Geodesics and the geometry of manifolds

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Abstract. In this paper contains some remarks regarding projectively related (metric) connections on a 4-dimensional manifold, that is, Levi-Civita connections which yield the same paths for their (unparametrised) geodesics. Some techniques for such an investigation are briefly outlined. The signature of the metric is arbitrary but special emphasis is laid on the Lorentz case and the connection with the Einstein principle of equivalence in general relativity theory. The results are based on joint work with David Lonie and Zhixiang Wang in Aberdeen.

1 Introductory remarks on projective relatedness

On the Euclidean plane E (similar comments apply to higher dimensions) the collection of maps defined on the whole of E and which map straight lines to straight lines is the 6-parameter affine group of *affine* maps: $(x, y) \to (ax + by + c, dx + ey + f)$ for constants a, b, c, d, e and f, with $ae - bd \neq 0$. However, there are other maps which also map straight lines to straight lines but which are not necessarily defined on the whole of E. These constitute the 8-parameter projective group of *projective* maps; $(x, y) \to (\frac{ax+by+c}{px+qy+r}, \frac{dx+ey+f}{px+qy+r})$ for constants a, b, c, d, e, f, p, q and r, with $ae - bd \neq 0 \neq px + qy + r$. The latter transformations arise by first extending E by means of the usual line at infinity to obtain the projective plane and then considering the "straight line" preserving maps on this latter space and projecting them down to E. Thus the line $px + qy + r = 0$ is mapped onto the line at infinity and the affine group is recovered from these transformations by setting $p = q = 0, r = 1$. One can view this in another way; if $f : E \to E$ is an affine map, its pullback, f^*, preserves the Levi-Civita connection D on E arising from the Euclidean metric on E. This is a consequence of the fact that such a map preserves affine parameters on the straight lines (geodesics of D) in E. However, if f is a projective map, f does not necessarily preserve affine parameters and so f^* does not necessarily preserve D but rather "replaces" D by another (distinct) connection D' which is projectively related to D, that is, two distinct connections D and D' arise which agree as to their (unparametrised)

geodesics. The leads to the following question; is it "general" for a manifold to admit distinct connections having the same (unparametrised) geodesics?

Let M be a connected, smooth manifold admitting (smooth) metrics g and g' with associated Levi-Civita connections ∇ and ∇'. The signature of g and of g' is arbitrary. Then ∇ and ∇' (or (M, g) and (M, g')) are called *projectively related* if the unparametrised geodesics associated with ∇ and ∇' have identical paths, (but not necessarily the same affine parameters). This turns out to be equivalent to the condition that, in any coordinate system on M, the Christoffel symbols arising from ∇ and ∇' satisfy [2, 19]

$$(1.1) \qquad \Gamma'^{a}_{bc} = \Gamma^{a}_{bc} + \delta^{a}_{b}\psi_c + \delta^{a}_{c}\psi_b,$$

where ψ is a global, smooth 1-form on M which, since ∇ and ∇' are metric connections, can be shown to be exact; $\psi = d\chi$ for some global (smooth) real valued function χ on M [2].

Another equivalent condition to projective relatedness for (M, g) and (M, g') is (regarding (M, g) as given) that g' satisfies

$$(1.2) \qquad g'_{ab;c} = 2g'_{ab}\psi_c + g'_{ac}\psi_b + g'_{bc}\psi_a,$$

where a semi-colon denotes a ∇-covariant derivative. Thus the problem is; given g on M solve (1.2) for g' and ψ. There is a transformation due to Sinjukov [18] which can simplify this problem. One replaces the second order, non-degenerate, symmetric tensor g' by the second order, non-degenerate, symmetric tensor a and the exact 1-form ψ by the exact 1-form λ according to

$$(1.3) \qquad a_{ab} = e^{2\chi}g'^{cd}g_{ac}g_{bd} \qquad \lambda_a = -e^{2\chi}\psi_b g'^{bc}g_{ac} \; (\Rightarrow \lambda_a = -a_{ab}\psi^b),$$

where an abuse of notation has been used in that g'^{ab} denotes the contravariant components of g' (and not the tensor g'_{ab} with indices raised using g) so that $g'_{ac}g'^{cb} = \delta^b_a$. Then (1.3) may be inverted to give

$$(1.4) \qquad g'^{ab} = e^{-2\chi}a_{cd}g^{ac}g^{bd} \qquad \psi_a = -e^{-2\chi}\lambda_b g^{bc}g'_{ac}.$$

