Curves with a node in projective spaces with good postulation

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Abstract. Fix integers d, g, r such that $r \geq 3$, $g > 0$ and $d \geq g + r$. Here we prove the existence of an integral non-special curve C in an r-dimensional projective space such that $\deg(C) = d$, $p_a(C) = g$, C has exactly one node and C has maximal rank (i.e. it has the expected postulation), i.e., the general non-special embedding of a general curve with a single node has maximal rank.

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1 Introduction

Let $X \subset \mathbb{P}^r$ be a closed subscheme. We say that X has *maximal rank* if for all integers $t \geq 1$ the restriction map $\rho_{r,X,t} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(X, \mathcal{O}_X(t))$ has maximal rank, i.e. it is either injective or surjective. Now assume that X is a reduced and connected curve of degree d and arithmetic genus g, spanning $\mathbb{P}^r$ and with $h^1(X, \mathcal{O}_X(1)) = 0$. Riemann-Roch gives $d \geq g + r$. If $d = r$ (and hence X is a rational normal curve) then we say that X has *critical value* 1 and that 1 is the critical value of the triple $(r, 0, r)$. Now assume $d > r$. Let k be the minimal integer ≥ 2 such that $\binom{r+k}{r} \geq kd + 1 - g$. We say that k is the *critical value* of X and of the triple (d, g, r). X has maximal rank if and only if $h^0(\mathcal{I}_X(t)) = 0$ for all $t < k$ and $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$. Since $k \geq 2$, we have $h^1(X, \mathcal{O}_X(k-1)) = 0$. Hence Castelnuovo-Mumford's lemma says that if $h^1(\mathcal{I}_X(k)) = 0$, then $h^1(\mathcal{I}_X(t)) = 0$ for all $t > k$. Hence X has maximal rank if and only if $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(k)) = 0$.

For all integers d, g, r such that $r \geq 0$, $g \geq 0$ and $d \geq g + r$ let $H(d, g, r)$ denote the open subset of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^r)$ parametrizing the smooth and non-degenerate curves $C \subset \mathbb{P}^r$ such that $p_a(C) = g$, $\deg(C) = d$ and $h^1(C, \mathcal{O}_C(1)) = 0$. The set $H(d, g, r)$ is a smooth and irreducible quasi-projective variety (here we use in an essential way that we only take non-special embeddings, because the Hilbert scheme of non-degenerate smooth curves of degree d and genus g may be reducible

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even when d is very near to $2g-2$ ([4], [7], [8] and references therein). Let $H(d,g,r)'$ denote the closure of $H(d,g,r)$ in $\mathrm{Hilb}(\mathbb{P}^r)$.

For any integer $g \geq 2$ set $\Delta_0(g) := \{C \in \overline{\mathcal{M}}_g : C$ is irreducible and with a unique node$\}$. The closure $\Delta_0(g)'$ of $\Delta_0(g)$ in $\overline{\mathcal{M}}_g$ is the irreducible divisor of $\overline{\mathcal{M}}_g$ usually denoted with Δ_0. Hence $\Delta_0(g)$ is non-empty, quasi-projective, irreducible and of dimension $3g-4$. Let $\Delta_0(1)$ denote a set with as its unique element the only integral nodal curve with arithmetic genus 1. Set $H(d,g,r)_1 := \{C \in H(d,g,r)' : C \in \Delta_0(g)$ and $h^1(C, \mathcal{O}_C(1)) = 0\}$. Set $H(d,g,r)'_1 : \{C \in H(d,g,r)' : h^1(C, \mathcal{O}_C(1)) = 0\}$. Notice that $H(d,g,r)_1$ is a non-empty and irreducible codimension one algebraic subset of $H(d,g,r)'$. In this paper we extend [5], [1], [2], [3] to general non-special embeddings of a general element of $\Delta_0(g)$ and prove the following result.

Theorem 1.1. *Fix integers $r \geq 3$, $g \geq 1$ and $d \geq g+r$. Let $X \subset \mathbb{P}^r$ be a general embedding of degree d of a general element of $\Delta_0(g)$. Then X has maximal rank.*

Theorem 1.1 is equivalent to say that a general element of $H(d,g,r)_1$ has maximal rank.

2 Preliminaries

For any curve $Y \subset \mathbb{P}^r$ with only nodes as singularities let N_Y denote its normal bundle. The sheaf N_Y is a rank $r-1$ vector bundle on Y and $\deg(N_Y) = (r+1)\deg(Y) + 2p_a(Y) - 2$. For any smooth variety W and any nodal curve $T \subset W$ let $N_{Y,W}$ denote the normal bundle of Y in $W.N_{Y,W}$ is a rank $(dim(W)-1)$ vector bundle on Y with degree $-\deg(\omega_W) + 2p_a(T) - 2$.

Fix a reduced curve $Y \subset \mathbb{P}^r$. We say that a line D is 1-secant (resp. 2-secant) to Y if $\sharp(Y \cap D) = 1$ (resp. $\sharp(Y \cap D) = 2$), Y is smooth at each point of $Y \cap D$ and D is not a tangent line of Y at one of the points of $Y \cap D$.

Lemma 2.1. *Let W be a smooth projective variety and F, R smooth and connected curves in W. Assume that R is a smooth and rational, that R intersects F at a single point, P, and quasi-transversal. Assume $h^1(F, N_{F,W}) = 0$ and that $N_{R,W}$ is trivial. Then $h^1(F \cup R, N_{F\cup R,W}) = 0$ and $F \cup R$ is smoothable in W.*

Proof. Set $r := \dim(W)$. The vector bundle $N_{F\cup R,W}|F$ is obtained from $N_{F,W}$ making a positive elementary transformation supported by P ([6], §2). Hence we have $h^1(F, N_{F\cup R,W}|F) = 0$. The vector bundle $N_{F\cup R,W}|R$ is obtained from $N_{R,W}$ making a positive elementary transformation supported by P ([6], §2). Hence $N_{F\cup R,W}|R$ is a direct sum of a line bundle of degree 1 and $r-2$ line bundles of degree 0. Hence $h^1(R, N_{F\cup R,W}|R(-P)) = 0$. Hence $h^1(F \cup R, N_{F\cup R,W}) = 0$ and $F \cup R$ is smoothable in W ([6], Theorem 4.1 and its proof). □

