A class of almost tangent structures in generalized geometry

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Abstract. A generalized almost tangent structure on the big tangent bundle $T^{big}M$ associated to an almost tangent structure on M is considered and several features of it are studied with a special view towards integrability. Deformation under a β- or a B-field transformation and the compatibility with a class of generalized Riemannian metrics are discussed. Also, a notion of tangentomorphism is introduced as a diffeomorphism f preserving the (generalized) almost tangent geometry and some remarkable subspaces are proved to be invariant with respect to the lift of f.

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1 Introduction

Almost tangent structures were introduced by R. S. Clark and M. Bruckheimer [4] and H. A. Eliopoulos [10] around 1960 and have been investigated by several authors, see [3], [5]-[8], [19], [25]. As is well-known, the tangent bundle of a manifold carries a canonical integrable almost tangent structure, hence the name. This almost tangent structure plays an important role in the Lagrangian description of analytical mechanics, [7]-[8], [11], [18].

Our aim is to consider this type of structure in *generalized geometry*, a theory introduced by N. Hitchin [13] in order to unify complex and symplectic geometry; Hitchin's suggestion was continued by M. Gualtieri whose PhD thesis [12] is an outstanding work on this subject. More precisely, we consider various versions of almost tangent structures on the big tangent bundles $T^{big}M$ and as main example we associate a generalized almost tangent structure $\mathcal{J}_J$ to a given almost tangent one J on the base manifold M. Let us note that under various names, the notion of generalized almost tangent structure was already considered by I. Vaisman in [22]-[24].

The content of paper is as follows. After a short survey in almost tangent geometry and the construction of $\mathcal{J}_J$ we study its invariance under β- and B-field transformations, respectively, and discuss the compatibility with generalized Riemannian metrics

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of $T^{big}M$ induced by usual Riemannian metrics. Under the name of *tangentomorphisms* we consider the diffeomorphisms f between two almost tangent manifolds preserving their almost tangent structures and consider the same problem on the big tangent bundles. Some remarkable subspaces are associated with a fixed tangentomorphism and their invariance with respect to $\mathcal{J}_J$ is proved. Since integrability is an important issue in a geometry induced by a tensor field of $(1,1)$-type, we study simultaneously integrability of two generalized almost tangent structures $\mathcal{J}_j$ by means of simultaneous integrability of J_1, J_2 of M. The last Section is devoted to the interplay between $\mathcal{J}_J$ and the covariant derivative induced by the Levi-Civita connection of the base manifold M.

2 Almost tangent geometry revisited

Let M be a smooth, m-dimensional real manifold for which we denote: $C^\infty(M)$-the real algebra of smooth real functions on M, $\Gamma(TM)$-the Lie algebra of vector fields on M, $T^r_s(M)$-the $C^\infty(M)$-module of tensor fields of (r,s)-type on M. An element of $T^1_1(M)$ is usually called *vector 1-form* or *affinor*.

Recall the concept of almost tangent geometry:

Definition 2.1. $J \in T^1_1(M)$ is called *almost tangent structure* on M if it has a constant rank and:

$$imJ = \ker J. \tag{2.1}$$

The pair (M, J) is an *almost tangent manifold*.

The name is motivated by the fact that (2.1) implies the nilpotence $J^2 = 0$ exactly as the natural tangent structure of tangent bundles. Denoting $rankJ = n$ it results $m = 2n$. If in addition, we suppose that J is integrable i.e.:

$$N_J(X,Y) := [JX, JY] - J[JX,Y] - J[X,JY] + J^2[X,Y] = 0, \tag{2.2}$$

then J is called *tangent structure* and (M, J) is called *tangent manifold*.

From [20, p. 3246] we get some features of tangent manifolds:
(i) the distribution $imJ\,(= \ker J)$ defines a foliation denoted by $V(M)$ and called *the vertical distribution*.

Example 2.2. $M = \mathbb{R}^2, J_e(x,y) = (0,x)$ is a tangent structure with $\ker J_e$ the Y-axis, hence the name. The subscript e comes from "Euclidean", see also Example 7.4.

(ii) there exists an atlas on M with local coordinates $(x,y) = \left(x^i, y^i\right)_{1\leq i\leq n}$ such that $J = \frac{\partial}{\partial y^i} \otimes dx^i$ i.e.:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{2.3}$$

We call *canonical coordinates* the above (x,y) and the change of canonical coordinates $(x,y) \to (\widetilde{x}, \widetilde{y})$ is given by:

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x) \\ \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^a} y^a + B^i(x). \end{cases} \tag{2.4}$$

It results an alternative description in terms of G-structures. Namely, a tangent structure is a G-structure with:

$$G = \{C = \begin{pmatrix} A & O_n \\ B & A \end{pmatrix} \in GL(2n, \mathbb{R}); \quad A \in GL(n, \mathbb{R}), B \in gl(n, \mathbb{R})\} \tag{2.5}$$

and G is the invariance group of matrix $J = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$, i.e., $C \in G$ if and only if $C \cdot J = J \cdot C$.