The condition (1.2) for projective relatedness can now be shown, from (1.3) and (1.4), to be equivalent to *Sinjukov's equation* [18]

$$(1.5) \qquad a_{ab;c} = g_{ac}\lambda_b + g_{bc}\lambda_a.$$

Thus, given (M, g), one tries to solve (1.5) for a and λ and then uses (1.3) and (1.4) to recover g' and ψ. It is noted that the condition $\psi \equiv 0$ on M, the condition $\lambda \equiv 0$ on M and the condition $\nabla = \nabla'$ are equivalent.

The general idea for the remainder of this paper is, firstly, to summarise the techniques and results for this problem *in the case when dimM = 4 but with g of arbitrary signature.* They were given in [12] (for g of positive definite signature), [9, 10, 11] (for g of Lorentz signature) and [20] (for g of neutral $(+, +, -, -)$ signature). The final section will concentrate on the applications to general relativity theory.

Some notation can be usefully given at this point. The tangent space to M at $m \in M$ is denoted by $T_m M$ whilst the 6-dimensional vector space of 2-forms (bivectors) at m is denoted by $\Lambda_m M$. Since a metric is involved the tensor type (that is

$(2,0)$, $(1,1)$ or $(0,2)$) of members of $\Lambda_m M$ will be ignored due to the natural isomorphisms between them obtained through the metric (raising and lowering indices). Then $\Lambda_m M$ is a Lie algebra under the usual matrix commutation (denoted $[,]$) and admits a metric $<,>$ given, for $F, G \in \Lambda_m M$ by $< F, G >= F^{ab} G_{ab}$. This metric has signature $(+,+,+,+,+,+)$, $(+,+,+,-,-,-)$ and $(+,+,-,-,-,-)$ when g has signature $(+,+,+,+)$, $(+,+,+,-)$ and $(+,+,-,-)$, respectively. A member $F \in \Lambda_m M$ has matrix rank an even integer and if this integer is 2, F is called *simple*. If F is simple it may be written as $F^{ab} = p^a q^b - q^a p^b$ for $p, q \in T_m M$. The 2-space of $T_m M$ spanned by such a pair p and q is independent of the choice of p and q and is called the *blade* of F and one sometimes denotes F, or its blade, by $p \wedge q$. If F is simple and $< F, F >$ is positive (respectively, negative) F is called *spacelike* (respectively, *timelike*). If F has matrix rank 4, it is called *non-simple*. The usual Hodge duality operator (a linear map $\Lambda_m M \to \Lambda_m M$) is denoted by $*$ and then F is simple if and only if $\overset{*}{F}$ is simple. One can then define the subspaces $\overset{+}{S}_m$ and $\overset{-}{S}_m$ of $\Lambda_m M$ by $\overset{+}{S}_m \equiv \{F \in \Lambda_m M : \overset{*}{F} = F\}$ and $\overset{-}{S}_m \equiv \{F \in \Lambda_m M : \overset{*}{F} = -F\}$ and also the set $\tilde{S}_m \equiv \overset{+}{S}_m \cup \overset{-}{S}_m$.

2 The algebra of 2-forms and the curvature tensor

The techniques used in [12, 9, 10, 11, 20] were based on holonomy theory and this in turn requires knowledge of the Lie (orthogonal) algebras of the signatures involved. This requires some discussion of the algebra of 2-forms for these signatures and these will be outlined now.