Lemma 2.2. *Let $Y \subset \mathbb{P}^r$ be a nodal curve. Set $g := p_a(Y)$ and $d := \deg(Y)$. Then N_Y is a rank $r-1$ vector bundle on Y and $\deg(Y) = (r+1)d + 2g - 2$. If $h^1(Y, \mathcal{O}_Y(1)) = 0$, then $h^1(Y, N_Y) = 0$.*

Proof. Look at the Euler's sequence of $T\mathbb{P}^r$

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}^{\oplus(r+1)}(1) \to T\mathbb{P}^r \to 0. \qquad (2.1)$$

Since Y is a curve, we have $h^2(Y,\mathcal{O}_Y)=0$. Restricting (2.1) to Y we get $h^1(Y,T\mathbb{P}^r|Y)=0$. There is a morphism $\eta : T\mathbb{P}^r|Y \to N_Y$ whose cokernel is supported by $\mathrm{Sing}(Y)$. Since Y is a curve, we have $h^2(Y,\mathrm{Ker}(\eta))=0$. Since $h^1(T,\mathbb{P}^r|Y)=0$, the exact sequence

$$0\to \mathrm{Ker}(\eta)\to T\mathbb{P}^r|Y\to \mathrm{Im}(\eta)\to 0$$

gives $h^1(Y,\mathrm{Im}(\eta))=0$. Since $\mathrm{Coker}(\eta)$ is supported by a finite set, we obtain that $h^1(Y,\mathrm{Coker}(\eta))=0$. Hence the exact sequence

$$0\to \mathrm{Im}(\eta)\to N_Y\to \mathrm{Coker}(\eta)\to 0$$

gives $h^1(Y,N_Y)=0$. □

Lemma 2.3. *Let $C\subset\mathbb{P}^r$, $r\ge 3$, be a smooth and non-degenerate curve such that $h^1(C,\mathcal{O}_C(1))=0$. Fix a line $D\subset\mathbb{P}^r$ such that $\sharp(D\cap C)=2$ and D is not tangent to C. Fix $O\in D\cap C$. Set $Y:=C\cup D$. Then $h^1(Y,N_Y)=0$, $Y\in H(d,g,r)'_1$ and Y is a flat limit of a flat family of elements of $H(d,g,r)_1$ whose singular point goes to O at the limit.*

Proof. By [6], Remark 4.1.1 and Corollary 4.2, or [9], Theorem 5.2, we have $h^1(Y,N_Y)$ and $Y\in H(d,g,r)'$. Since C is smooth, N_C is a quotient of $T\mathbb{P}^r|C$. Hence N_C is spanned. Hence $h^0(C,N_C(-O))=h^0(C,N_C)-\mathrm{rank}(N_C)$. Since $h^1(C,N_C)=0$, Riemann-Roch gives $h^1(C,N_C(-O))=0$. Let $\pi:\Pi\to\mathbb{P}^r$ be the blowing up of O and Y' (resp. C', resp. D') the strict transform of Y (resp. C, resp. D). Since C and D are smooth, the morphism π induces $u: C'\to C$ and $D'\cong D$. Y' is nodal and we call P its unique singular point. Since $N_Y|C$ is obtained from N_C making two positive elementary transformations, $N_{Y'}|C$ is obtained from $u^*(N_C(-O))$ making some positive elementary transformations. By assumption $h^1(C,N_C)=0$. Hence $h^1(N_{Y'}|C)=0$. Since N_D is a direct sum of $r-1$ line bundles of degree 1, $N_D(-O)$ is trivial. Hence $N_{Y'}|D'$ is obtained from a trivial vector bundle making some positive elementary transformation. Hence $h^1(D',N_{D'}(-P))=0$. Hence Y' is smoothable in Π ([6], Theorem 4.1.1 and its proof) . □

Remark 2.1. Fix integers $r\ge 3$, $g>0$ and $d\ge g+r$. It is easy to prove the existence of a non-degenerate curve $X\subset\mathbb{P}^r$ such that $h^1(X,\mathcal{O}_X(1))=0$, X is irreducible and X has an ordinary node as its unique singularity. Since X is non-degenerate, we have $h^0(\mathcal{I}_X(1))=0$. Applying Riemann-Roch on X we get $h^1(\mathcal{I}_X(1))=d-g-r$. Hence if $d=g+r$, then $h^1(\mathcal{I}_X(1))=0$.

Lemma 2.4. *Fix integers d,g,r such that $r\ge 3$, $g>0$, $d\ge g+r$ and $2d+1-g\le\binom{r+2}{2}$. Then there is $X\in H(d,g,r)_1$ such that $h^0(\mathcal{I}_X(1))=0$ and $h^1(\mathcal{I}_X(t))=0$ for all $t\ge 2$.*

Proof. In all cases we construct a certain non-degenerate irreducible curve $X\subset\mathbb{P}^r$. Hence the curve X we will construct will also have $h^0(\mathcal{I}_X(1))=0$. By Castelnuovo-Mumford's lemma it is sufficient to find $X\in H(d,g,r)_1$ such that $h^1(\mathcal{O}_X(2))=0$. Fix a general $C\in H(d-1,g-1,r)$. Since C has maximal rank ([1], [2], [3]) and $2(d-1)+1-(g-1)<\binom{r+2}{2}$, we have $h^0(C,\mathcal{I}_C(1))=0$ and $h^1(\mathcal{I}_C(2))>0$. Let $Q\subset\mathbb{P}^r$ be any quadric hypersurface containing C. Let $D\subset\mathbb{P}^r$ be a general 2-secant line of C. Since C is non-degenerate and the singular locus of a quadric is a linear space, Q is smooth at a general $P \in C$. Since D is general, we may assume that $P \in C \cap D$ is a smooth point of Q. Since C is non-degenerate and Q is not a cone with vertex containing P, $D \nsubseteq Q$. Hence $h^0(\mathcal{I}_{C\cup D}(2)) < h^0(C, \mathcal{I}_C(2))$. Since $h^0(C \cup D, \mathcal{O}_{C\cup D}(2)) = h^0(C, \mathcal{O}_C(2)) + 1$, we get $h^1(\mathcal{I}_{C\cup D}(2)) = 0$. Apply Lemma 2.3. □

