On Tzitzeica equation and spectral properties of related Lax operators

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Abstract. We discuss the Tzitzeica equation and the spectral properties associated with its Lax operator L. We prove that the continuous spectrum of L is rotated with respect to the contour of the Riemann-Hilbert problem with angle $\pi/6$. We also show that the poles of the dressing factors and their inverses are discrete eigenvalues of L.

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1 Introduction

The famous Tzitzeica equation:

$$2\frac{\partial^2 \phi}{\partial \xi \partial \eta} = e^{2\phi} - e^{-4\phi}, \tag{1.1}$$

was discovered more than a century ago [15, 16] and was first used to analyze special surfaces in differential geometry for which the ratio K/d^4 is constant, see also [18, 17]. Here K is the Gauss curvature of the surface and d is the distance from the origin to the tangent plane at the given point. At the end of 1970-is eq. (1.1) was established to have higher integrals of motion [4]. Next Zhiber and Shabat [22] proved that it is completely integrable Hamiltonian system. Finally Mikhailov constructed its Lax pair [10, 11] which possesses highly nontrivial symmetry, known today as the group of reductions. In fact along with the sine-Gordon eq., Tzitzeica equation (1.1) is one of the simplest representatives of the well known by now 2-dimensional Toda field theories [10, 11].

The present paper proposes a study of the Lax representation of (1.1) and of the spectral properties of the relevant Lax operator. In Section 2 we start with preliminaries concerning the well known facts about the Lax representation and the reductions, proposed by Mikhailov, used to pick it up from the generic Lax operators. In the next Section 3 we construct the fundamental analytic solution of L. We prove that it has analyticity properties with respect to λ in each of the sectors Ω_ν, $\nu =$

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$0, \dots, 5$, see eq. (3.4) and Figure 1. In the next Section 4, starting from a dressing factor inspired by Zakaharov-Shabat-Mikhailov [21, 11], we outline the construction of the generic N-soliton solutions of Tzitzeica eq. In Section 5 we analyze the spectral properties of the Lax pair. We construct the kernel of the resolvent in terms od the FAS of L, see eq. (5.2) below. The theorem 5.1 demonstrates that the continuous spectrum of L is on the rays b_ν (5.4) and is rotated with respect to the contour of the RHP on angle $\pi/6$. We prove that the poles of the dressing factors and their inverse are discrete eigenvalues of L.

2 Preliminaries

We start with the Lax representation of Tzitzeica equation found by Mikhailov [10, 11].

$$
\begin{aligned}
L_1\Psi &\equiv \frac{\partial \Psi}{\partial \xi} - (U_0 + \lambda U_1)\Psi(\xi,\eta,\lambda) = 0, \\
L_2\Psi &\equiv \frac{\partial \Psi}{\partial \eta} - (V_0 + \lambda^{-1} V_1)\Psi(\xi,\eta,\lambda) = 0,
\end{aligned} \tag{2.1}
$$

where

$$
\begin{aligned}
U_0 &= -\begin{pmatrix} \phi_{1,\xi} & 0 & 0 \\ 0 & \phi_{2,\xi} & 0 \\ 0 & 0 & \phi_{3,\xi} \end{pmatrix}, \qquad
U_1 = \begin{pmatrix} 0 & e^{\phi_1-\phi_2} & 0 \\ 0 & 0 & e^{\phi_2-\phi_3} \\ e^{\phi_1-\phi_3} & 0 & 0 \end{pmatrix}, \\
V_0 &= \begin{pmatrix} \phi_{1,\eta} & 0 & 0 \\ 0 & \phi_{2,\eta} & 0 \\ 0 & 0 & \phi_{3,\eta} \end{pmatrix}, \qquad
V_1 = \begin{pmatrix} 0 & 0 & e^{\phi_1-\phi_3} \\ e^{\phi_1-\phi_2} & 0 & 0 \\ 0 & e^{\phi_2-\phi_3} & 0 \end{pmatrix}.
\end{aligned} \tag{2.2}
$$

It is easy to check that the compatibility conditions of L_1 and L_2 gives the equation:

$$
2\frac{\partial^2 \phi_\alpha}{\partial \xi \partial \eta} = e^{2(\phi_\alpha - \phi_{\alpha+1})} - e^{2(\phi_{\alpha-1} - \phi_\alpha)}, \qquad \alpha = 1, 2, 3, \tag{2.3}
$$

where $\alpha \pm 1$ should be taken mod 3, which generalizes the Tzitzeica equation.

Following Mikhailov, we impose reductions of the Lax pair [10, 11]. We notice that the Lax pair above satisfies identically a $\mathbb{Z}_3$-reduction of the form:

$$
Q^{-1}\Psi(\xi,\eta,\lambda)Q = \Psi(\xi,\eta,q\lambda), \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad q = e^{2\pi i/3}. \tag{2.4}
$$

We also impose two $\mathbb{Z}_2$-reductions, as follows.