2.1 Case g has positive definite signature $(+,+,+,+)$

In this case each simple $F \in \Lambda_m M$ is spacelike and for any such F an orthogonal basis x, y, z, w for $T_m M$ exists so that F is proportional to $x \wedge y$ and $\overset{*}{F}$ to $z \wedge w$. If F is non-simple, a similar basis exists such that $F = \alpha(x \wedge y) + \beta(z \wedge w)$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0 \neq \beta$. If, in addition, $\alpha \neq \pm\beta$, F determines the 2-spaces $x \wedge y$ and $z \wedge w$ uniquely. Each non-zero $F \in \tilde{S}_m$ is non-simple. In fact, a non-simple $F \in \Lambda_m M$ takes the above form with $\alpha = \beta$ if and only if $F \in \overset{+}{S}_m$ and with $\alpha = -\beta$ if and only if $F \in \overset{-}{S}_m$. It can be checked that if $F \in \overset{+}{S}_m$ and $G \in \overset{-}{S}_m$ then $[F, G] = 0$ and $< F, G >= 0$ and that $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$ is a product of Lie algebras since each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is isomorphic to the Lie algebra $o(3)$. For this signature, each $F \in \Lambda_m M$ satisfies the condition that the double dual $\overset{**}{F}$ of F is equal to F.

2.2 Case g has Lorentz signature $(+,+,+,-)$

In this case $\overset{**}{F} = -F$ for each $F \in \Lambda_m M$ and the subspaces $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are trivial. From a pseudo-orthogonal basis x, y, z, t at m with $g_m(x,x) = g_m(y,y) = g_m(z,z) = -g_m(t,t) = 1$ one may construct a null basis l, n, y, z at m with $\sqrt{2}l = x + t$ and $\sqrt{2}n = x - t$ so that l and n are null vectors at m. One may choose a null basis

in which a spacelike bivector F is proportional to $y \wedge z$ and one, where a timelike bivector may be written as a multiple of $t \wedge x$ or $l \wedge n$. A simple $F \in \Lambda_m M$ satisfying $< F, F >= 0$ is called *null* and a null basis as above may be chosen so that F is s multiple of $l \wedge y$ and $\overset{*}{F}$ of $l \wedge z$. If $F \in \Lambda_m M$ is non-simple, a null basis may be chosen so that $F = \alpha(l \wedge n) + \beta(y \wedge z)$ for non-zero real numbers α and β and the 2-spaces $(l \wedge n)$ and $(y \wedge z)$ are uniquely determined by F.

2.3 Case g has neutral signature (+,+,-,-)

In this case one has $\overset{**}{F} = F$ for each $F \in \Lambda_m M$. For $F \in \overset{+}{S}_m$ and $G \in \overset{-}{S}_m$, $[F, G] = 0$ and $< F, G >= 0$ and the Lie algebra $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$ with each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ being isomorphic to the Lie algebra $o(1, 2)$. In this case one can choose an orthonormal basis x, y, s, t with $g_m(x, x) = g_m(y, y) = -g_m(s, s) = -g_m(t, t) = 1$ and then a null basis of null vectors l, n, L, N, where $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ and so $g_m(l, n) = g_m(L, N) = 1$ with all other inner products between these null basis members equal to zero. For this signature, a simple $F \in \Lambda_m M$ is called *null* if $< F, F >= 0$ *and* $F \notin \tilde{S}_m$ and *totally null* if $< F, F >= 0$ *and* $F \in \tilde{S}_m$. It follows that the blade of any totally null simple bivector consists of null vectors any two of which are orthogonal. If the above orthonormal basis is chosen so that the duals of $x \wedge y$, $x \wedge t$ and $x \wedge s$ are, respectively, $s \wedge t$, $s \wedge y$ and $y \wedge t$, then $\overset{+}{S}_m$ is spanned by $l \wedge n - L \wedge N$, $l \wedge N$ and $n \wedge L$ and a similar basis can be constructed for $\overset{-}{S}_m$. It then follows that $\overset{+}{S}_m$ and $\overset{-}{S}_m$ consist only of totally null (simple) bivectors and non-simple bivectors.

2.4 The curvature map

The curvature tensor $Riem$ gives rise to a linear map $\Lambda_m M \rightarrow \Lambda_m M$ called the *curvature map* and given by $F^{ab} \rightarrow R^{ab}{}_{cd} F^{cd}$ [4]. The algebraic properties of this map, together with its rank, range space and kernel are rather important as will be seen in the next section.