Remark 2.2. Fix a closed subscheme $W \subset \mathbb{P}^r$ and an effective Cartier divisor D of $\mathbb{P}^r$. Set $a := \deg(D)$. We will take as D a hyperplane if $r \geq 4$ and a smooth quadric surface if $r = 3$. Let $\mathrm{Res}_D(W)$ be the residual scheme of W with respect to H, i.e. the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_W : \mathcal{I}_D$ as its ideal sheaf. If W is reduced, then $\mathrm{Res}_D(W)$ is the union of the irreducible components of W not contained in H. For any $t \in \mathbb{Z}$ we have the following exact sequence of coherent sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_D(W)}(t-a) \to \mathcal{I}_W(t) \to \mathcal{I}_{W\cap D,D}(t) \to 0. \tag{2.2}$$

From (2.2) we get

$$h^i(\mathcal{I}_W(t)) \leq h^i(\mathcal{I}_{\mathrm{Res}_D(W)}(t-a)) + h^i(D, \mathcal{I}_{W\cap D,D}(t)),$$

for all $i \geq 0$ and all $t \in \mathbb{Z}$.

Remark 2.3. Fix a flat family $\{Y_\lambda\}_{\lambda\in\Delta}$ of curves $Y_\lambda \subset \mathbb{P}^r$, where Δ is a connected affine curve. Call $u : \mathcal{Y} \to \Delta$ the corresponding family. Fix $o \in \Delta$ and take a line $D \subset \mathbb{P}^r$ which is 2-secant to Y_o. Taking a finite covering of Δ if necessary, we may assume that u has two disjoint section s_1, s_2 with $\{s_1(o), s_2(o)\} = Y \cap D$. For any $t \in \Delta$ let D_t be the line spanned by $s_1(t)$ and $s_2(t)$. There is an open neighborhood Δ' of o in Δ instead of Δ we reduce to the case in which $\sharp(Y_t \cap D_t) = 2$ for all t and D_t is 2-secant to Y_t for all $t \in \Delta'$.

3 Proof of Theorem 1.1

For all integers $m \geq 3$ and $t \geq 2$ define the integers $a_{m,t}$ and $b_{m,t}$ by the relations

$$(t-1)\cdot a_{m,t} + 1 + tr + b_{m,t} = \binom{m+t}{m},\ 0 \leq b_{m,t} \leq t-2. \tag{3.1}$$

Set $a_{m,0} = a_{m,1} = m$ and $b_{m,0} = b_{m,1} = 0$. Fix integers d, g, r such that $r \geq 3$, $g > 0$ and $d \geq g + r$. Let k be the critical value of the triple (d, g, r). By the semicontinuity theorem for cohomology to be of maximal rank is an open condition among non-special embeddings of curves. Recall that $H(d, g, r)'_1$ is irreducible. Hence it is sufficient to prove the existence of $X_i \in H(d, g, r)'_1$, $i = 1, 2$, such that $h^1(\mathcal{I}_{X_1}(k)) = 0$ and $h^0(\mathcal{I}_{X_2}(k-1)) = 0$. Notice that if $kd + 1 - g = \binom{r+k}{r}$, then any X_1 as above satisfies $h^0(\mathcal{I}_{X_1}(k)) = 0$ and hence in this particular case we do not need to check the existence of X_2. For the case $k = 1$ see Remark 2.1. The case $k = 2$ is true by Lemma 2.4. From now on we assume $k \geq 3$. In the case $r \geq 4$ we only write the proof of the existence of X_1, since the proof of the existence of X_2 is similar (and trivial for $k = 2$). In the case $r = 3$ we only write the proof of the existence of X_2.

Define the integers $u_{r,y,g-1}$ and $v_{r,y,g-1}$ by the relations

$$(3.2) \qquad y \cdot u_{r,y,g-1} + 1 - (g-1) + v_{r,y,g-1} = \binom{r+y}{r}, \; 0 \le v_{r,y,g-1} \le y-1.$$

Claim 1: To prove the existence of X_1 (resp. X_2) it is sufficient to find $Y \in H(d-1,g-1,r)'$ and a line D 2-secant to Y such that $h^1(\mathcal{I}_{Y\cup D}(k)) = 0$ (resp. $h^0(\mathcal{I}_{Y\cup D}(k-1)) = 0$).

Proof of Claim 1: By semicontinuity it is sufficient to prove that $Y \cup D \in H(d,g,r)'_1$. Take a smoothing of Y inside $\mathbb{P}^r$, say $\{Y_t\}_{t\in\Delta}$, $o \in \Delta$, and $Y_t \in H(d-1,g-1,r)$ for all $t \in \Delta$, and call $u : \mathcal{Y} \to \Delta$ the corresponding family. Taking a finite covering of Δ if necessary, we may assume that u has two disjoint sections s_1, s_2 with $\{s_1(o), s_2(o)\} = Y \cap D$. For any $t \in \Delta$ let D_t be the line spanned by $s_1(t)$ and $s_2(t)$. Taking an open neighborhood of o in Δ instead of Δ we reduce to the case in which $\sharp(Y_t \cap D_t) = 2$ for all t and $Y_t \cup D_t$ is nodal. Use the flat family $\{Y_t \cup D_t\}_{t\in\Delta} \subset H(d,g,r)'$ and apply Lemma 2.3.