The natural almost tangent structure J of $M = TN$ is an example of tangent structure having exactly the expression (2.3) if (x^i) are the coordinates on N and (y^i) are the coordinates in the fibers of $TN \to N$. Also, J_e of Example 2.2 has the above expression (2.3) with $n = 1$, whence it is integrable. A third class of examples is obtained by duality: if J is an (integrable) endomorphism with $J^2 = 0$ then its dual $J^* : \Gamma(T^*M) \to \Gamma(T^*M)$, given by $J^*\alpha := \alpha \circ J$ for $\alpha \in \Gamma(T^*M)$, is (integrable) endomorphism with $(J^*)^2 = 0$. Let us call this type of endomorphisms a *weak almost tangent structure*.

3 Generalized almost tangent structures

Fix now a smooth manifold M of dimension m not necessary even. The framework of this work is provided by the manifold $T^{big}M := TM \oplus T^*M$. This manifold is the total space of a vector bundle $\pi : T^{big}M \to M$; so $T^{big}M$ is called *the big tangent bundle* of M [21] and the C^∞-module of its sections $\Gamma(T^{big}M)$ has the elements $\mathcal{X} = (X, \alpha) = X + \alpha$, where $X \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$. $T^{big}M$ is endowed with *the Courant structure* $(<,>, [,])$, [6]:
1. the (neutral) inner product (of signature (m, m)):

$$g_{big}\left((X, \alpha), (Y, \beta)\right) = \frac{1}{2}\left(\beta(X) + \alpha(Y)\right); \tag{3.1}$$

2. the (skew-symmetric) Courant bracket:

$$[(X, \alpha), (Y, \beta)]_C = \left([X, Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}d(\beta(X) - \alpha(Y))\right). \tag{3.2}$$

The same manifold $TM \oplus T^*M$ is called sometimes *the Pontryagin bundle* of M (in [14]) or *generalized tangent bundle* of M (in [17]).

Inspired by the first Section we introduce:

Definition 3.1. i) A *weak classical generalized almost tangent structure* on M is an endomorphism $\mathcal{J}$ of the big tangent bundle $T^{big}M$ satisfying:

$$\mathcal{J}^2 = 0. \tag{3.3}$$

If, moreover, $\mathcal{J}$ satisfies:

$$\ker \mathcal{J} = im\mathcal{J}, \tag{3.4}$$

then $\mathcal{J}$ is a *classical generalized almost tangent structure*.

ii) ([23, p. 278]) If $\mathcal{J}$ satisfies in addition the property of skew-symmetry with respect to g_{big}:

$$g_{big}(\mathcal{J}\mathcal{X},\mathcal{Y})+g_{big}(\mathcal{X},\mathcal{J}\mathcal{Y})=0, \tag{3.5}$$

then we call it (weak) *generalized almost tangent structure*. Moreover, if $\mathcal{J}$ is integrable i.e. its Nijenhuis tensor vanishes:

$$\mathcal{N}_{\mathcal{J}}(\mathcal{X},\mathcal{Y}):=[\mathcal{J}\mathcal{X},\mathcal{J}\mathcal{Y}]_C-\mathcal{J}[\mathcal{X},\mathcal{J}\mathcal{Y}]_C-\mathcal{J}[\mathcal{J}\mathcal{X},\mathcal{Y}]_C+\mathcal{J}[\mathcal{X},\mathcal{Y}]_C=0, \tag{3.6}$$

then $\mathcal{J}$ is called (weak) *generalized tangent structure*.

iii) If $\mathcal{J}(TM)\subset TM$ and $\mathcal{J}(T^*M)\subset T^*M$ then $\mathcal{J}$ is called (weak) *splitting generalized (almost) tangent structure*.

Remark 3.2. The interest in such types of endomorphisms comes from the theory of Dirac structures, a concept introduced in [6] in order to give a geometric theory of constrained (physical) systems; for other details see [1]. More precisely, as is pointed out in [24], for a weak generalized tangent structure $\mathcal{J}$ its image $im\mathcal{J}:=\mathcal{D}_{\mathcal{J}}$ is a Dirac structure.

Recall after [12] that an arbitrary endomorphism $\mathcal{J}$ can be represented in the matrix form:

$$\mathcal{J}=\begin{pmatrix} A & \sharp_\pi \\ \flat_\sigma & B\end{pmatrix}, \tag{3.7}$$

where:

$$\begin{cases} A:\Gamma(TM)\to\Gamma(TM), \quad A:=p_{TM}\circ\mathcal{J}\circ i_{TM} \\ \sharp_\pi:\Gamma(T^*M)\to\Gamma(TM), \quad \sharp_\pi:=p_{TM}\circ\mathcal{J}\circ i_{T^*M} \\ \flat_\sigma:\Gamma(TM)\to\Gamma(T^*M), \quad \flat_\sigma:=p_{T^*M}\circ\mathcal{J}\circ i_{TM} \\ B:\Gamma(T^*M)\to\Gamma(T^*M), \quad B:=p_{T^*M}\circ\mathcal{J}\circ i_{T^*M}\end{cases}$$

with p_* the projection and i_* the inclusion map. The condition (3.5) yields that:

i) $\sharp_\pi$ is defined by a bivector π by $\sharp_\pi(\alpha):=i_\alpha\pi$, for $\alpha\in\Gamma(T^*M)$,

ii) $\flat_\sigma$ is defined by a 2-form σ by $\flat_\sigma(X):=i_X\sigma$, for $X\in\Gamma(TM)$,

iii) $B=-A^*$.