3 General techniques

The general procedure for a (partial) solution of this problem consists of several steps and depends on the signature of g. Only a very brief summary can be given here with more details available in the literature quoted. The main idea is to consider the holonomy group Φ of (M, g) for which the associated holonomy algebra ϕ is a subalgebra of the appropriate orthogonal algebra the latter being, in the above cases, either $o(4)$, $o(1, 3)$ or $o(2, 2)$. One can utilise the bivector representation of these orthogonal algebras from $\Lambda_m M$ and construct all possible subalgebras. This is reasonably straightforward for the positive definite signature but more complicated for Lorentz signature and even more so for neutral signature. Such subalgebras are available in the literature (see [17] for Lorentz signature and [3] for neutral signature) but lists tailored to the present needs can be found in [12, 4, 20]. One then notes that the range of the curvature map is a subspace of the *infinitesimal holonomy algebra* this

latter being a subalgebra of ϕ. Thus, having chosen a particular holonomy algebra (in bivector representation) to study, the range of the curvature map is a subspace of it. (The Ambrose-Singer theorem [1] is useful here and for details of this and of holonomy theory in general, see [14].)

Now starting with (M, g), with $\dim M = 4$ and g of arbitrary signature, suppose that (M, g') is projectively related to it, so that the results of section 1 apply. If $F \in \Lambda_m M$ lies in the kernel of the curvature map obtained from ∇, so that $R^{ab}{}_{cd}F^{cd} = 0$, then if F is simple, say $F = p \wedge q$ for $p, q \in T_m M$, it turns out [9] that $p \wedge q$ is an eigenspace of the symmetric, second order tensor $\nabla \lambda$ (that is, of $\lambda_{a;b}$) with respect to g. Thus one immediately has information on λ. Similar comments, depending on the signature of g, apply if F is non-simple. If this kernel is such that $T_m M$ becomes an eigenspace of $\nabla \lambda$ with respect to g at each $m \in M$ then [9]

$$(3.1) \qquad (a)\, \nabla \lambda = cg, \qquad (b)\, \lambda_d R^d{}_{abc} = 0, \qquad (c)\, a_{ae} R^e{}_{bcd} + a_{be} R^e{}_{acd} = 0$$

holds on M. It follows from (3.1)(a) that either λ vanishes on M (and so $\nabla = \nabla'$) or λ is a non-trivial *homothetic* (co)vector field on M which is non-zero over some open, dense subset of M [4]. Thus one has a kind of "rigidity" result that if λ vanishes on some non-empty open subset $U \subset M$ (equivalently, $\nabla = \nabla'$ on U) then $\nabla = \nabla'$ on M. From another angle this result says that either $\nabla = \nabla'$ or (M, g) is sufficiently special to admit a non-trivial homothety. Result (3.1)(b) shows that the rank of the curvature map is at most 3 and that λ lies in a nice way with respect to the curvature tensor *Riem* from ∇ and can be found or, at least, restricted if the algebraic structure of *Riem* (through the curvature map) is known. Result (3.1)(c) then gives an algebraic relation between the curvature *Riem* and the Sinjukov tensor a; a relation that can be solved algebraically for a if the algebraic structure of *Riem* is known [4]. (Of course, g is a solution of (3.1)(c) but there may be others.) These partial results for λ and a can then be substituted into Sinjukov's equation (1.5) for the final solution. If the kernel of the curvature map does not lead to $T_m M$ being an eigenspace of $\nabla \lambda$ with respect to g at each $m \in M$ other, more direct methods are needed (and can be found in the literature quoted).

The final results of such a programme (for (M, g') projectively related to (M, g)) are that, for g of signature $(+, +, +, +)$, if the holonomy algebra ϕ is a proper subalgebra of $o(4)$, either $\nabla = \nabla'$ or examples can be constructed for which ∇ and ∇' are distinct but where g and g' can still be found. If $\nabla = \nabla'$, holonomy theory can be used to find how g' is related to g. The signature of g' may differ from that of g. More details can be found in [12]. If g has signature $(+, +, -, -)$, the situation is, not surprisingly, more complicated. But one still finds that if ϕ is a proper subalgebra, but not the 5-dimensional subalgebra, of $o(2,2)$ then in many cases $\nabla = \nabla'$ (and the relationship between g' and g can be found) and significant information can be uncovered in the other cases. Again, g and g' may differ in signature. In all of this work the algebra of the members of $\Lambda_m M$ briefly discussed in section 2 is crucial. The situation when g has Lorentz signature will be discussed more fully in the next section. [It is briefly remarked that in all cases one may relate the work discussed here to the problem of studying projective symmetries on the manifold in question.]