Let m be the maximal integer $x \ge 0$ such that $a_{r,x} \le g-1$. Since $d \ge g+r$, we have $m \le k$. $r > 3$. Consider the following assertion: $E_{r,x}$, $r \ge 3$, $x \ge 2$. Fix integers u, q such that $xu + 1 - q + 2x \le \binom{r+x}{r}$. Then there exists (C,D) such that $C \in H(u,q,r)$, D is a line, $\sharp(C \cap D) = 1$, $C \cup D$ is nodal and $h^1(\mathcal{I}_{C\cup D}(x)) = 0$.

Lemma 3.1. *$E_{r,x}$ is true for all integers $r \ge 3$ and $x \ge 2$.*

Proof. Let e be the critical value of $(u+2,q,r)$. By assumption we have $e \le x$. Notice that $e \ge 2$. Castelnuovo-Mumford's lemma shows that it is sufficient to find (C,D) such that $C \in H(u,q,r)$, D is a line, $\sharp(C\cap D) = 1$, $C \cup D$ is nodal and $h^1(\mathcal{I}_{C\cup D}(e)) = 0$. We follow the proofs in [1], [2] and [3] for the genus $g := q$ and the integer $d := u+1$, but we need to modify the very last step of the proofs in the quoted papers.

(a) First assume $r = 3$. In this case we take the proof of [2], Lemma V.2, for the critical value e, i.e. starting with a certain curve, Y, with $h^i(\mathcal{I}_Y(e-2)) = 0$, $i = 0,1$. In the quadric surface Q one of the added lines, D', is linked to the remaining lines or to C only at one point. We get (Y', D') with $Y' \in H(u,q,3)'$, $\sharp(Y' \cap D') = 1$, $Y' \cup D'$ nodal and with $h^1(\mathcal{I}_{Y'\cup D'}(e)) = 0$; here contrary to [2], Lemma V.2, we don't need to distinguish several subcases, because $h^0(Y' \cup D', \mathcal{O}_{Y'\cup D'}(e)) = (u+1)e + 1 - q$ and our numerical assumptions give $\binom{e+3}{3} - (u+1)e - 1 + q \ge e$. We smooth Y' to some $Y \in H(u,q,3)$, say $\{Y_\lambda\}$ and follow this deformation with a family of lines $\{D_\lambda\}$ with D_λ 1-secant line of Y_λ (Remark 2.3).

(b) From now on we assume $r \ge 4$. Let $H \subset \mathbb{P}^r$ be a hyperplane. Assume for the moment $r \ge 5$, but also assume that the lemma is true in $\mathbb{P}^{r-1}$. We follow [3], §5, (with $j := e$) but in the last step we add in a hyperplane H a curve $Y_1 \cup D_1 \subset H$ with D_1 1-secant to Y_1. Let ρ be the maximal integer t such that $a_{r,t} \le q$ (ρ is called r in [3], §2). To see that this construction is possible, we need to check in each subcase (b1), (b2) and (b3) the numerical obstructions stated in [3]. Set $a := \deg(Y_1)$ and $y := p_a(Y_1)$. We have $y \le q$ and $(e-1)a + 1 - y = \binom{r+e-1}{r}$.

(b1) First assume $e = \rho$. Since $e(a_{r,e} + r) + 1 - a_{r,e} + b_{r,e} = \binom{r+e}{e}$, $b_{r,e} \le e-2$, $q \ge a_{r,e}$, $u - q \ge q - a_{r,e}$ and $eu + 1 - q + 2e \le \binom{r+e}{r}$, this case is impossible.

(b2) Now assume $e = \rho + 1$. Take $W \in H(a_{r,e-1} + r, a_{r,e-1}, r)$ with maximal rank. Hence $h^1(\mathcal{I}_W(e-1)) = 0$ and $h^0(\mathcal{I}_W(e-1)) = v_{r,e-1,q} \le e-2$. First assume $q \ge a_{r,e-1} + r - 1$. Take a general smooth curve $U \subset H$ such that $\deg(U) = u - a_{r,e-1}$, $\sharp(U \cap W) = r$ and $p_a(W) = q - a_{r,e-1} - (r-1)$. Let $T \subset H$ be a general line meeting T. Hence $W \cup U \in H(u,q,r)'$ and T is 1-secant to $W \cup U$. Hence (moving if necessary T as in Remark 2.3) it is sufficient to prove $h^1(\mathcal{I}_{W\cup U\cup T}(e)) = 0$. Since $h^1(\mathcal{I}_W(e-1)) = 0$, it is sufficient to prove $h^1(H, \mathcal{I}_{U\cup T\cup(W\cap H),H}(e)) = 0$. Since $a_{r,e-1} \le e-2 \le \binom{r+e}{r} - e(u+1) - 1 + q$, we have $\sharp(W \cap H) - \sharp(W \cap U) + h^0(U \cup T, \mathcal{O}_{U\cup T}(e)) \le \binom{r+e-1}{r-1}$. By the inductive assumption in $\mathbb{P}^{r-1}$ we have $h^1(H, \mathcal{I}_{U\cup T,H}(e)) = 0$. Hence it is sufficient to prove that the points in $W \cap (H \setminus U)$ give independent conditions to $H^0(H, \mathcal{I}_{U\cup T,H}(e))$. We want to apply [3], Lemma 1.6, with $e = 0$, i.e. $s = r$, $g'' \ge 0$ and hence $(s - r - 2 - (d'' - g'' - r + 1)) < 0 \le g''$. Now assume $q \le a_{r,e-1} + r - 2$. In this case we may take $U \subset H$ smooth and rational and meeting W at $q + 1 - a_{r,e-1}$ points.