and hence the condition (3.3) means:

$$A^2=-\sharp_\pi\circ\flat_\sigma, \quad \pi(A^*\alpha,\beta)=\pi(\alpha,A^*\beta), \quad \sigma(AX,Y)=\sigma(X,AY). \tag{3.8}$$

The second relation (3.8) reads π *is compatible with* A while the third part of (3.8) is expressed as σ *is compatible with* A. The first relation (3.8) means that: $A^2X=-i_{i_X\sigma}\pi$ for every vector field $X\in\Gamma(TM)$; therefore $\beta(A^2X)=-\pi(i_X\sigma,\beta)=-\pi(\sigma(X,\cdot),\beta)$ for any $\beta\in\Gamma(T^*M)$.

Example 3.3. An almost tangent structure J yields a classical generalized almost tangent structure $\mathcal{J}_J$ with:

$$\mathcal{J}_J:=\begin{pmatrix} J & 0 \\ 0 & -J^*\end{pmatrix}, \tag{3.9}$$

since $\mathcal{J}_J^2 = 0$ and also $\mathcal{J}_J$ satisfies (3.4). Moreover, we have (3.5) and then we call it *the generalized almost tangent structure* induced by J. Note that $\mathcal{J}_J$ is a splitting generalized almost tangent structure.

With respect to integrability we have:

Proposition 3.1. *The generalized almost tangent structure $\mathcal{J}_J$ is integrable if and only if J is integrable. The associated Dirac structure is $\mathcal{D}_{\mathcal{J}_J} = V(M) \oplus V^*(M)$ where $V^*(M)$ is the foliation generated by the weak tangent structure J^*.*

Proof. We have: $N_{\mathcal{J}}(\mathcal{X} = X + \alpha, \mathcal{Y} = Y + \gamma) = Z + \eta$ where $Z = [JX, JY] - J[X, JY] - J[JX, Y]$ and:

$$\eta(V) = \alpha(N_J(Y, V)) - \gamma(N_J(X, V)), \tag{3.10}$$

for any $V \in \Gamma(TM)$. In other words:

$$N_{\mathcal{J}}(\mathcal{X} = X + \alpha, \mathcal{Y} = Y + \gamma) = (N_J(X, Y), \alpha \circ N_J(\cdot, Y) - \gamma \circ N_J(X, \cdot)) \tag{3.11}$$

and the conclusion follows directly. The second part is a direct application of Remark 3.2. □

More generally, if $a, b \in \mathbb{R}^*$ then the pencil:

$$\mathcal{J}_{J,a,b} := \begin{pmatrix} aJ & 0 \\ 0 & -bJ^* \end{pmatrix} \tag{3.12}$$

is a splitting weak generalized almost tangent structure and $\mathcal{J}_J = \mathcal{J}_{J,1,1}$.

4 Compatibility with generalized Riemannian metrics induced by usual metrics

Recall after [24] that a *generalized Riemannian metric* on the big tangent bundle $T^{big}M$ can be produced by an endomorphism $\mathcal{G}$ on this manifold such that:
1. $\mathcal{G}^2 = I_{T^{big}M}$ i.e. $\mathcal{G}$ is an almost product structure on $T^{big}M$,
2. $g_{big}(\mathcal{G}\mathcal{X}, \mathcal{G}\mathcal{Y}) = g_{big}(\mathcal{X}, \mathcal{Y})$ i.e. $\mathcal{G}$ is a g_{big}-orthogonal transformation.

Representing $\mathcal{G}$ as:

$$\mathcal{G} = \begin{pmatrix} \varphi & \sharp_{g_1} \\ \flat_{g_2} & \varphi^* \end{pmatrix} =: \mathcal{G}_{\varphi, g_1, g_2}, \tag{4.1}$$

where φ is an endomorphism of the tangent bundle TM, φ^* its dual map, $\flat_{g_i}(X) := i_X g_i$, $X \in \Gamma(TM)$ and $\sharp_{g_i} := \flat_{g_i}^{-1}$, $i \in \{1, 2\}$ for g_1, g_2 Riemannian metrics on M, the above two conditions are equivalent to:

$$\varphi^2 = I - \sharp_{g_1} \circ \flat_{g_2}, \quad g_i(X, \varphi Y) = -g_i(\varphi X, Y), \tag{4.2}$$

for any X, $Y \in \Gamma(TM)$ and $i \in \{1, 2\}$.

Fix now (J, g) a pair (almost tangent structure, Riemannian metric) on M and for $\varepsilon = \pm 1$ say that J is *ε-compatible with* g if $g(JX, Y) = \varepsilon g(X, JY)$, for any X, $Y \in \Gamma(TM)$. Consider also on $T^{big}M$ the generalized Riemannian metric $\mathcal{G}_g = \mathcal{G}_{0,g,g}$ induced by g. A natural question is if the induced generalized almost tangent structure $\mathcal{J}_J$ is compatible with this generalized Riemannian metric.