4   Lorentz signature and Space-Times

For this rather important signature, there are, in the classification scheme of [17], thirteen possible, proper subalgebras of $o(1,3)$ one of which (the R_5 case in the notation of [17]) can not be a candidate for a holonomy subalgebra. Thus twelve such subalgebras remain and it turns out that in seven of these cases, necessarily one finds $\nabla = \nabla'$ and in three more one again has $\nabla = \nabla'$ except in rather degenerate cases. Again, g and g' can be easily related and may differ in signature. Only two proper subalgebras cause problems and these are discussed in [9, 10, 11]. In each of these cases, the relationship between g' and g can be found.

One particular case (which is of major importance in general relativity theory) is when the original pair (M, g) is a *vacuum* (that is, a *Ricci-flat*) *space-time*. This situation was investigated (in the generalised situation when (M, g) is an Einstein space) many years ago in [15] and more recently in [13] and specifically in the vacuum case in [6, 8]. This importance arises from the well-known Einstein principle of equivalence for space-times. Einstein suggested that the path of a freely falling, neutral, etc, test particle in his theory should be a (timelike) geodesic. Thus one poses the following question; suppose one starts with a *vacuum* space-time (M, g) with Levi-Civita connection ∇ and puts another Lorentz metric g' on M with Levi-Civita connection ∇' insisting that at each $m \in M$ the g-timelike ∇-geodesics from m whose initial direction lies in some non-empty open subset of the set of all directions at m are also g'-timelike geodesics with respect to ∇'. How are ∇ and ∇' (and also g and g') related? It is easily checked that the above conditions lead to (M, g) and (M, g') being projectively related and, much less obviously, that ∇ and ∇' are necessarily equal (implying that (M, g') is also vacuum) and that, with one exceptional case, that g and g' are related by a constant conformal factor, $g' = cg$ for $0 \neq c \in \mathbb{R}$. (The techniques used in [6, 8] either do not use holonomy theory or use it only sparingly.) Thus, with this exceptional case disregarded (perhaps, not surprisingly, it is a (vacuum) pp-wave space-time [6, 8]) the geodesic structure of a vacuum space-time uniquely determines the metric up to units. In fact, one need not assume that the original g-timelike geodesics are g'-timelike, or in fact that g' has Lorentz signature; it is sufficient to assume only that they are ∇'-geodesics. Further, the conclusion that $\nabla = \nabla'$ shows that ∇ and ∇' automatically agree as to what constitutes an affine parameter and thus agree on the concept of proper time in Einstein's theory. So Einstein's principle of equivalence together with the vacuum condition (and disregarding the pp-waves) uniquely determines the space-time metric up to units of measurement.

The more general investigation of the Lorentz case using holonomy theory and described in the first paragraph of this section allows *non-vacuum* space-times to be considered. In particular, one interesting case is when the original space-time (M, g) is a standard Friedmann-Robertson-Walker-Lemaitre (FRWL) cosmological model. Here, the holonomy algebra is, generically, isomorphic to $so(1, 3)$ and can be handled by techniques using Killing and projective symmetry without recourse to holonomy theory. In this case, if (M, g') is projectively related to (M, g), ∇ and ∇' need not be equal and g' and g have a more complicated relationship. However, (M, g') is still an FRWL model which, in general, has a different cosmic time function from (M, g) but whose (constant curvature) space sections of constant cosmic time agree with those of (M, g). In addition, these space sections have the same sign

(or zero) for their (constant) curvature as those of (M, g). Further and from the physical viewpoint, (M, g) and (M, g') have the same Hubble "constant" but different deceleration parameters [7].

Another result of some generality is the following rigidity consequence [11]. First it is noted that one can achieve a convincing definition of a "generic" space-time (or a generic region of a space-time) using the Whitney C^∞ topology [16]. If (M, g) is a space-time and $U \subset M$ is a non-empty, connected, open subset such that, with the metric h induced on it from g, (U, h) is a generic space-time, then if (M, g') is projectively related to (M, g) and the Levi-Civita connections of these space-times agree on U they agree on M.