(b3) Now assume $e \ge \rho + 2$. Take $W \in H(u_{r,e-1,q}, q, r)$ with maximal rank. Hence $h^1(\mathcal{I}_W(e-1)) = 0$ and $h^0(\mathcal{I}_W(e-1)) = v_{r,e-1,q} \le e-2$. Let $U \subset H$ be a general smooth rational curve of degree $u - u_{r,e-1,q}$ and T a general line meeting W at exactly one point and with $T \cap W \in H$. Since $W \cup T \in H(u,q,r)'$, to prove the lemma in this case it is sufficient to prove $h^1(\mathcal{I}_{W\cup U\cup T}(e)) = 0$ for general (U,T). Since $h^1(\mathcal{I}_W(e-1)) = 0$, it is sufficient to prove $h^1(H, \mathcal{I}_{U\cup T\cup(W\cap H,H}(e)) = 0$. We have $\sharp(W \cap H) = u_{r,e-1,q} \ge e+1$, because, $q \le u_{r,e-1,q} - r$ and hence $(e-1)u_{r,e-1,q} \ge \binom{r+e-1}{r} + r - (e-1)$. By the case $q = 0$ in $\mathbb{P}^{r-1}$ there is a pair (U,T) in $\mathbb{P}^{r-1}$ such that $h^1(H, \mathcal{I}_{U\cup T,H}(e)) = 0$. Since $\binom{r+e_1}{r-1} - e(u+1) - 1 + q \ge 2e > h^0(\mathcal{I}_W(e-1))$, we have $\sharp(W \cap H) - 1 \le \binom{r+e-1}{e-1} - h^0(H, \mathcal{I}_{U\cup T,H}(e))$. To get $h^1(H, \mathcal{I}_{U\cup T\cup(W\cap H),H}(e)) = 0$ we want to apply [3], Lemma 1.6, with S a single point (a case even easier than the one in [3], Lemma 1.6, where $\sharp(S) \ge r$).

(c) Now assume $r = 4$. Here the situation is simpler, because to control the postulation of $T \cap H$, $T \subset \mathbb{P}^4$ a sufficiently general curve and H a hyperplane, we may use [1], Lemma 1.4, to control $T \cap H$ and hence we could even prove Lemma 3.1 by induction on e starting with a pair (Y_{e-1}, D) for the critical value $e-1$ and arriving to the pair (Y_e, D) for the critical value e. □

3.1 Case $r = 3$ of Theorem 1.1

In this subsection we conclude the proof of Theorem 1.1 in the case $r = 3$. We fixed the integer $g > 0$ and called m the maximal integer such that $a_{3,m} \le g - 1$. For any $P \in \mathbb{P}^3$ let $\chi(P)$ denote the first infinitesimal neighborhood of P in $\mathbb{P}^3$, i.e. the closed subscheme of $\mathbb{P}^3$ with $(\mathcal{I}_P)^2$ as its ideal sheaf. The scheme $\chi(P)$ has dimension zero, $\deg(\chi(P)) = 3$ and $\chi(P)_{red} = \{P\}$. We call $\chi(P)$ the nilpotent with P as its support.

We only prove the existence of X_2, i.e. of a pair (C,D) with $C \in H(d-1, g-1, 3)$, D a 2-secant line of C and $h^0(\mathcal{I}_{C\cup D}(k-1)) = 0$. The triple $(d-1, g, 3)$ has either critical value k or critical value $k-1$.

(a) In this step we assume that $(d-1, g, 3)$ has critical value $k-1$. Since $(d, g, 3)$ has not critical value $k-1$, we have

$$\binom{k+2}{3} < kd + 1 - g \le \binom{k+2}{3} + k. \tag{3.3}$$

(a1) First assume $k \geq m+3$ and $d-2-u_{3,k-3,g-1} \geq v_{3,k-3,g-1}$ (by the first inequality in (3.3) it is sufficient to assume $u_{3,k-1,g-1} - u_{3,k-3,g-1} \geq v_{3,k-3,g-1}$). As in [2], Lemma VI.4, take (Y, Q, D, D', S, S') satisfying $R(k-1)$ for the genus $g-1$ with respect to the integer $x = 0$. Hence $Y \in H(u_{3,k-3,g-1}, g-1, 3)$, Q is a smooth quadric surface intersecting transversally Y, D and D' are disjoint 1-secant lines of Y contained in Y, $S' = \emptyset$, $\sharp(S) = v_{3,k-3,g-1}$, $S \subset D \setminus Y \cap D$. Deforming Y we may also assume that no line of Q is 2-secant to Y. Let E_i, $0 \leq i \leq d-1-u_{3,k-1,g-1}$, be lines of Q intersecting D, not containing the point $D \cap Y$ and such that $E_i \cap Y \neq 0$ if and only if $0 \leq i \leq v_{3,k-3,g-1}$. Let Z be the union of Y, D, the lines E_i, $1 \leq i \leq d-1-u_{3,k-1,g-1}$ and the $v_{3,k-3,g-1}$ nilpotents $\chi(P)$, $P \in D \cap E_i$, $1 \leq i \leq v_{3,k-3,g-1}$. We have $Z \in H(d-1, g-1, 3)'$ ([2], Corollary 1.4) and E_0 is a 2-secant line of Z. The scheme $\mathrm{Res}_Q(Z \cup E_0)$ is the union of Y and the points P, $P \in D \cap E_i$, $1 \leq i \leq v_{3,k-3,g-1}$. Since $\sharp(Y \cap (Q \setminus D)) > k+1$, we see as in [2], lines 12–16 of the proof of Lemma VI.1) that $h^0(\mathcal{I}_{\mathrm{Res}_Q(Z \cup E_0)}(k-1))) = 0$. Hence it is sufficient to prove $h^0(Q, \mathcal{I}_G(k-1, d-1-u_{3,k+1,g-1})) = 0$, where $G := Y \cap (Q \setminus (D \cup E_0 \cup \cdots E_{v(3,k+1,g-1)}))$. We apply [2], Lemma VIII.8.