Proposition 4.1. *If J is ε-compatible with g then the generalized tangent structure $\mathcal{J}$ induced by J is $(-\varepsilon)$-compatible with the generalized Riemannian metric $\mathcal{G}_g$:*

$$\mathcal{G}_g \circ \mathcal{J}_J = -\varepsilon \mathcal{J}_J \circ \mathcal{G}_g. \tag{4.3}$$

Proof. We have:

$$\mathcal{G}_g := \begin{pmatrix} 0 & \sharp_g \\ \flat_g & 0 \end{pmatrix} \tag{4.4}$$

and then:

$$\mathcal{G}_g \circ \mathcal{J}_J = \begin{pmatrix} 0 & -\sharp_g \circ J^* \\ \flat_g \circ J & 0 \end{pmatrix}, \quad \mathcal{J}_J \circ \mathcal{G}_g = \begin{pmatrix} 0 & J \circ \sharp_g \\ -J^* \circ \flat_g & 0 \end{pmatrix}.$$

The hypothesis means $\flat_g \circ J = \varepsilon J^* \circ \flat_g$ yielding then $\sharp_g \circ J^* = \varepsilon J \circ \sharp_g$. Comparing the previous relations it results the required equality. □

5 Deformation under B-field and β-field transformations

Besides the diffeomorphisms, the Courant bracket admits some other symmetries, namely the *B-field transformations.* Now we are interested in what happens if we apply to the generalized almost tangent structure $\mathcal{J}_J$ a B-field transformation.

Let B be a 2-form on M viewed as a map $B : \Gamma(TM) \to \Gamma(T^*M)$ and consider the B-transform:

$$e^B := \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}.$$

We define $\mathcal{J}_{B,J} := e^B \mathcal{J}_J e^{-B}$ which has the expression:

$$\mathcal{J}_{B,J} = \begin{pmatrix} J & 0 \\ BJ + J^*B & -J^* \end{pmatrix}. \tag{5.1}$$

$\mathcal{J}_{B,J}$ coincides with $\mathcal{J}_J$ if and only if $BJ + J^*B = 0$ which means the skew-symmetry:

$$B(JX, Y) = -B(X, JY), \tag{5.2}$$

for any X, $Y \in \Gamma(TM)$.

Example 5.1. Let (J, g) be an *almost tangent metric structure* which means that J is (-1)-compatible with g. We consider the associated 2-form $B(X, Y) := g(JX, Y)$ for X, $Y \in \Gamma(TM)$ and then $B(JX, Y) = -B(X, JY)$ since both expressions are equal to 0. In conclusion $\mathcal{J}_{B,J}$ is just $\mathcal{J}_J$.

Proposition 5.1. *For any 2-form B the endomorphism $\mathcal{J}_{B,J}$ is a classical generalized almost tangent structure which is a generalized almost tangent structure if and only if B satisfies the skew-symmetry condition* (5.2).

Proof. Indeed, $\mathcal{J}_{B,J}^2 = e^B \mathcal{J}_J^2 e^{-B} = 0$, so $im\mathcal{J}_{B,J} \subseteq \ker \mathcal{J}_{B,J}$. Let $\mathcal{X} = X + \alpha \in \ker \mathcal{J}_{B,J}$. Then $JX = 0$ so that $X \in \ker J = imJ$ and $J^*(\alpha - B(X)) = 0$ so that $\alpha - B(X) \in \ker J^* = imJ^*$. Take $X = JY$ and $\alpha = B(X) + J^*\gamma$. It follows $X + \alpha = \mathcal{J}_{B,J}(Y + B(Y) - \gamma) \in im\mathcal{J}_{B,J}$ and we have the second part of conclusion, $\ker \mathcal{J}_{B,J} \subseteq im\mathcal{J}_{B,J}$. □

Remark 5.2. In the general case, if $\mathcal{J}$ is represented as $\mathcal{J} = \begin{pmatrix} J & \beta \\ B & -J^* \end{pmatrix}$, then its B-transform:

$$\mathcal{J}_B = \begin{pmatrix} J - \beta B & \beta \\ BJ + J^*B + B - B\beta B & -J^* + B\beta \end{pmatrix} \tag{5.3}$$

defines also a weak classical generalized almost tangent structure.

Similarly we shall see what happens if we apply to the endomorphism $\mathcal{J}_J$ a β-field transformation. Let β be a bivector field on M viewed as a map $\beta : \Gamma(T^*M) \to \Gamma(TM)$ and consider the β-transform:

$$e^\beta := \begin{pmatrix} I & \beta \\ 0 & I \end{pmatrix}. \tag{5.4}$$

We can define $\mathcal{J}_{\beta,J} := e^\beta \mathcal{J}_J e^{-\beta}$ which has the expression:

$$\mathcal{J}_{\beta,J} = \begin{pmatrix} J & -J\beta - \beta J^* \\ 0 & -J^* \end{pmatrix}, \tag{5.5}$$

which means that for $\mathcal{X} = X + \alpha \in \Gamma(T^{big}M)$, we have:

$$\mathcal{J}_{\beta,J}(\mathcal{X}) = (JX - J(\beta(\alpha)) - \beta(J^*\alpha), -J^*\alpha).$$

If the bivector field β satisfies the skew-symmetry $\beta \circ J^* = -J \circ \beta$ then $\mathcal{J}_{\beta,J}$ coincides with $\mathcal{J}_J$.