1. The first $\mathbb{Z}_2$-reduction is

$$
\begin{gathered}
\Psi^*(\xi,\eta,\lambda^*) = \Psi(\xi,\eta,\lambda), \\
U_0 = U_0^*, \qquad U_1 = U_1^*, \qquad V_0 = V_0^*, \qquad V_1 = V_1^*,
\end{gathered} \tag{2.5}
$$

i.e., the fields $\phi_k = \phi_k^*$ are real functions.

2. The second $\mathbb{Z}_2$-reduction

$$A_0^{-1}\Psi^\dagger(\xi,\eta,-\lambda^*)A_0 = \Psi^{-1}(\xi,\eta,\lambda), \qquad A_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

$$A_0^{-1}U_k^\dagger A_0 = (-1)^{k+1}U_k, \qquad A_0^{-1}V_k^\dagger A_0 = (-1)^{k+1}V_k, \qquad k=1,2. \tag{2.7}$$

These conditions lead to:

$$\phi_1 = -\phi_3 = \phi, \qquad \phi_2 = 0, \tag{2.8}$$

and

$$U_1 = \begin{pmatrix} 0 & e^{\phi} & 0 \\ 0 & 0 & e^{\phi} \\ e^{-2\phi} & 0 & 0 \end{pmatrix} \qquad V_1 = \begin{pmatrix} 0 & 0 & e^{-2\phi} \\ e^{\phi} & 0 & 0 \\ 0 & e^{\phi} & 0 \end{pmatrix}. \tag{2.9}$$

After the last reduction Tzitzeica equation acquires its classical form (1.1). There is another form of (1.1) which we will call Tzitzeica II:

$$2\frac{\partial^2\phi}{\partial\xi\partial\eta} = -e^{2\phi} + e^{-4\phi}, \tag{2.10}$$

which is obtained from (1.1) by replacing $\xi \to i\xi$ and $\eta \to i\eta$.

In what follows we will construct the fundamental analytic solutions (FAS) of the Lax pair. For the sake of convenience we will apply to the Lax pair a simple gauge transformation after which the new Lax operator takes the form:

$$L\chi \equiv i\frac{\partial\chi}{\partial\xi} + (Q(\xi) - \lambda J)\,\chi(\xi,\lambda) = 0, \tag{2.11}$$

where we have replaced $i\lambda$ by λ and

(2.12)

$$\chi(\xi,\lambda) = f_0 e^{-\phi H_1}\Psi(\xi,\lambda), \qquad Q(\xi) = -2\frac{\partial\phi}{\partial\xi}(\mathcal{J} - \mathcal{J}^T), \quad J = \operatorname{diag}(q,1,q^2),$$

$$\mathcal{J} = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad f_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} q & 1 & q^2 \\ 1 & 1 & 1 \\ q^2 & 1 & q \end{pmatrix}.$$

3 The FAS of the Lax operators with $\mathbb{Z}_n$-reduction.

The idea for the FAS for the generalized Zakharov-Shabat (GZS) system has been proposed by Shabat [14]. However for the GZS J is with real eigenvalues, while our Lax operator has complex eigenvalues.

The Jost solutions of eq. (2.11) are defined by:

$$\lim_{\xi\to-\infty}\chi_+(\xi,\lambda)e^{i\lambda J\xi} = \mathbb{1}, \qquad \lim_{\xi\to\infty}\chi_-(\xi,\lambda)e^{i\lambda J\xi} = \mathbb{1}. \tag{3.1}$$

They satisfy the integral equations:

$$Y_{\pm}(\xi,\lambda) = \mathbb{1} + \int_{\pm\infty}^{\xi} dy e^{-i\lambda J(\xi-y)} Q(y) Y_{\pm}(y,\lambda) e^{i\lambda J(\xi-y)}, \tag{3.2}$$

where $Y_{\pm}(\xi,\lambda) = \chi_{\pm}(\xi,\lambda)e^{i\lambda J\xi}$. Unfortunately, with our choice for $J = \operatorname{diag}(q,1,q^2)$ this integral equations have no solutions. The reason is that the factors $e^{i\lambda J(\xi-y)}$ in the kernel in (3.2) can not be made to decrease simultaneously.

Following the ideas of Caudrey, Beals and Coifman, see [3, 2, 8] we start with the Jost solutions for potentials on compact support, i.e. assume that $Q(\xi) = 0$ for $\xi < -L_0$ and $\xi > L_0$. Then the integrals in (3.2) converge and one can prove the existence of $Y_{\pm}(\xi,\lambda)$.

