{g}(A_{_N} X, N) = 0. \tag{2.12}$$

From (2.11), the operator A^*_ξ is $S(TM)$-valued self-adjoint such that $A^*_\xi \xi = 0$.

Definition 2.1. [2, 3, 4, 5, 6]. We say that M is *totally umbilical* if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth function β such that

$$B(X,Y) = \beta\, g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

We say that M is *totally geodesic* if $B = 0$ on $\mathcal{U}$. We also say that $S(TM)$ is *totally geodesic* in M if $C = 0$ on $\mathcal{U}$.

Example. In the case $\dim M = 2$, we have the following example. The lightlike cone Λ^2_0 of R^3_1 is a 2-dimensional totally umbilical lightlike hypersurface [2]. Except for this example, there are many examples of 2-dimensional totally umbilical 1-lightlike submanifolds. About it, see Example 1 and 2 in [3] and Example 6 in [4].

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{2.13}$$

for any $X,\,Y,\,Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM). \tag{2.14}$$

But the connection ∇^* on $S(TM)$ is metric. Using (2.6), (2.7) and (2.8), (2.9), for all $X, Y, Z \in \Gamma(TM)$, we get the Gauss-Codazzi equations of M and $S(TM)$

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_{N}Y - B(Y,Z)A_{N}X \\ &\quad + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,\end{aligned}\tag{2.15}$$

$$\begin{aligned}\bar{R}(X,Y)N &= -\nabla_X(A_{N}Y) + \nabla_Y(A_{N}X) + A_{N}[X,Y] + \tau(X)A_{N}Y \\ &\quad - \tau(Y)A_{N}X + \{B(Y,A_{N}X) - B(X,A_{N}Y) + 2d\tau(X,Y)\}N\,;\end{aligned}\tag{2.16}$$

$$\begin{aligned}R(X,Y)\xi &= -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] - \tau(X)A^*_\xi Y \\ &\quad + \tau(Y)A^*_\xi X + \{C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\tau(X,Y)\}\xi.\end{aligned}\tag{2.17}$$

A lightlike hypersurface $M = (M, g, \nabla)$ equipped with a degenerate metric g and a linear connection ∇ is said to be of *constant curvature* c if there exists a constant c such that the curvature tensor R of ∇ satisfies

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).\tag{2.18}$$

The induced Ricci type tensor $R^{(0,\,2)}$ of (M, g, ∇) is defined by

$$R^{(0,\,2)}(X,Y) = trace\{Z \longmapsto R(Z,X)Y\}, \quad \forall\, X,\, Y \in \Gamma(TM).$$

In general, $R^{(0,\,2)}$ is not symmetric [2, 4, 5]. A tensor field $R^{(0,\,2)}$ of M is called its *induced Ricci tensor*, denote Ric, of M if it is symmetric. It is well known that $R^{(0,\,2)}$ is symmetric if and only if the 1-form τ is closed, i.e., $d\tau = 0$ on TM [2].

For any $X \in \Gamma(TM)$, let $\nabla^\perp_X N = Q(\bar{\nabla}_X N)$, where Q is the projection morphism of $T\bar{M}$ on $tr(TM)$ with respect to the decomposition (2.5). Then $\nabla^\perp$ is a linear connection on the transversal vector bundle $tr(TM)$ of M. We say that $\nabla^\perp$ is the *transversal connection* of M. We define the curvature tensor $R^\perp$ of $tr(TM)$ by

$$R^\perp(X,Y)N = \nabla^\perp_X\nabla^\perp_Y N - \nabla^\perp_Y\nabla^\perp_X N - \nabla^\perp_{[X,Y]}N, \quad \forall\, X,\, Y \in \Gamma(TM).\tag{2.19}$$

If $R^\perp$ vanishes identically, then the transversal connection $\nabla^\perp$ is said to be *flat* [7].