Proposition 5.2. *For any bivector field β the endomorphism $\mathcal{J}_{\beta,J}$ is a classical generalized almost tangent structure.*

Proof. Indeed, $\mathcal{J}_{\beta,J}^2 = e^\beta \mathcal{J}_J^2 e^{-\beta} = 0$ so $im\mathcal{J}_{\beta,J} \subseteq \ker \mathcal{J}_{\beta,J}$. Let $X + \alpha \in \ker \mathcal{J}_{\beta,J}$. Then $J^*\alpha = 0$ so that $\alpha \in \ker J^* = imJ^*$ and $J(X - \beta(\alpha)) = 0$ so that $X - \beta(\alpha) \in \ker J = imJ$. Take $\alpha = J^*\gamma$ and $X = \beta(\alpha) + JY$. It follows $X + \alpha = \mathcal{J}_\beta(Y - \beta(\gamma) - \gamma) \in im\mathcal{J}_{\beta,J}$ and we have the other inclusion, too, $\ker \mathcal{J}_{\beta,J} \subseteq im\mathcal{J}_{\beta,J}$. □

Remark 5.3. In the general case, if $\mathcal{J}_J$ is represented $\mathcal{J} = \begin{pmatrix} J & \beta \\ B & -J^* \end{pmatrix}$ then its β-transform:

$$\mathcal{J}_{\beta,J} = \begin{pmatrix} J + \beta B & -J\beta - \beta J^* + \beta - \beta B \beta \\ B & -J^* - B\beta \end{pmatrix}$$

defines also a weak classical generalized almost tangent structure.

6 Tangentomorphisms and invariant subspaces

We shall prove that a diffeomorphism between two almost tangent manifolds preserving the almost tangent structures induces an isomorphism between their generalized tangent bundles which preserves the associated generalized almost tangent structures.

Definition 6.1. Let (M_1, J_1) and (M_2, J_2) be two almost tangent manifolds. We say that the diffeomorphism $f : M_1 \to M_2$ is a (J_1, J_2)-*tangentomorphism* if it satisfies:

$$J_2 \circ f_* = f_* \circ J_1. \tag{6.1}$$

Lemma 6.1. *If $f : (M_1, J_1) \to (M_2, J_2)$ is a tangentomorphism then $J_1^* \circ f^* = f^* \circ J_2^*$.*

Proof. For $X \in \Gamma(TM_1)$ and $\alpha \in \Gamma(T^*M_2)$ we have:

$$[(J_1^* \circ f^*)(\alpha)](X) = (f^*\alpha)(J_1X) = \alpha(f_*(J_1X))$$

and respectively:

$$[(f^* \circ J_2^*)(\alpha)](X) = (J_2^*\alpha)(f_*X) = \alpha(J_2(f_*X)) = \alpha(f_*(J_1X)),$$

which means the conclusion. □

Proposition 6.2. *Let $f : (M_1, J_1) \to (M_2, J_2)$ be a tangentomorphism. Then it induces an endomorphism between the generalized tangent bundles $f^{big} : T^{big}M_1 \to T^{big}M_2$ given by:*

$$f^{big}(\mathcal{X}) := f_*X + (f^{-1})^*\alpha. \tag{6.2}$$

It satisfies:

$$\mathcal{J}_{J_2} \circ f^{big} = f^{big} \circ \mathcal{J}_{J_1}. \tag{6.3}$$

Proof. Using the previous lemma we obtain for any $\mathcal{X} = X + \alpha \in \Gamma(T^{big}M_1)$:

$$\mathcal{J}_{J_2} \circ f^{big}(\mathcal{X}) = \mathcal{J}_{J_2}(f_*X + (f^{-1})^*\alpha) = (J_2 \circ f_*(X), -J_2^* \circ (f^{-1})^*(\alpha)) =$$

$$= (f_* \circ J_1(X), -(f^{-1})^* \circ J_1^*\alpha) = f^{big}(J_1X - J_1^*\alpha)$$

and the last term is $f^{big} \circ \mathcal{J}_{J_1}(X + \alpha)$ which means the required equality. □

Extending this definition, we say that two generalized almost tangent structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are *isomorphic* if there exists an endomorphism $F : \Gamma(T^{big}M_1) \to \Gamma(T^{big}M_2)$ such that $\mathcal{J}_2 \circ F = F \circ \mathcal{J}_1$.