Our next step will be to determine the continuous spectrum of L. As we shall show below, the continuous spectrum of L consists of those points λ, for which

$$\operatorname{Im} \lambda(J_k - J_j) = \operatorname{Im} \lambda(q^{2-k} - q^{2-j}) = 0. \tag{3.3}$$

It is easy to check that for each pair of indices $k \neq j$ eq. (3.3) has a solution of the form $\arg\lambda = \mathrm{const}_{kj}$. The solutions for all choices of the pairs k,j fill up a pair of rays l_ν and $l_{\nu+3}$ which are given by:

$$l_\nu\colon\ \arg(\lambda) = \frac{\pi(2\nu+1)}{6}, \qquad \Omega_\nu\colon \frac{\pi(2\nu+1)}{6} \leq \arg\lambda \leq \frac{\pi(2\nu+3)}{6}, \tag{3.4}$$

where $\nu = 0,\dots,5$, see Fig. 1.

Thus the analyticity regions of the FAS are the 6 sectors Ω_ν, $\nu = 0,\dots,6$ split up by the set of rays l_ν, $\nu = 0,\dots,5$, see Fig. 1. Now we will outline how one can construct a FAS in each of these sectors.

Obviously, if $\operatorname{Im}\lambda\alpha(J) = 0$ on the rays $l_\nu \cup l_{\nu+3}$, then $\operatorname{Im}\lambda\alpha(J) > 0$ for $\lambda \in \Omega_\nu \cup \Omega_{\nu+1} \cup \Omega_{\nu+2}$ and $\operatorname{Im}\lambda\alpha(J) < 0$ for $\lambda \in \Omega_{\nu-1} \cup \Omega_{\nu-2} \cup \Omega_{\nu-3}$; of course all indices here are understood modulo 6. As a result the factors $e^{-i\lambda J(\xi-y)}$ will decay exponentially if $\operatorname{Im}\alpha(J) < 0$ and $\xi - y > 0$ or if $\operatorname{Im}\alpha(J) > 0$ and $\xi - y < 0$. In eq. (3.5) below we have listed the signs of $\operatorname{Im}\alpha(J)$ for each of the sectors Ω_ν.

To each ray one can relate the root satisfying $\operatorname{Im}\lambda\alpha(J) = 0$, i.e.

$$\begin{array}{llllll} l_0, & \pm(e_1 - e_2) & \Omega_0 & \alpha_1 < 0, & \alpha_2 > 0 & \alpha_3 > 0 \\ l_1, & \pm(e_1 - e_3) & \Omega_1 & \alpha_1 > 0, & \alpha_2 > 0 & \alpha_3 < 0 \\ l_2, & \pm(e_2 - e_3) & \Omega_2 & \alpha_1 < 0, & \alpha_2 < 0 & \alpha_3 < 0. \end{array} \tag{3.5}$$

There are two fundamental regions: Ω_0 and Ω_1. The transition from Ω_0 and Ω_1 to the other sectors is realized by the automorphism C_0:

$$C_0\Omega_\nu \equiv \Omega_{\nu+2}, \qquad C_0 l_\nu \equiv l_{\nu+2}, \quad \nu = 0,1,2. \tag{3.6}$$

The next step is to construct the set of integral equations for FAS which will be analytic in Ω_ν. They are different from the integral equations for the Jost solutions (3.2) because for each choice of the matrix element (k,j) we specify the lower limit of the integral so that all exponential factors $e^{i\lambda(J_k - J_j)(\xi-y)}$ decrease for $\xi, y \to \pm\infty$,

$$X^\nu_{kj}(\xi,\lambda) = \delta_{kj} + i\int_{\epsilon_{kj}\infty}^{x} dy e^{-i\lambda(J_k-J_j)(\xi-y)} \sum_{p=1}^{h} Q_{kp}(y) X^\nu_{pj}(y,\lambda), \tag{3.7}$$

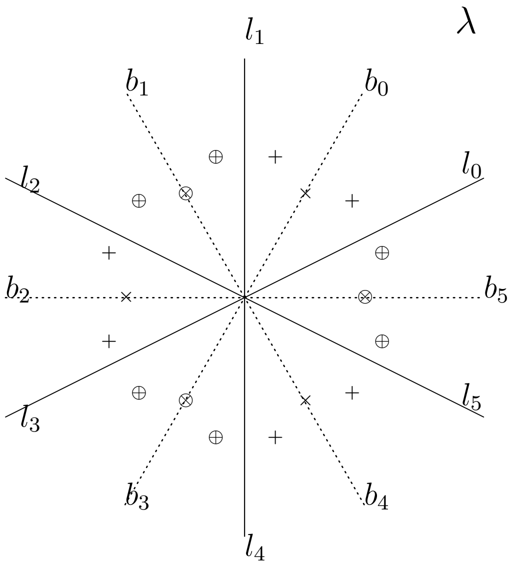

Figure 1: The contour of the RHP with $\mathbb{Z}_3$-symmetry fills up the rays l_ν, $\nu = 1, \dots, 6$. By $\times$ and $\otimes$ (resp. by $+$ and $\oplus$) we have denoted the locations of the discrete eigenvalues corresponding to a soliton of first type (resp. of second type).

where the signs ϵ_{kj} for each of the sectors Ω_ν are collected in the table 1, see also [19, 7, 9]. We also assume that for $k = j$ $\epsilon_{kk} = -1$.