Theorem 2.1. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. The following assertions are equivalent:*

(1) *The transversal connection of M is flat, i.e., $R^\perp = 0$.*
(2) *The 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*
(3) *The Ricci type tensor $R^{(0,\,2)}$ is an induced Ricci tensor of M.*

Proof. From (2.7) and the definition of the transversal connection $\nabla^\perp$, we have

$$\nabla^\perp_X N = \tau(X)N, \quad \forall\, X \in \Gamma(TM).$$

Substituting this equation into the right side of (2.19), we get

$$R^\perp(X,Y)N = 2d\tau(X,Y)N, \quad \forall\, X,\, Y \in \Gamma(TM).$$

From this result we deduce our assertion. □

3 Proof of Theorems

Proof of Theorem 1.1

Case (1): *Step 1.* Let ζ be tangent to M. It is well known [1] that if ζ is tangent to M, then it belongs to $S(TM)$. Replacing Y by ζ to (2.6) and using (2.2), we have

$$\nabla_X \zeta = -X + \theta(X)\zeta, \quad B(X,\zeta) = 0, \quad \forall\, X \in \Gamma(TM). \tag{3.1}$$

Substituting the first equation of (3.1) [denote $(3.1)_1$] into the right side of

$$R(X,Y)\zeta = \nabla_X \nabla_Y \zeta - \nabla_Y \nabla_X \zeta - \nabla_{[X,\,Y]}\zeta, \quad \forall\, X,\, Y \in \Gamma(TM)$$

and using (2.15), (3.1) and the fact ∇ is torsion-free, we have

$$\bar{R}(X,Y)\zeta = R(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta, \; \forall\, X,\, Y \in \Gamma(TM).$$

Taking the scalar product with ζ to this equation and using $g(\bar{R}(X,Y)\zeta,\zeta) = 0$ and (2.1), we show that the 1-form θ is closed on TM, i.e., $d\theta = 0$ on TM. Thus we get

$$R(X,Y)\zeta = \theta(X)Y - \theta(Y)X, \;\; \forall\, X,\, Y \in \Gamma(TM). \tag{3.2}$$

Applying $\bar{\nabla}_X$ to $\theta(Y) = g(Y,\zeta)$ and using (2.2), (2.6) and $\bar{g}(\zeta, N) = 0$, we have

$$(\nabla_X \theta)(Y) = -g(X,Y) + \theta(X)\theta(Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.3}$$

Step 2. Assume that M is locally symmetric. Apply ∇_Z to (3.2), we have

$$R(X,Y)\nabla_Z\zeta = (\nabla_Z\theta)(X)Y - (\nabla_Z\theta)(Y)X, \;\; \forall\, X,\, Y \in \Gamma(TM).$$

Substituting $(3.1)_1$ and (3.3) in this equation and using (3.2), we obtain

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{3.4}$$

Thus M is a space of constant negative curvature -1.

Applying ∇_U to (3.4) and using (3.4) and the fact $\nabla_U R = 0$, we have

$$(\nabla_U g)(X,Z)Y = (\nabla_U g)(Y,Z)X, \quad \forall\, X,\, Y,\, Z,\, U \in \Gamma(TM).$$

Taking $Z = Y = \xi$ to this equation and using (2.10) and (2.13), we have

$$B(X,Y) = 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Thus M is totally geodesic. By (2.13), ∇ is a torsion-free metric connection of M. Consider quasi-orthonormal frame fields $F = \{\xi,\, N,\, W_a\}$ and $F' = \{\xi',\, N',\, W'_a\}$ of $T\bar{M}$ induced on $\mathcal{U} \subset M$ by $\{S(TM),\, ltr(TM)\}$ and $\{S'(TM),\, ltr'(TM)\}$ respectively. By straightforward calculations [2, 5], we obtain the relationship between ∇ and ∇' induced by the Gauss and Weingarten equations with respect to $S(TM)$ and $S'(TM)$ as follows:

$$\nabla'_X Y = \nabla_X Y + B(X,\, Y) \left\{ \frac{1}{2} \left(\sum_{a=1}^{m} \epsilon_a (\mathbf{f}_a)^2 \right) \xi - \sum_{a=1}^{m} \mathbf{f}_a\, W_a \right\},$$

for all $X,\, Y \in \Gamma(TM)$, where ϵ_a is signature of W_a for each a and $\mathbf{f}_a$ are smooth functions on $\mathcal{U}$ such that $\mathbf{f}_a = \bar{g}(N',\, W_a)$. From this results we show that the induced connection ∇ of M is a unique torsion-free metric connection on M because of $B = 0$.