(a2) Now assume $k \geq m+3$ and $d-2-u_{3,k-3,g-1} < v_{3,k-3,g-1}$. Take (Y, Q, D, D', S, S') satisfying $R(k-3)$ for the genus $g-1$ with respect to the integer $x := v_{3,k-3,g-1} - (d-3-u_{3,k+1,g-1})$. Here we use [2], Lemma VII.2, which says that $0 \leq 2x \leq v_{3,k-3,g-1}$. Deforming Y we may assume that Y is transversal to Q and that Q contains no 2-secant line of Y. Fix $d-2-u_{3,k+1,g-1}$ lines E_i, $0 \leq i \leq d-3-u_{3,k+1,g-1}$, in the linear system of lines in Q intersecting D with the only condition that E_i intersects $Y \cap (Q \setminus (D \cup D'))$ if and only if $1 \leq i \leq x$. Let Z be the union of Y, D, D', the lines E_i, $i \neq 0$, and the nilpotents $\chi(P)$, $P \in D \cap E_i$, $1 \leq i \leq v_{3,k+1,g-1} - x$, and $P \in D' \cap E_i$, $1 \leq i \leq x$. We have $Z \in H(d-1, g-1, 3)'$, E_0 is a 2-secant line of Z (it intersects D and D', but not Y) and $Z \cup E_0 \in H(d, g, 3)'_1$ (Lemma 2.3).

(a3). Now assume $k \leq m+2$, i.e. $k \in \{m, m+1, m+2\}$. We use the assertion $H(k-3)$ of [2] instead of the assertion $R(k-3)$. Here need to distinguish four subcases. In every subcase we start with a solution (Y, Q, D, S) of H_{k-3}. Let $(1,0)$ be the system of lines on Q containing D. Deforming if necessary Y we may assume that Q is transversal to Y and that D is the only 2-secant line of Y contained in Q.

(a3.1) Assume $g-1 = a_{3,k-3}$ (it implies $k = m+2$). Since $b_{3,k-3} \leq (k-3)/3$ ([2], III.1), we have $b_{3,k-3} \leq d-2-a_{3,k-3}$. Take a line D' of type $(1,0)$ on Q and 1-secant to Y. Let E_i, $0 \leq i \leq d-2-a_{3,k}$, be lines of type $(0,1)$ on Q such that $D' \cap Y \notin E_i$ for any i and $E_i \cap Y \neq \emptyset$ if and only if $0 \leq\leq b_{3,k-3}$. Let Z be the union of Y, D', E_i, $i \geq 1$, and the nilpotents $\chi(D'' \cap E_i)$, $1 \leq i \leq b_{3,k-3}$. We have $Z \in H(d-1, g-1, 3)'$ and E_0 is a 2-secant line of Z.

(a3.2) Assume $g-1 \geq a_{3,k-3}+1$ and $b_{3,k-3} \leq d-2-a_{3,k-3}-(g-2)$. Let E_i, $0 \leq i \leq d-2-a_{3,k-3}$, be lines of type $(0,1)$ on Q such that $D \cap Y \notin E_i$ for any i and $E_i \cap Y \neq \emptyset$ if and only if $0 \leq i \leq g-2-a_{3,k-3}+b_{3,k-3}$. Let Z be the union of Y, D, E_i, $i \geq 1$, and $\chi(E_j \cap D)$, $1 \leq i \leq b_{3,k-3}$. We have $Z \in H(d-1, g-1, 3)'$ and E_0 is a 2-secant line of Z.

(a3.3) Assume $b_{3,k-3} \geq d+1-a_{3,k-3}-g$ and $b_{3,k-3} + (g-3-a_{3,k-3}) \leq 3(d-3-a_{3,k-3})$. Since $b_{3,k-3} \leq (k-3)/3$ ([2], III.1) and $g-1 > a_{3,k-1}$, we have $g-1 \geq a_{3,k-3}+2$. Let D' be a general 2-secant line of Y. Instead of Q we take a general quadric surface Q' containing $D \cup D'$, say as lines of type $(1,0)$. Let E_i, $0 \le i \le d-3-a_{3,k-3}$ be lines of type $(0,1)$ on Q', not intersecting $Y \cap (D \cup D')$ and with $E_i \cap D_i \neq \emptyset$ if and only if $0 \le i \le g-3-a_{3,k-3}+b_{3,k-3}-2(d-3-a_{3,k-3})$. Let Z be the union of Y, D, D', the lines L_i, $i \ge 1$, the nilpotents $\chi(D \cap E_i)$, $i \ge 1$, and the nilpotents $\chi(D' \cap E_i)$, $1 \le i \le x$. we have $Z \in H(d-1, g-1, e)'$ and E_0 is 2-secant to Z.

(a3.4) Assume $b_{3,k-3} \ge d+1-b_{3,k-3}-g$ and $b_{3,k-3}+(g-3-a_{3,k-3}) > 3(d-3-a_{3,k-3})$. Since $b_{3,k-3} \le (k-3)/3$ ([2], III.1), $d-1 \ge a_{3,k-1}+3$ and $a_{3,k-1}-a_{3,k-3} > 2(k-1)$ ([2], III.1), this case cannot occur.

(b) Now assume that $(d-1, g, 3)$ has critical value k. Let x be the maximal integer $x > 0$ such that $(x, g, 3)$ has critical value $\le k-1$. It is easy to check that $x \ge g+3$ and that $x < d$. We proved the existence of a pair (C, D) such that $C \in H(x, g-1, 3)$, D is a 2-secant line of C, Let $E \subset \mathbb{P}^3$ be any smooth rational curve such that $\deg(E) = d-x$, $\sharp(C \cap E) = 1$, $E \cap D = \emptyset$ and E meets quasi-transversally C (e.g., take as E a general smooth rational curve of degree $d-x$ intersecting C). Set $X_2 := (C \cup E) \cup D$.

3.2 End of the proof of Theorem 1.1 for $r \ge 4$

From now on we assume $r \ge 4$. We define the following assertions $H_{r,x}$, $x \ge 1$, $R_{r,y,g-1}$, $y \ge m$, and $R'_{r,m+1,g-1}$ (only if $r \ge 5$ and $g-1 \ge v_{r,m,g-1}$).

$H_{r,x}$: A general $C \in H(a_{r,x}+r, a_{r,x}-b_{r,x}, r)$ satisfies $h^i(\mathcal{I}_C(x)) = 0$, $i = 0, 1$.