The solution of the integral equations (3.7) will be the FAS of L in the sector Ω_ν. The asymptotics of $X^\nu(x,\lambda)$ and $X^{\nu-1}(x,\lambda)$ along the ray l_ν can be written in the form [8, 9]:

(3.8)
$$\begin{aligned}
\lim_{x\to-\infty} e^{i\lambda Jx} X^\nu(x,\lambda e^{i0}) e^{-i\lambda Jx} &= S_\nu^+(\lambda), && \lambda \in l_\nu,\\
\lim_{x\to\infty} e^{i\lambda Jx} X^\nu(x,\lambda e^{i0}) e^{-i\lambda Jx} &= T_\nu^-(\lambda) D_\nu^+(\lambda), && \lambda \in l_\nu,\\
\lim_{x\to-\infty} e^{i\lambda Jx} X^{\nu-1}(x,\lambda e^{-i0}) e^{-i\lambda Jx} &= S_\nu^-(\lambda), && \lambda \in l_\nu,\\
\lim_{x\to\infty} e^{i\lambda Jx} X^{\nu-1}(x,\lambda e^{-i0}) e^{-i\lambda Jx} &= T_\nu^+(\lambda) D_\nu^-(\lambda), && \lambda \in l_\nu,
\end{aligned}$$

where the matrices $S_\nu^\pm$ and $T_\nu^\pm$ belong to $su(2)$ subgroups of $sl(3)$. More specifically from the integral equations (3.7) we find:

(3.9)
$$\begin{aligned}
S_0^+(\lambda) &= \mathbb{1} + s_{0;21}^+ E_{21}, & T_0^-(\lambda) &= \mathbb{1} + \tau_{0;12}^- E_{12},\\
S_0^-(\lambda) &= \mathbb{1} + s_{0;12}^+ E_{12}, & T_0^+(\lambda) &= \mathbb{1} + \tau_{0;21}^+ E_{21},\\
D_0^+(\lambda) &= d_{0;1}^+ E_{11} + \frac{1}{d_{0;1}^+} E_{22} + E_{33}, & D_0^-(\lambda) &= \frac{1}{d_{0;1}^-} E_{11} + d_{0;1}^- E_{22} + E_{33}.
\end{aligned}$$

(k,j)	(1,2)	(1,3)	(2,3)	(2,1)	(3,2)	(3,1)
Ω_0	$-$	$+$	$+$	$+$	$-$	$-$
Ω_1	$-$	$+$	$-$	$+$	$-$	$+$
Ω_2	$-$	$+$	$-$	$+$	$-$	$+$
Ω_3	$+$	$+$	$+$	$-$	$-$	$-$
Ω_4	$-$	$+$	$-$	$+$	$-$	$+$
Ω_5	$-$	$+$	$-$	$+$	$+$	$-$

Table 1: The set of signs ϵ_{kj} for each of the sectors Ω_ν.

and

$$
\begin{aligned}
&S_1^+(\lambda) = \mathbb{1} + s_{1;31}^+ E_{31}, && T_1^-(\lambda) = \mathbb{1} + \tau_{1;13}^- E_{13}, \\
&S_1^-(\lambda) = \mathbb{1} + s_{1;13}^+ E_{13}, && T_1^+(\lambda) = \mathbb{1} + \tau_{1;31}^+ E_{31}, \\
&D_1^+(\lambda) = d_{1;1}^+ E_{11} + E_{22} + \frac{1}{d_{1;1}^+} E_{33}, && D_1^-(\lambda) = \frac{1}{d_{1;1}^-} E_{11} + E_{22} + d_{1;1}^- E_{33},
\end{aligned} \tag{3.10}
$$

where by E_{kj} we mean a 3×3 matrix with matrix elements $(E_{kj})_{mn} = \delta_{um}\delta_{jn}$.