As $B = 0$, we have $A^*_\xi = 0$ due to (2.11). From (2.17), we get $R(X,Y)\xi = -2d\tau(X,Y)\xi$. Replacing Z by ξ to (3.4), we have $R(X,Y)\xi = 0$. This results imply $d\tau = 0$ on TM. We also obtain the relationship between τ and $\tau\,'$ induced by the Gauss and Weingarten equations with respect to $S(TM)$ and $S'(TM)$ as follows:

$$\tau\,'(X) = \tau(X) + B(X,\, N' - N), \quad \forall\, X \in \Gamma(TM).$$

Thus we have $d\tau = d\tau\,'$. Consequently we show that the Ricci type tensor $R^{(0,\,2)}$ is an induced symmetric Ricci tensor on M.

Case (2): *Step 1.* In case ζ is tangent to M: By Călin [1], ζ belongs to $S(TM)$. If $S(TM)$ is totally geodesic in M, then we have $A_N = 0$ due to (2.12). Applying $\bar{\nabla}_X$ to $g(\zeta, N) = 0$ with $X \in \Gamma(TM)$ and using (2.2) and (2.7), we have $\eta(X) = 0$. It is a contradiction to $\eta(\xi) = 1$. Thus $S(TM)$ is not totally geodesic in M.

In case ζ is not tangent to M: By the decomposition (2.5), ζ is decomposed by

$$\zeta = W + fN, \tag{3.5}$$

where W is a smooth non-vanishing vector field on M and $f = \theta(\xi) \neq 0$ is a smooth function. Applying $\bar{\nabla}_X$ to (3.5) and using (2.2), (2.6) and (2.7), we have

$$\nabla_X W = -X + \theta(X)W + fA_N X, \quad \forall\, X \in \Gamma(TM), \tag{3.6}$$

$$Xf + f\tau(X) + B(X,\, W) = f\theta(X), \quad \forall\, X \in \Gamma(TM). \tag{3.7}$$

Substituting (3.7) into $[X,Y]f = X(Yf) - Y(Xf)$ and using (3.6) and (3.7), we have

$$\begin{aligned}(\nabla_X B)(Y,W) - (\nabla_Y B)(X,W) + \tau(X)B(Y,W) - \tau(Y)B(X,W)&\\ + f\{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X,Y)\} \;=\; 2fd\theta(X,Y),&\end{aligned} \tag{3.8}$$

for all $X,\, Y \in \Gamma(TM)$. Using (2.15), (2.16) and (3.5), the equation (3.8) reduce to

$$2fd\theta(X,Y) = \bar{g}(\bar{R}(X,Y)\zeta,\, \xi), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.9}$$

Substituting (3.6) into $R(X,Y)W = \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,\,Y]}W$ and using (2.15), (2.16), (3.5), (3.6), (3.7), (3.9) and the fact ∇ is torsion-free, we have

$$\bar{R}(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.10}$$

Taking the scalar product with ζ to (3.10) and using $g(\bar{R}(X,Y)\zeta,\zeta) = 0$ and (2.1), we show that the 1-form θ is closed on TM, i.e., $d\theta = 0$ on TM.

Step 2. Assume that $S(TM)$ is totally geodesic in M. Substituting (2.15) with $Z = W$ and (2.16) into (3.10) and using (3.5), (3.8) and $d\theta = 0$, we have

$$R(X,Y)W = \theta(X)Y - \theta(Y)X, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.11}$$

Applying $\bar{\nabla}_X$ to $\theta(Y) = g(Y,\zeta)$ and using (2.2) and (2.6), we have

$$(\nabla_X\theta)(Y) = eB(X,Y) - g(X,Y) + \theta(X)\theta(Y), \quad \forall\, X,\, Y \in \Gamma(TM), \tag{3.12}$$

where $e = \bar{g}(\zeta, N)$. Assume that $e = 0$. Applying $\bar{\nabla}_X$ to $g(\zeta, N) = 0$ with $X \in \Gamma(TM)$ and using (2.2) and (2.7), we have $\eta(X) = 0$. It is a contradiction to $\eta(\xi) = 1$. Thus e is non-vanishing function.