$R_{r,x,g-1}$, $x \ge m$: There exists a triple (X, Z, T) such that

(i) $X = Z \cup T$, $Z \cap T = \emptyset$ and $h^i(\mathcal{I}_X(x)) = 0$, $i = 0, 1$;

(ii) $Z \in H(u_{r,x,g-1}-v_{r,x,g-1}, g-1, r)$ and T is a union of $v_{r,x,g-1}$ disjoint lines.

$R'_{r,m+1,g-1}$ (under the assumptions $r \ge 5$ and $g-1 \ge v_{r,m,g-1}$): There is $Y \in H(u_{r,m+1,g-1}, g-1-v_{r,m+1,g-1}, r)$ such that $h^i(\mathcal{I}_Y(m+1)) = 0$, $i = 0, 1$.

Of course, to see that $H_{r,x}$ (resp. $R_{r,x,g-1}$) makes sense for $x \ge 1$ (resp. $x \ge m$) we need to check that $a_{r,x} \ge b_{r,x}$ for all $x \ge 1$ (resp. $u_{r,x,g-1}-v_{r,x,g-1} \ge g-1+r$ for all $x \ge m$). These inequalities are true for the following reasons. A stronger form of the inequality $a_{4,x} \ge b_{4,x}+4$ is [1], Lemma 2, plus that $b_{4,1} = 0$. We have $u_{4,m,g-1}-v_{4,m,g-1} \ge g-1+4$ by [1], Lemma 9. We have $u_{4,x,g-1}-v_{4,x,g-1} \ge g-1+4$ for all $x > m$ by [1], Lemma 5, and the inequality $v_{4,x,g-1} \le x-1$. More restrictive inequalities are proved in [3], §5, for the case $r \ge 5$. Granted this, for any $C \in H(a_{r,x}+r, a_{r,x}-b_{r,x}, r)'$ we have $h^1(\mathcal{I}_C(x)) = h^0(\mathcal{I}_C(x))$ by the equation in (3.2). The equation in (3.1) gives $h^1(\mathcal{I}_{Z \sqcup T}(x)) = h^0(\mathcal{I}_{Z \sqcup T}(x))$ for any $Z \sqcup T$ with $Z \in H(u_{r,x,g-1}-v_{r,x,g-1}, g-1, r)$ and T a union of $v_{r,x,g-1}$ disjoint lines such that $Z \cap T = \emptyset$. Similarly, if $g-1 \ge v_{r,m+1,g-1}$ and $Y \in H(u_{r,m+1,g-1}, g-1-v_{r,m+1,g-1}, r)'$, then $h^1(\mathcal{I}_Y(m+1)) = h^0(\mathcal{I}_Y(m+1))$. To prove one of these assertions $H_{r,x}$, $R_{r,x,g-1}$ or $R'_{r,m+1,g-1}$ it is sufficient to find a " solution " which is smoothable (by semicontinuity). For instance, to prove $H_{r,x}$ it is sufficient to prove the existence of $C \in H(a_{r,x}+r, a_{r,x}-b_{r,x}, r)'$ such that $h^1(\mathcal{I}_C(x)) = 0$. The assertion $H_{r,x}$, $r \ge 5$ and $x \ge 1$, are true by [3], Lemma 1. If $R'_{r,m+1,g-1}$ is defined and $r \ge 5$, then $R'_{r,m+1,g-1}$ is true ([3], Lemma 3.2). For $y \ge m+1$ $R_{r,y,g-1}$ implies $R_{r,y+1,g-1}$ ([1], Lemma 8, for $r=4$, [3], Lemma 3.6, for $r \ge 5$. If $r \ge 5$ and $R'_{r,m+1,g-1}$ is not defined, then $R_{r,m+1,g-1}$ is true ([3], Lemma 3.3). $R_{4,m+1,g-1}$ is true ([1], Lemma 10). If $r \ge 5$ and $R'_{r,m+1,g-1}$ is defined, then $R_{r,m+2,g-1}$ is true ([3], Lemma 3.5). Hence we may use all $H_{r,x}$ and all $R_{r,y,g-1}$, except $R_{r,y+1,g-1}$ when $r \ge 5$ and $R'_{r,m+1,g-1}$ is defined. In the latter case we may use $R'_{r,m+1,g-1}$. Fix a hyperplane H of $\mathbb{P}^r$.

(a) Here we assume $m=k$. Since $k \ge 3$, $g \ge a_{r,m}$, $d \ge g+r$ and $kd+1-g \le \binom{m+r}{r}$, we get $g = a_{r,m}$ and $d = a_{r,m}+r$. Take a solution C of $H_{r,k-1}$. Hence $C \in H(a_{r,k-1}+r, a_{r,k-1}-b_{r,k-1}, r)$ and $h^i(\mathcal{I}_C(k-1)) = 0$, $i=0,1$. First assume $d \ge a_{r,k-1}+r+(g-a_{r,k-1}+b_{r,k-1})$. Since $(m-2)a_{r,m-1}+r(m-1)+m-3 \ge \binom{r+m-1}{r}$, we have $a_{r,m-1}-2 \ge 2m$. Hence Lemma 3.1 gives the existence of (U,T) with $U \cup T \subset H$, $U \in H(d-a_{r,k-1}, g-1, r-1)$, $\sharp(U \cap C) =$, T a 2-secant line of $W \cup U$ and with $h^1(H, \mathcal{I}_{U\cup T,H}(e)) = 0$. By Remark 2.2 to prove the existence of X_1 it is sufficient to prove $h^1(H, \mathcal{I}_{U\cup T\cup(C\cap H),H}(m)) = 0$. Since $kd+1-g \le \binom{r+k}{r}$, the case $t = k-1$ of (3.1) gives

$$h^0(U\cup T, \mathcal{O}_{U\cup T}(k)) \le \binom{r+k-1}{r-1} - \sharp(C\cap H) + \sharp(C \cap (U\cup T)).$$

The curve $U \subset H$ is general in $H(d-a_{r,k-1}, g-1, r-1)$ by [3], Lemmas 1.5 applied to the integer $r-1$. Hence Lemma 3.1 and the generality of $U\cup T$ gives $h^1(H, \mathcal{I}_{U\cup T}(k)) = 0$. Apply [3], Lemma 1.6.