The corresponding factors for the asymptotics of $X^\nu(x, \lambda e^{i0})$ for $\nu > 1$ are obtained from eqs. (3.9), (3.10) by applying the automorphism C_0. If we consider potential on finite support, then we can define not only the Jost solutions $\Psi_\pm(x,\lambda)$ but also the scattering matrix $T(\lambda) = \chi_-(x,\lambda)\chi_+^{-1}(x,\lambda)$. The factors $S_\nu^\pm(\lambda)$, $T_\nu^\pm(\lambda)$ and $D_\nu^\pm(\lambda)$ provide an analog of the Gauss decomposition of the scattering matrix with respect to the ν-ordering, i.e.:

$$
T_\nu(\lambda) = T_\nu^-(\lambda) D_\nu^+(\lambda) \hat{S}_\nu^+(\lambda) = T_\nu^+(\lambda) D_\nu^-(\lambda) \hat{S}_\nu^-(\lambda), \qquad \lambda \in l_\nu. \tag{3.11}
$$

The $\mathbb{Z}_n$-symmetry imposes the following constraints on the FAS and on the scattering matrix and its factors:

$$
\begin{aligned}
C_0 X^\nu(x, \lambda\omega) C_0^{-1} &= X^{\nu-2}(x,\lambda), & C_0 T_\nu(\lambda\omega) C_0^{-1} &= T_{\nu-2}(\lambda), \\
C_0 S_\nu^\pm(\lambda\omega) C_0^{-1} &= S_{\nu-2}^\pm(\lambda), & C_0 D_\nu^\pm(\lambda\omega) C_0^{-1} &= D_{\nu-2}^\pm(\lambda),
\end{aligned} \tag{3.12}
$$

where the index $\nu - 2$ should be taken modulo 6. Consequently we can view as independent only the data on two of the rays, e.g. on l_0 and l_1; all the rest will be recovered using the reduction conditions.

If in addition we impose the $\mathbb{Z}_2$-symmetry, then we will have also:

$$
\begin{aligned}
&\text{a)} \quad K_0^{-1}(X^\nu(x, -\lambda^*))^\dagger K_0 = \hat{X}^{N+1-\nu}(x,\lambda), && K_0^{-1}(S_\nu^\pm(-\lambda^*)) K_0 = \hat{S}_{N+1-\nu}^\mp(\lambda), \\
&\text{b)} \quad K_0^{-1}(X^\nu(x, \lambda^*))^* K_0 = \hat{X}^\nu(x,\lambda), && K_0^{-1}(S_\nu^\pm(\lambda^*)) K_0 = \hat{S}_{N+1-\nu}^\mp(\lambda),
\end{aligned} \tag{3.13}
$$

where by ‘hat’ we denote the inverse matrix. Analogous relations hold true for $T_\nu^\pm(\lambda)$ and $D_\nu^\pm(\lambda)$. One can prove also that $D_\nu^+(\lambda)$ (resp. $D_\nu^-(\lambda)$) allows analytic extension for $\lambda \in \Omega_\nu$ (resp. for $\lambda \in \Omega_{\nu-1}$. Another important fact is that $D_\nu^+(\lambda) = D_{\nu+1}^-(\lambda)$ for all $\lambda \in \Omega_\nu$.

The next important step is the possibility to reduce the solution of the ISP for the GZSs to a (local) RHP. More precisely, we have:

$$
\begin{aligned}
&X^{\nu}(x,\eta,\lambda) = X^{\nu-1}(x,\eta,\lambda)G_{\nu}(x,\eta,\lambda), && \lambda\in l_{\nu},\\
&G_{\nu}(x,\eta,\lambda) = e^{i\lambda J\xi-\lambda^{-1}V_2t}G_{0,\nu}(\lambda)e^{-i\lambda J\xi+\lambda^{-1}V_2t}, && G_{0,\nu}(\lambda) = \hat{S}_{\nu}^{-}S_{\nu}^{+}(\lambda)\Big|_{t=0}.
\end{aligned}
\tag{3.14}
$$

The collection of all these relations for $\nu = 0,1,\dots,5$ together with

$$
\lim_{\lambda\to\infty} X^{\nu}(x,\eta,\lambda) = \mathbb{1},
\tag{3.15}
$$

can be viewed as a local RHP posed on the collection of rays $\Sigma \equiv \{l_{\nu}\}_{\nu=1}^{2N}$ with canonical normalization. Rather straightforwardly we can prove that if $X^{\nu}(x,\lambda)$ is a solution of the RHP then $\chi^{\nu}(x,\lambda) = X^{\nu}(x,\lambda)e^{-i\lambda J\xi}$ is a FAS of L with potential

$$
Q(\xi,t) = \lim_{\lambda\to\infty} \lambda\left(J - X^{\nu}(\xi,\eta,\lambda)J\hat{X}^{\nu}(\xi,\eta,\lambda)\right).
\tag{3.16}
$$

4 The dressing method and the N-soliton solutions

There are several methods for effective calculations of soliton solutions for Tzitzeica eq., see [12, 20, 13]. It is also well known that Tzitzeica eq. has two types of one-soliton solutions, see below. The dressing method that we will use below [21, 11, 10] allows us also to find how the spectral properties of L change due to the dressing.