Let (J_i, g_i) be almost tangent metric structures on M_i, $i \in \{1, 2\}$ and $f : (M_1, J_1, g_1) \to (M_2, J_2, g_2)$ a tangentomorphism. For $i \in \{1, 2\}$, consider:

$$\mathcal{S}_i := \{\mathcal{X} = X + \alpha \in \Gamma(T^{big}M_i) \mid i_X g_i = \alpha\}, \tag{6.4}$$

(6.5)

$$\check{\mathcal{S}}_1^f := \{\mathcal{X} = X + f^*(\alpha) \in \Gamma(T^{big}M_1) \mid i_X g_1 = f^*(\alpha), X \in \Gamma(TM_1), \alpha \in \Gamma(T^*M_2)\},$$

(6.6)
$$\hat{\mathcal{S}}_2^f := \{\mathcal{X} = f_*(X) + \alpha \in \Gamma(T^{big}M_2) \mid i_{f_*(X)}g_2 = \alpha, X \in \Gamma(TM_1), \alpha \in \Gamma(T^*M_2)\}.$$

A straightforward computation gives:

(6.7)
$$\mathcal{J}_{J_i}(\mathcal{S}_i) \subset \mathcal{S}_i,\ \mathcal{J}_{J_1}(\check{\mathcal{S}}_1^f) \subset \check{\mathcal{S}}_1^f,\ \mathcal{J}_{J_2}(\hat{\mathcal{S}}_2^f) \not\subseteq \hat{\mathcal{S}}_2^f.$$

Therefore, a more interesting case is the coincidence of above almost tangent structures:

Proposition 6.3. *Let f be a tangentomorphism on the almost tangent metric manifold (M, J, g). Then the following subspaces of $\Gamma(T^{big}M)$ are invariant by $\mathcal{J}_J$:*

(6.8)
$$\check{\mathcal{S}}^f := \{X + f^*(\alpha) \mid i_X g = f^*(\alpha), X + \alpha \in \Gamma(T^{big}M)\},$$

(6.9)
$$\hat{\mathcal{S}}^f := \{f_*(X) + \alpha \mid i_{f_*(X)}g = \alpha, X + \alpha \in \Gamma(T^{big}M)\},$$

(6.10)
$$\bar{\mathcal{S}}^f := \{f_*(X) + f^*(\alpha) \mid i_{f_*(X)}g = f^*(\alpha), X + \alpha \in \Gamma(T^{big}M)\}.$$

Proof. Fix $Y \in \Gamma(TM)$.
i) For $X + f^*(\alpha) \in \check{\mathcal{S}}^f$ we have $\mathcal{J}_J(X + f^*(\alpha)) := JX - J^*(f^*(\alpha))$. Then:

$$(i_{JX}g)(Y) = g(JX, Y) = -g(X, JY) = -(i_X g)(JY) = -(f^*(\alpha))(JY) = (-J^*(f^*(\alpha)))(Y).$$

ii) For $f_*(X) + \alpha \in \hat{\mathcal{S}}^f$ we have $\mathcal{J}(f_*(X) + \alpha) = J(f_*(X)) - J^*\alpha = f_*(JX) - J^*\alpha$. Then:

$$i_{f_*(JX)}g(Y) = g(f_*(JX), Y) = g(J(f_*(X)), Y) = -g(f_*X, JY) = -i_{f_*X}g(JY) =$$
$$= -J^*(i_{f_*X}g)(Y) = -J^*\alpha(Y).$$

iii) For $f_*(X) + f^*(\alpha) \in \bar{\mathcal{S}}^f$ we have $\mathcal{J}(f_*(X) + f^*(\alpha)) := J(f_*(X)) - J^*(f^*(\alpha)) = f_*(JX) - f^*(J^*\alpha)$. Then:

$$i_{f_*(JX)}g(Y) = g(f_*(JX), Y) = g(J(f_*X), Y) = -g(f_*X, JY) = -i_{f_*X}g(JY) =$$
$$= -f^*\alpha(JY) = -J^*f^*\alpha(Y)$$

and the last term is $-f^*(J^*\alpha)(Y)$, which gives the conclusion. □

7 Simultaneously integrability of two generalized almost tangent structures

Two skew-commuting almost tangent structures J_1 and J_2 on a $4k$-dimensional manifold M satisfying:

(7.1)
$$\dim(\ker J_1 \cap \ker J_2) = k$$

are simultaneously integrable if [15]-[16]:

(7.2)
$$N_{J_1,J_1} = 0, \quad N_{J_1,J_2} = 0, \quad N_{J_2,J_2} = 0,$$

where the Nijenhuis tensor field of the pair (J_1, J_2) is generally defined as:

$$\begin{aligned} 2N_{J_1,J_2}(X,Y) = \; & [J_1X, J_2Y] - J_1[J_2X, Y] - J_2[X, J_1Y] + [J_2X, J_1Y] \\ & -J_2[J_1X, Y] - J_1[X, J_2Y] + (J_1J_2 + J_2J_1)[X, Y]. \end{aligned} \tag{7.3}$$

From these conditions follows that both J_1 and J_2 are integrable but conversely not.