Step 3. Assume that M is locally symmetric. Applying ∇_Z to (3.11), we have

$$R(X, Y)\nabla_Z W = (\nabla_Z \theta)(X)Y - (\nabla_Z \theta)(Y)X, \quad \forall\, X, Y \in \Gamma(TM).$$

Substituting (3.6) and (3.12) in this equation and using (3.11), we obtain

$$R(X, Y)Z = \{g(X, Z) - eB(X, Z)\}Y - \{g(Y, Z) - eB(Y, Z)\}X, \tag{3.13}$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing Z by W to (3.13) and then, comparing this result with (3.11) and using the fact $\theta(X) = g(X, W) + f\eta(X)$, we have

$$\{f\eta(X) + eB(X, W)\}Y = \{f\eta(Y) + eB(Y, W)\}X, \quad \forall\, X, Y \in \Gamma(TM).$$

Replacing Y by ξ to this equation and using the fact $X = PX + \eta(X)\xi$, we have

$$f\, PX = e\, B(X, W)\xi, \quad \forall\, X \in \Gamma(TM).$$

The left term of this equation belongs to $S(TM)$ and the right term belongs to $TM^{\perp}$. This imply $fPX = 0$ and $eB(X, W) = 0$ for all $X \in \Gamma(TM)$. From the first equation of this results we deduce $f = 0$. It is contradiction to $f \neq 0$. Thus $S(TM)$ is not totally geodesic in M. □

Corollary 3.1. *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\bar{M}$. Then the structure 1-form θ is closed on TM, i.e., we have $d\theta = 0$ on TM.*

Proof of Theorem 1.2

Case (1): Let ζ be tangent to M. Then we can use all equations and results of Step 1 in (1) of Theorem 1.1. Applying ∇_Z to (3.2) and using $(3.1)_1$ and (3.3), we have

$$(\nabla_Z R)(X, Y)\zeta = R(X, Y)Z - g(X, Z)Y + g(Y, Z)X. \tag{3.14}$$

Substituting (3.14) into $(R(U, Z)R)(X, Y)\zeta = 0$ and using $(3.1)_1$ and (3.14), we have

$$\begin{aligned} 0 \;=\; (R(U, Z)R)(X, Y)\zeta \;=&\; \theta(Z)(\nabla_U R)(X, Y)\zeta - \theta(U)(\nabla_Z R)(X, Y)\zeta \\ &+ \{B(U, Y)\eta(Z) - B(Z, Y)\eta(U)\}X - \{B(U, X)\eta(Z) - B(Z, X)\eta(U)\}Y, \end{aligned} \tag{3.15}$$

for all $X, Y, Z, U \in \Gamma(TM)$. Replacing U by ζ to (3.15) and using $(\nabla_\zeta R)(X, Y)\zeta = 0$ due to (3.2) and (3.14), we have $(\nabla_Z R)(X, Y)\zeta = 0$. From this and (3.14), we get

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X, \quad \forall\, X, Y, Z \in \Gamma(TM). \tag{3.16}$$

Thus M is a space of constant negative curvature -1. Replacing U by ξ to (3.15) and using (2.10), (3.16) and $(\nabla_Z R)(X, Y)\zeta = 0$, we have

$$B(Y, Z)X = B(X, Z)Y, \quad \forall\, X, Y, Z \in \Gamma(TM).$$

Replacing Y by ξ to this equation and using (2.10), we have

$$B(X,Y)=0, \quad \forall\, X,\, Y\in\Gamma(TM).$$

Thus M is totally geodesic. Therefore we show that ∇ is a unique torsion-free metric connection on M by (2.13). As $B=0$, we have $A^*_\xi=0$ due to (2.11). From (2.19), we get $R(X,Y)\xi=-2d\tau(X,Y)\xi$ for all $X,\,Y\in\Gamma(TM)$. Replacing Z by ξ to (3.16), we have $R(X,Y)\xi=0$. This results imply $d\tau=0$. Thus the transversal connection ∇^ℓ is flat and $R^{(0,\,2)}$ is an induced symmetric Ricci tensor on M.

Case (2): Let $S(TM)$ be totally geodesic in M. Then we can use all equations and results of Step 1 and 2 in (2) of Theorem 1.1. Thus $f=\bar g(\zeta,\xi)$ and $e=\bar g(\zeta,N)$ are non-vanishing functions. Substituting (3.5) into (3.10) and using (2.17), we have

$$\bar R(X,Y)W=\theta(X)Y-\theta(Y)X-2fd\tau(X,Y)N, \quad \forall\, X,\, Y\in\Gamma(TM). \tag{3.17}$$

Taking the scalar product with W to this equation and using the facts $g(W,N)=e$, $g(X,W)=\theta(X)-f\eta(X)$ and $g(\bar R(X,Y)W,\,W)=0$, we have

$$2e\,d\tau(X,Y)=\theta(Y)\eta(X)-\theta(X)\eta(Y), \quad \forall\, X,\, Y\in\Gamma(TM). \tag{3.18}$$