Let us consider dressing factor of the following form:

$$
u(\xi,\eta,\lambda) = \mathbb{1} + \sum_{s=0}^{2}\left(\sum_{l=1}^{N_1}\frac{Q^{-s}A_lQ^{s}}{\lambda-\lambda_l q^{s}} + \sum_{r=N_1+1}^{N}\frac{Q^{-s}A_rQ^{s}}{\lambda-\lambda_r q^{s}} + \sum_{r=N_1+1}^{N}\frac{Q^{-s}A_r^{*}Q^{s}}{\lambda-(\lambda_r^{*})q^{s}}\right),
\tag{4.1}
$$

with $3N_1+6N_2$ poles and λ_p is real if $p\in\overline{1,N_1}$ and complex if $p\in\overline{N_1+1,N_1+N_2}$.

Then we write down the residues $A_k(\xi,\eta)$ as degenerate matrices of the form:

$$
A_k(\xi,\eta) = |n_k(\xi,\eta)\rangle\langle m_k^{T}(\xi,\eta)|, \qquad (A_k)_{ij}(\xi,\eta) = n_{ki}(\xi,\eta)m_{kj}(\xi,\eta).
\tag{4.2}
$$

Thus $u(\xi,\eta,\lambda)$ has 9 poles located at λ_1q^k with λ_1 real and λ_2q^k, $\lambda_2^{*}q^k$, with $k=0,1,2$ and λ_2 complex. From the second $\mathbb{Z}_2$-reduction, $A_0^{-1}u^{\dagger}(\xi,\eta,-\lambda^{*})A_0 = u^{-1}(\xi,\eta,\lambda)$, after taking the limit $\lambda\to\lambda_k$, we obtain algebraic equation for $|n_k\rangle$ in terms of $\langle m_k^{T}|$:

$$
|\nu\rangle = \mathcal{M}^{-1}|\mu\rangle.
\tag{4.3}
$$

Below for simplicity we write down the matrix $\mathcal{M}$ for $N_1=N_2=1$:

$$
|\nu\rangle = \left(\begin{array}{c} |n_1\rangle \\ \hline |n_2\rangle \\ |n_2^{*}\rangle \end{array}\right), \qquad |\mu\rangle = \left(\begin{array}{c} A_0|m_1\rangle \\ \hline A_0|m_2\rangle \\ A_0|m_2^{*}\rangle \end{array}\right), \qquad \mathcal{M} = \left(\begin{array}{c|cc} A & B & B^{*} \\ \hline B^{*} & D & E \\ B & E^{*} & D^{*} \end{array}\right),
\tag{4.4}
$$

$$
\begin{aligned}
&A = \frac{1}{2\lambda_1^3}\operatorname{diag}(Q^{(1)}, Q^{(2)}, Q^{(3)}), && B = \frac{1}{\lambda_1^3+\lambda_2^3}\operatorname{diag}(P^{(1)}, P^{(2)}, P^{(3)}), \\
&D = \frac{1}{2\lambda_2^3}\operatorname{diag}(P^{(1)}, P^{(2)}, P^{(3)}), && E = \frac{1}{\lambda_2^3+\lambda_2^{*,3}}\operatorname{diag}(K^{(1)}, K^{(2)}, K^{(3)}), \\
&Q^{(j)} = \langle m_1^T | \Lambda_{11}^{(j)}(\lambda_l, \lambda_1) | m_1 \rangle, && K^{(j)} = \langle m_2^{*,T} | \Lambda_{12}^{(j)}(\lambda_1, \lambda_2^*) | m_1 \rangle, \\
&P^{(j)} = \langle m_2^T | \Lambda_{21}^{(j)}(\lambda_2, \lambda_1) | m_l \rangle,
\end{aligned}
\tag{4.5}
$$

with

$$
\Lambda_{lp}^{(j)} = -\lambda_l\lambda_p E_{1+j,3-j} + \lambda_l^2 E_{2+j,2-j} + \lambda_p^2 E_{3+j,1-j}, \quad j = 1,2,3. \tag{4.6}
$$

For example, in order to obtain the 2-soliton solution of the Tzitzeica equation we take the limit $\lambda \to 0$ in the equations satisfied by the dressing factor $u(\xi, \eta, \lambda)$ and integrate. The result is:

$$
\phi_{\rm Ns}(\xi, \eta) = -\frac{1}{2}\ln\left| 1 - \frac{n_{1,1}m_{1,1}}{\lambda_1} - \frac{n_{2,1}m_{2,1}}{\lambda_2} - \frac{n_{2,1}^* m_{2,1}^*}{\lambda_2^*} \right|. \tag{4.7}
$$

The above formulae can be easily generalized for any N_1 and N_2.