Let us remark that the generalized almost tangent structures $\mathcal{J}_{J_1}$, $\mathcal{J}_{J_2}$ are skew-commuting if and only if the almost tangent structures J_1 and J_2 are skew-commuting. Inspired by the result above we introduce:

Definition 7.1. Two generalized almost tangent structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on the $4k$-dimensional manifold M satisfying $\dim(\ker \mathcal{J}_1 \cap \ker \mathcal{J}_2) = 2k$ are said to be *simultaneously integrable* if:

$$N_{\mathcal{J}_1,\mathcal{J}_1} = 0, \quad N_{\mathcal{J}_1,\mathcal{J}_2} = 0, \quad N_{\mathcal{J}_2,\mathcal{J}_2} = 0, \tag{7.4}$$

where the Nijenhuis tensor field of the pair $(\mathcal{J}_1, \mathcal{J}_2)$ is:

$$\begin{aligned} 2N_{\mathcal{J}_1,\mathcal{J}_2}(\mathcal{X},\mathcal{Y}) = \; & [\mathcal{J}_1\mathcal{X}, \mathcal{J}_2\mathcal{Y}]_C - \mathcal{J}_1[\mathcal{J}_2\mathcal{X}, \mathcal{Y}]_C - \mathcal{J}_2[\mathcal{X}, \mathcal{J}_1\mathcal{Y}]_C + [\mathcal{J}_2\mathcal{X}, \mathcal{J}_1\mathcal{Y}]_C \\ & -\mathcal{J}_2[\mathcal{J}_1\mathcal{X}, \mathcal{Y}]_C - \mathcal{J}_1[\mathcal{X}, \mathcal{J}_2\mathcal{Y}]_C + (\mathcal{J}_1\mathcal{J}_2 + \mathcal{J}_2\mathcal{J}_1)[\mathcal{X}, \mathcal{Y}]_C. \end{aligned} \tag{7.5}$$

Remark that these conditions yields that both $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable but not conversely.

Proposition 7.1. *Let two skew-commuting almost tangent structures J_1 and J_2 be given on the $4k$-dimensional manifold M satisfying* $\dim(\ker J_1 \cap \ker J_2) = k$. *Then the generalized almost tangent structures $\mathcal{J}_{J_1}$ and $\mathcal{J}_{J_2}$ are simultaneously integrable if and only if J_1 and J_2 are simultaneously integrable.*

Proof. Since we have

$$\dim(\ker \mathcal{J}_{J_1} \cap \ker \mathcal{J}_{J_2}) = 2\dim(\ker J_1 \cap \ker J_2) + 2\dim(M) - [\dim(\ker J_1) + \dim(\ker J_2)] \tag{7.6}$$

and from the condition $\ker J_i = imJ_i$, $i \in \{1, 2\}$, we deduce that $\dim(\ker J_i) = \dim(M) = 4k$. The relation between the intersection of the kernels becomes:

$$\dim(\ker \mathcal{J}_{J_1} \cap \ker \mathcal{J}_{J_2}) = 2\dim(\ker J_1 \cap \ker J_2) = 2k. \tag{7.7}$$

Similar to the formula (3.11) we have that $N_{\mathcal{J}_{J_1},\mathcal{J}_{J_2}}(\mathcal{X} = X + \alpha, \mathcal{Y} = Y + \gamma) = Z + \eta$, where $Z = N_{J_1,J_2}(X, Y)$ and:

$$\eta(V) = \alpha(N_{J_1,J_2}(Y, V)) - \gamma(N_{J_1,J_2}(X, V)), \tag{7.8}$$

for any $V \in \Gamma(TM)$. In conclusion, $N_{\mathcal{J}_{J_i},\mathcal{J}_{J_j}} = 0$, $i \in \{1, 2\}$, if and only if $N_{J_i,J_j} = 0$, $i \in \{1, 2\}$. □

Example 7.2. For any $a, b \in \mathbb{R}^*$ define now the family $(J_{a,b})$ with $J_{a,b} := a \cdot J_1 + b \cdot J_2$. A straightforward calculus gives that $J_{a,b}$ defines an almost tangent structure if and only if $J_1J_2 + J_2J_1 = 0$. Similar, consider the family $(\mathcal{J}_{a,b})$ defined by $\mathcal{J}_{a,b} := a \cdot \mathcal{J}_1 + b \cdot \mathcal{J}_2$. In fact:

$$\mathcal{J}_{a,b} := \begin{pmatrix} a \cdot J_1 + b \cdot J_2 & 0 \\ 0 & -(a \cdot J_1 + b \cdot J_2)^* \end{pmatrix} = \begin{pmatrix} J_{a,b} & 0 \\ 0 & -J^*_{a,b} \end{pmatrix}. \tag{7.9}$$

It results that $\mathcal{J}_{a,b}$ is a weak generalized almost tangent structure if and only if $J_1J_2 + J_2J_1 = 0$.

In order to have a class of examples we introduce:

Definition 7.3. Let g be a non-degenerate 2-form on M. Two almost tangent structures (J_1, J_2) form *a dual pair with respect to* g if $\ker J_1 \perp_g \ker J_2$.

Since $\ker J_i = imJ_i$, $i \in \{1, 2\}$, the condition of the previous definition is equivalent to $g(J_1X, J_2Y) = 0$ for any $X, Y \in \Gamma(TM)$. In the same way can be defined a *dual pair* of (weak) generalized almost tangent structures $\mathcal{J}_1$, $\mathcal{J}_2$ with respect to a non-degenerate 2-form g of $T^{big}M$.