Applying ∇_Z to (3.11) and using (3.6), (3.11) and (3.12), we have

$$\begin{aligned}(\nabla_Z R)(X,Y)W=&\ R(X,Y)Z+\{g(Y,Z)-eB(Y,Z)\}X\\ &-\{g(X,Z)-eB(X,Z)\}Y, \quad \forall\, X,\, Y,\, Z,\, U\in\Gamma(TM).\end{aligned} \tag{3.19}$$

Applying $\bar\nabla_X$ to $e=\bar g(\zeta,N)$ with $X\in\Gamma(TM)$ and using (2.2) and (2.7), we have

$$Xe=e\{\theta(X)+\tau(X)\}-\eta(X), \quad \forall\, X\in\Gamma(TM). \tag{3.20}$$

Substituting (3.19) into $(R(U,Z)R)(X,Y)W=0$ and using (2.13), (3.6), (3.19), (3.20) and the fact $\bar R(U,Z)X=\bar R(X,Z)U+\bar R(U,X)Z$ for all $X,\,Z,\,U\in\Gamma(TM)$, we have

$$\begin{aligned}0\ =&\ \theta(Z)\{R(X,Y)U+g(Y,U)X-g(X,U)Y\}\\ -&\ \theta(U)\{R(X,Y)Z+g(Y,Z)X-g(X,Z)Y\}\\ +&\ e\{\bar g(\bar R(X,Z)U+\bar R(U,X)Z,\ \xi)Y-\bar g(\bar R(Y,Z)U+\bar R(U,Y)Z,\ \xi)X\},\end{aligned} \tag{3.21}$$

for all $X,\,Y,\,Z,\,U\in\Gamma(TM)$. Taking $U=\xi$ and $Z=W$ to (3.21) and using (3.17), (3.18) and the fact $\bar g(\bar R(X,Y)\xi,\,\xi)=0$, we have

$$\theta(W)R(X,Y)\xi=f\{\theta(X)Y-\theta(Y)X\}, \ \ \forall\, X,\, Y\in\Gamma(TM).$$

Taking the scalar product with N to this equation and using (2.19), we have

$$2\theta(W)d\tau(X,Y)=f\{\theta(Y)\eta(X)-\theta(X)\eta(Y)\}, \ \ \forall\, X,\, Y\in\Gamma(TM). \tag{3.22}$$

From the facts $\theta(W)=\bar g(\zeta,W)=g(W,W)+ef$ and $1=\bar g(\zeta,\zeta)=g(W,W)+2ef$, we have $\theta(W)=1-ef$. Substituting $\theta(W)=1-ef$ and (3.18) into (3.22), we have

$$d\tau(X,Y)=f\{\theta(Y)\eta(X)-\theta(X)\eta(Y)\}, \ \ \forall\, X,\, Y\in\Gamma(TM). \tag{3.23}$$

Comparing (3.18) and (3.23), we have $2ef = 1$, i.e., $g(W, W) = 0$. Thus the projection W of the structure vector field ζ on M is a null vector field.

If the transversal connection $\nabla^{\perp}$ is flat, then, by Theorem 2.1, we get $d\tau = 0$ on TM. Replacing Y by ξ to (3.18) with $d\tau = 0$, we also have

$$g(X, W) = 0, \quad \forall\, X \in \Gamma(TM).$$

This implies $W = e\xi$ and $B(X, W) = 0$. Thus ζ is decomposed by $\zeta = e\xi + fN$ and $2ef = 1$. Applying $\bar{\nabla}_X$ to $g(Y, W) = 0$ and using (2.6) and (3.6), we have

$$eB(X, Y) = g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Thus M is totally umbilical with $\beta = 2f$. Using this, (3.12), (3.19) and (3.21) reduce

$$(\nabla_X\theta)(Y) = \theta(X)\theta(Y), \quad (\nabla_Z R)(X, Y)W = R(X, Y)Z,$$
$$(R(U, Z)R)(X, Y)W = \theta(Z)R(X, Y)U - \theta(U)R(X, Y)Z = 0, \tag{3.24}$$

for all $X,\, Y,\, Z \in \Gamma(TM)$. Replacing U by W to (3.24) and using $\theta(W) = \frac{1}{2}$, we have

$$R(X, Y)Z = 2\theta(Z)\{\theta(X)Y - \theta(Y)X\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \qquad \square$$

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