For the sake of brevity we skip the details, which allow one to obtain the explicit form of the N-soliton solutions. We just mention that along with the explicit expressions for the vectors $|n_k\rangle$ in terms of $\langle m_j|$ that follow from (4.3)–(4.6) and take into account that $|m_j\rangle$ are solutions of the 'naked' Lax operator with vanishing potential $\phi = 0$.

5 The resolvent of the Lax operator

The FAS can be used to construct the kernel of the resolvent of the Lax operator L. In this section by $\chi^\nu(\xi, \lambda)$ we will denote:

$$
\chi^\nu(\xi, \lambda) = u(\xi, \lambda)\chi_0^\nu(\xi, \lambda), \tag{5.1}
$$

where $\chi_0^\nu(\xi, \lambda)$ is a regular FAS and $u(\xi, \lambda)$ is a dressing factor of general form (4.1).

Remark 5.1. The dressing factor $u(\xi, \lambda)$ has $3N_1 + 6N_2$ simple poles located at $\lambda_l q^p$, $\lambda_r q^p$ and $\lambda_r^* q^p$ where $l = 1, \dots, N_1$, $r = 1, \dots, N_2$ and $p = 0, 1, 2$. Its inverse $u^{-1}(\xi, \lambda)$ has also $3N_1 + 6N_2$ poles located $-\lambda_l q^p$, $-\lambda_r q^p$ and $-\lambda_r^* q^p$. In what follows for brevity we will denote them by λ_j, $-\lambda_j$ for $j = 1, \dots, 3N_1 + 6N_2$.

Let us introduce

$$
R^\nu(\xi, \xi', \lambda) = \frac{1}{i}\chi^\nu(\xi, \lambda)\Theta_\nu(\xi - \xi')\hat{\chi}^\nu(\xi', \lambda), \tag{5.2}
$$

$$
\Theta_\nu(\xi - \xi') = \operatorname{diag}\left(\eta_\nu^{(1)}\theta(\eta_\nu^{(1)}(\xi - \xi')), \eta_\nu^{(2)}\theta(\eta_\nu^{(2)}(\xi - \xi')), \eta_\nu^{(3)}\theta(\eta_\nu^{(3)}(\xi - \xi')) \right), \tag{5.3}
$$

where $\theta(\xi - \xi')$ is the step-function and $\eta_\nu^{(k)} = \pm 1$, see the table 2.

Theorem 5.1. *Let $Q(\xi)$ be a Schwartz-type function and let $\lambda_j^\pm$ be the simple zeroes of the dressing factor $u(\xi, \lambda)$ (4.1). Then*

	Υ_0	Υ_1	Υ_2	Υ_3	Υ_4	Υ_5
$\eta_\nu^{(1)}$	$-$	$-$	$-$	$+$	$+$	$+$
$\eta_\nu^{(2)}$	$+$	$+$	$-$	$-$	$-$	$+$
$\eta_\nu^{(3)}$	$-$	$+$	$+$	$+$	$-$	$-$

Table 2: The set of signs $\eta_\nu^{(k)}$ for each of the sectors Υ_ν (5.4).

1. *The functions $R^\nu(\xi,\xi',\lambda)$ are analytic for $\lambda\in\Upsilon_\nu$ where*

$$(5.4)\qquad b_\nu\colon\ \arg\lambda=\frac{\pi(\nu+1)}{3},\qquad \Upsilon_\nu\colon \frac{\pi(\nu+1)}{3}\leq\arg\lambda\leq\frac{\pi(\nu+2)}{3},$$

having pole singularities at $\pm\lambda_j^\pm$;

2. *$R^\nu(\xi,\xi',\lambda)$ is a kernel of a bounded integral operator for $\lambda\in\Upsilon_\nu$;*

3. *$R^\nu(\xi,\xi',\lambda)$ is uniformly bounded function for $\lambda\in b_\nu$ and provides a kernel of an unbounded integral operator;*