Consider now $(\mathcal{J}_1, \mathcal{J}_2)$ a dual pair of (weak) generalized almost tangent structures with respect to the neutral metric g_{big}. Then $g_{big}(\mathcal{J}_1\mathcal{X}, \mathcal{J}_2\mathcal{Y}) = 0$. A step further is to suppose that $(\mathcal{J}_i, g_{big})$, $i \in \{1, 2\}$, are generalized almost tangent metric structures i.e.:

$$g_{big}(\mathcal{J}_i\mathcal{X}, \mathcal{Y}) = -g_{big}(\mathcal{X}, \mathcal{J}_i\mathcal{Y}).$$

Then the image of the endomorphisms $\mathcal{J}_1\mathcal{J}_2$, $\mathcal{J}_2\mathcal{J}_1$ is a subspace in the set of g_{big}-null sections of $T^{big}M$.

Proposition 7.2. *If the almost tangent structures J_1 and J_2 satisfy $J_1J_2 = J_2J_1 = 0$ then the generalized almost tangent structures $\mathcal{J}_{J_1}$ and $\mathcal{J}_{J_2}$ induced by them form a dual pair with respect to g_{big}.*

Proof. For $\mathcal{X} = X + \alpha, \mathcal{Y} = Y + \gamma \in \Gamma(T^{big}M)$ the relation:

$$g_{big}(\mathcal{J}_{J_1}(\mathcal{X}), \mathcal{J}_{J_2}(\mathcal{Y})) = -\frac{1}{2}[\alpha(J_1J_2Y) + \gamma(J_2J_1X)] = 0$$

gives the conclusion. □

Example 7.4. Returning to Example 2.2 it results that J_e and J_e^{dual} given by $J_e^{dual}(x, y) = (y, 0)$ form a dual pair with respect to the Euclidean metric of $\mathbb{R}^2$. We have:

$$J_eJ_e^{dual} + J_e^{dual}J_e = I. \tag{7.10}$$

A pair (J_1, J_2) of weak almost tangent structures satisfying $J_1J_2 + J_2J_1 = I$ is called *almost bitangent structure* in [9, p. 7].

8 Covariant derivatives on the generalized tangent bundle

Let ∇ be the Levi-Civita connection associated to a given Riemannian metric g on M and ∇' its extension to 1-forms [2, p. 28]:

$$(\nabla'_X\alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_XY), \tag{8.1}$$

with $X, Y \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$. Then we define the extension of ∇ to $T^{big}M$:

$$\nabla^{big}_{\mathcal{X}}\mathcal{Y} = \nabla^{big}_{X+\alpha}Y + \gamma := \nabla_XY + \nabla'_{\sharp_g\alpha}\gamma. \tag{8.2}$$

In general, ∇^{big} is not a linear connection on $T^{big}M$, but it satisfies the following properties:

i) is $\mathbb{R}$-bilinear,

ii) $\nabla^{big}_{f\mathcal{X}}\mathcal{Y} = f\nabla^{big}_{\mathcal{X}}\mathcal{Y}$ for any $f \in C^{\infty}(M)$,

iii) $\nabla^{big}_{\mathcal{X}} f\mathcal{Y} = f\nabla^{big}_{\mathcal{X}}\mathcal{Y} + X(f)\mathcal{Y}$.

If ∇ is J-invariant: $\nabla_X JY = J(\nabla_X Y)$ for any $X, Y \in \Gamma(TM)$, then ∇' is J^*-invariant: $\nabla'_X J^*\alpha = J^*(\nabla'_X\alpha)$ for any $\alpha \in \Gamma(T^*M)$. With respect to the big tangent bundle we have:

Proposition 8.1. *If ∇ is J-invariant then ∇^{big} is $\mathcal{J}_J$-invariant.*

Proof. From definitions it results:

$$\begin{aligned}\mathcal{J}_J(\nabla^{big}_{\mathcal{X}}\mathcal{Y}) &= \mathcal{J}_J(\nabla_X Y + \nabla'_{\sharp_g\alpha}\gamma) \\ &= J(\nabla_X Y) - J^*(\nabla'_{\sharp_g\alpha}\gamma) \\ &= \nabla_X JY - \nabla'_{\sharp_g\alpha}J^*\gamma = \nabla^{big}_{\mathcal{X}}\mathcal{J}_J\mathcal{Y},\end{aligned}$$

for any $\mathcal{X} = X + \alpha, \mathcal{Y} = Y + \gamma \in \Gamma(T^{big}M)$. □

Remark that ∇^{big} is a *natural operator*, that is, for any isometry $f : (M_1, g_1) \to (M_2, g_2)$ such that the isomorphism f^{big} satisfies $f^{big}(\mathcal{S}_1) \subseteq \mathcal{S}_2$ with respect to S_i from (6.4), the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}_1 \times \mathcal{S}_1 & \overset{\nabla^{big}_1}{\longrightarrow} & \mathcal{S}_1 \\ f^{big} \times f^{big} \downarrow & & \downarrow f^{big} \\ \mathcal{S}_2 \times \mathcal{S}_2 & \overset{\nabla^{big}_2}{\longrightarrow} & \mathcal{S}_2 \end{array}.$$

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