(b) Now assume $k = m+1$. First assume $kd+1-g > \binom{r+k}{k} - b_{r,m}$. In this case the proof of the case $m=k$ works verbatim, even without knowing the exact values of d and g. Now assume $kd+1-g \le \binom{r+k}{r} - b_{r,m}$ and $d \ge a_{r,m}+2r+1$. Since $d \ge g+r$, we have $d - a_{r,m} - r \ge g - a_{r,m}$. Take a general $C \in H(a_{r,m}+r, a_{r,m}, r)$. Since C has maximal rank ([1], [3]), we have $h^1(\mathcal{I}_C(k-1)) = 0$ and $h^0(\mathcal{I}_C(k-1)) = b_{r,k-1}$. We may assume that C is transversal to H. We claim the existence $U\cup T \subset H$ such that (U,T) satisfies the thesis of Lemma 3.1 and with $U \in H(d-1-a_{r,k}, g-1-a_{r,k}, r-1)$, $\sharp(U\cap C) = 1$ and T 2-secant to $C \cup U$. To check the claim it is sufficient to note that $a_{r,m-1}+r-1 \ge 2(m+1)$. Now assume $d \le a_{r,m}+2r$. Since $d \ge g+r \ge a_{r,m}+r$, we get $d \le g+2r$ and $kd+1-g \le \binom{r+k}{r} - 2k$. We start with a general $C' \in H(a_{r,m}+r-1, a_{r,m}-1, r)$ and add $U' \cup T \subset H$ with $U' \in H(d-a_{r,m}, d-a_{r,m}-r+1, r-1)$ with $\sharp(U'\cap C') = 1 + (g - a_{r,m})$.

(c) Now assume $k \ge m+2$. First assume $d \ge u_{r,k-1,g-1} + v_{r,k-1,g-1} + 1$. Take (C,A) satisfying $R_{r,k-1,g-1}$. Let $U \subset H$ be a general rational normal curve containing exactly one point of each connected component of $C \cup A$, i.e. containing the set $A \cap H$ and exactly one point of $C \cap H$ (C exists, because we assumed $d \ge u_{r,k-1,g-1} + v_{r,k-1,g-1} + 1$). Fix $P \in C \cap H$ with $P \notin U$ and take a general line T through P and intersecting C. For general C, A and U we may assume that T is a 2-secant line of $C\cup A\cup U$. By Lemma 2.3 it is sufficient to prove $h^1(\mathcal{I}_{C\cup A\cup U\cup T}(k)) = 0$, i.e. $h^1(H, \mathcal{I}_{U\cup T\cup(C\cap H)}(k)) = 0$. Since (d,g,r) has critical value k, we have

$$h^0(C\cup T, \mathcal{O}_{C\cup T}(k)) + \sharp(C\cap H) - \sharp(C\cap U) - \sharp(C \cap T) \le \binom{r+k-1}{r-1}.$$

Further, we have $\sharp(C\cap H) - \sharp(C\cap U) \ge 2k$, because $u_{r,k-1,g-1} \ge 3k$ by (3.2). Hence Lemma 3.1 implies $h^1(H, \mathcal{I}_{U\cup T}(k)) = 0$ Apply [3], Lemma 1.6. Now assume

$d \le u_{r,k-1,g-1} + v_{r,k-1,g-1}$. Take $Y \in H(u_{r,k-1,g-1}, g-1, r)$ with maximal rank. Hence $h^1(\mathcal{I}_Y(k-1)) = 0$. First assume $d \ge u_{r,k-1,g-1} + 2$. We add in H the curve $E \cup D$, where E is a smooth rational curve intersecting Y quasi-transversally and exactly one point and D is a 1-secant line of E passing through one of the points of $Y \cap (H \setminus E)$. By Lemma 3.1 we may assume $h^1(H, \mathcal{I}_{E\cup D}(k)) = 0$. Since D is a 2-secant line of $Y \cup E$, it is sufficient to apply Lemma 2.3 and Remark 2.3. Now assume $d \le u_{r,k-1,g-1} + 1$. In this case we have $kd + 1 - g \le \binom{r+k}{r} - 2k$. Take $Y' \in H(u_{r,k-1,g-1} - 1, g-1, r)$ with maximal rank and add $E \cup D \subset H$ with E smooth and rational and $\sharp(E \cap Y') = 1$. □

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References

[1] E. Ballico, Ph. Ellia, *On postulation of curves in* $\mathbb{P}^4$, Math. Z. 188, 2 (1985), 215-223.

[2] E. Ballico, Ph. Ellia, *The maximal rank conjecture for nonspecial curves in* $\mathbf{P}^3$, Invent. Math. 79, 3 (1985) 541-555.

[3] E. Ballico, Ph. Ellia, *The maximal rank conjecture for non-special curves in* $\mathbb{P}^n$, Math. Z. 196 (1987), 355-367.

[4] L. Ein, *The irreducibility of the Hilbert scheme of smooth space curves*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 83-87, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[5] R. Hartshorne, A. Hirschowitz, *Droites en position générale dans* $\mathbb{P}^n$, Algebraic Geometry, Proceedings, La Rábida 1981, 169-188, Lect. Notes in Math. 961, Springer, Berlin, 1982.

[6] R. Hartshorne, A. Hirschowitz, *Smoothing algebraic space curves*, Algebraic Geometry, Sitges 1983, 98-131, Lecture Notes in Math. 1124, Springer, Berlin, 1985.

[7] C. Keem, *Reducible Hilbert scheme of smooth curves with positive Brill-Noether number*, Proc. Amer. Math. Soc. 122, 2 (1994) 349-354.

[8] S. Kim, *On the irreducibility of the family of algebraic curves in complex projective space* $\mathbb{P}^r$, Comm. Algebra 29, 10 (2001), 4321-4331.

[9] E. Sernesi, *On the existence of certain families of curves*, Invent. Math. 75, 1 (1984), 25-57.

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