4. *$R^\nu(\xi,\xi',\lambda)$ satisfy the equation:*

$$(5.5)\qquad L(\lambda)R^\nu(\xi,\xi',\lambda)=1\!\!1\delta(\xi-\xi').$$

Idea of the proof. 1. First we shall prove that $R^\nu(\xi,\xi',\lambda)$ has no jumps on the rays l_ν. From Section 3 we know that $X^\nu(\xi,\lambda)$ and therefore also $\chi^\nu(\xi,\lambda)$ are analytic for $\lambda\in\Omega_\nu$. So we have to show that the limits of $R^\nu(\xi,\xi',\lambda)$ for $\lambda\to l_\nu$ from Υ_ν and $\Upsilon_{\nu-1}$ are equal. Let show that for $\nu=0$. From the asymptotics (3.8) and from the RHP (3.14) we have:

$$(5.6)\qquad \chi^0(\xi,\lambda)=\chi^1(\xi,\lambda)G_1(\lambda),\qquad G_1(\lambda)=\hat{S}_1^+(\lambda)S_1^-(\lambda),\quad \lambda\in l_1,$$

where $G_1(\lambda)$ belongs to an $SL(2)$ subgroup of $SL(3)$ and is such that it commutes with $\Theta_1(\xi-\xi')$. Thus we conclude that

$$(5.7)\qquad R_1(\xi,\xi',\lambda e^{+i0})=R_1(\xi,\xi',\lambda e^{-i0}),\qquad \lambda\in l_1.$$

Analogously we prove that $R_\nu(\xi,\xi',\lambda e^{+i0})$ has no jumps on the other rays l_ν.

The jumps on the rays b_ν appear because of two reasons: first, because of the functions $\Theta_\nu(\xi-\xi')$ and second, it is easy to check that for $\lambda\in b_\nu$ the kernel $R_\nu(\xi,\xi',\lambda)$ oscillates for ξ,ξ' tending to $\pm\infty$. Thus on these lines the resolvent is unbounded integral operator.

2. Assume that $\lambda\in\Upsilon_\nu$ and consider the asymptotic behavior of $R^\nu(\xi,\xi',\lambda)$ for $\xi,\xi'\to\infty$. From equations (3.8) we find that

$$(5.8)\qquad R^\nu_{ij}(\xi,\xi',\lambda)\ =\ \sum_{p=1}^{n}X^\nu_{ip}(\xi,\lambda)e^{-i\lambda J_p(\xi-\xi')}\Theta_{\nu;pp}(\xi-\xi')\hat{X}^\nu_{pj}(\xi',\lambda).$$

Due to the fact that $\chi_\nu(\xi,\lambda)$ has the special triangular asymptotics for $\xi \to \infty$ and $\lambda \in \Upsilon_\nu$ and for the correct choice of $\Theta_\nu(\xi-\xi')$ (5.3) we check that the right hand side of (5.8) falls off exponentially for $\xi \to \infty$ and arbitrary choice of ξ'. All other possibilities are treated analogously.

3. For $\lambda \in b_\nu$ the arguments of 2) can not be applied because the exponentials in the right hand side of (5.8) $\operatorname{Im}\lambda = 0$ only oscillate. Thus we conclude that $R^\nu(\xi,\xi',\lambda)$ for $\lambda \in b_\nu$ is only a bounded function and thus the corresponding operator $R(\lambda)$ is an unbounded integral operator.

4. The proof of eq. (5.5) follows from the fact that $L(\lambda)\chi_\nu(\xi,\lambda) = 0$ and

$$\frac{\partial \Theta(\xi-\xi')}{\partial \xi} = \mathbb{1}\delta(\xi-\xi'), \tag{5.9}$$

which concludes the proof. □

Lemma 5.2. *The poles of $R^\nu(\xi,\xi',\lambda)$ coincide with the poles of the dressing factors $u(\xi,\lambda)$ and its inverse $u^{-1}(\xi,\lambda)$.*

Proof. The proof follows immediately from the definition of $R^\nu(\xi,\xi',\lambda)$ and from Remark 5.1. □

Thus we have established that dressing by the factor $u(\xi,\lambda)$, we in fact add to the discrete spectrum of the Lax operator $6N_1 + 12N_2$ discrete eigenvalues; for $N_1 = N_2 = 1$ they are shown on Figure 1.

6 Conclusions

We have constructed the FAS of L which satisfy a RHP on the set of rays l_ν. We also constructed the resolvent of the Lax operator and proved that its continuous spectrum fills up the rays b_ν rather than l_ν. From Figure 1 we see that the eigenvalues corresponding to the solitons of first type lay on the continuous spectrum of L. This explains why the solitons of first type are singular functions.

Using the explicit form of the resolvent $R^\nu(\xi,\xi',\lambda)$ and the contour integration method one can derive the completeness relation of the FAS. One can derive also the soliton solutions of the other NLEE in Tzitzeica hierarchy [5, 6]. These equations also have Lax representation with the same Lax operator L, but with different M-operators; usually they are taken to be polynomial in λ. So in deriving their soliton solutions we will need to change only the η-dependence of the vectors m_k.

Similarly one can construct the N-soliton solutions also of the Tzitzeica-II equation and analyze the spectral properties of the relevant Lax operator. This, along with the details of calculating the N-soliton solutions will be published elsewhere.

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