5   The Weyl projective tensor

In [21], Weyl showed that on a manifold of dimension $n \geq 3$ two projectively related connections ∇ and ∇' necessarily have the same *Weyl projective tensor* W given as the type $(1, 3)$ tensor with components

$$(5.1) \qquad W^a{}_{bcd} = R^a{}_{bcd} - \frac{1}{n-1}(\delta^a{}_c R_{bd} - \delta^a{}_d R_{bc}),$$

where $R_{ab} \equiv R^c{}_{acb}$ are the components of the Ricci tensor. (For dimension 2 the tensor W is identically zero and for dimension 1 it is not defined.) One can then ask about the converse of this result, that is, suppose M is a manifold of dimension at least 3 and ∇ and ∇' are metric connections on M with compatible metrics g and g', respectively (only the metric case is considered here). Suppose also that the associated Weyl projective tensors W and W' from ∇ and ∇' are equal. Are ∇ and ∇' projectively related? The answer, for any such dimension and signature of g, is in the negative. This is discussed in detail in [5]. In spite of this negative result, the condition that two space-times have the same Weyl projective tensor still provides significant information for the relationship between g and g', at least in the vacuum case [6].

References

[1] W. Ambrose, I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. 75 (1953), 428-443.

[2] L. P. Eisenhart, *Riemannian Geometry*, Princeton Univ. Press 1966.

[3] R. Ghanam and G. Thompson, *The holonomy of Lie algebras of neutral metrics in dimension four*, J. Math. Phys. 42 (2001), 2266.

[4] G. S. Hall, *Symmetries and Curvature Structure in General Relativity*, World Scientific, 2004.

[5] G. S. Hall, *On the converse of Weyl's conformal and projective theorems*, Publications de l'Institut Mathematique, 94, (108) (2013), 55-65.

[6] G. S. Hall and D. P. Lonie, *The principle of equivalence and projective structure in spacetimes*, Class. Quant. Grav. 24 (2007), 3617-3636.

[7] G. S. Hall and D. P. Lonie, *The principle of equivalence and cosmological metrics*, J. Math. Phys. **49** (2008), 022502

[8] G. S. Hall and D. P. Lonie, *Geodesic equivalence of Einstein spaces in General Relativity*, Class. Quant. Grav. 26 (2009), 125009.

[9] G. S. Hall and D. P. Lonie, *Holonomy and projective equivalence in 4-dimensional Lorentz manifolds*, Sigma 5 (2009), 066, 23 pages.

[10] G. S. Hall and D. P. Lonie, *Projective structure and holonomy in four-dimensional Lorentz manifolds*, J. Geom. Phys. 61 (2011), 381-399.

[11] G. S. Hall and D. P. Lonie, *Projective structure and holonomy in General Relativity*, Class. Quant. Grav. 28 (2011), 083101.

[12] G. S. Hall and Z. Wang, *Projective structure in 4-dimensional manifolds with positive definite metrics*, J. Geom. Phys. 62 (2012), 449-463.

[13] V. Kiosak and V. Matveev, *Complete Einstein metrics are geodesically rigid*, Comm. Math. Phys. 289, 1 (2009), 383-400.

[14] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 1, Interscience Publ., 1963.

[15] A. Z. Petrov, *Einstein Spaces*, Pergamon Press 1969.

[16] A. D. Rendall, *Curvature of genericspace-times in General Relativity*, J. Math. Phys. 29 (1988), 1569-1574.

[17] J. F. Schell, *Classification of 4-dimensinal Riemannian spaces*, J. Math. Phys. 2 (1961), 202.

[18] N. S. Sinyukov, *Geodesic Mappings of Riemannian Spaces* (in Russian), "Nauka", Moscow, 1979.

[19] T. Y. Thomas, *Differential Invariants of Generalised Spaces*, Cambridge Univ. Press 1934.

[20] Z. Wang and G. S. Hall, *Projective structure in 4-dimensional manifolds with metric signature* $(+, +, -, -)$, J. Geom. Phys. 66 (2013), 37-49.

[21] H. Weyl, *Zur Infinitesimalgeometrie: Einordnung der projektiven und konformen Auffassung*, Nachrichten der Kniglichen Gesellschaft der Wissenschaften zu Göttingen; Mathematisch-physikalische Klasse, Göttinger Nachrichten, (1921), 99112.

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