Uniqueness of complete spacelike hypersurfaces in generalized Robertson-Walker spacetimes

Junhong Dong and Ximin Liu

Abstract. In this paper we study the uniqueness of complete noncompact spacelike hypersurfaces immersed in generalized Robertson-Walker (GRW) spacetimes. According to a suitable restriction on the higher order mean curvature and the norm of the gradient of the height function of the hypersurface, we obtain some rigidity theorems in GRW spacetimes. Besides, we establish nonparametric results on the entire vertical graph in such ambient spacetimes.

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1 Introduction

Spacelike hypersurface in spacetimes are objects of increasing interest in recent years. A basic question on this topic is the problem of uniqueness of spacelike hypersurface in certain spacetimes. For instance, in [7] L. J. Alías with A. Romero and M. Sánchez prove that in a generalized Robertson-Walker spacetime obeying the timelike convergence condition, every compact spacelike hypersurface with constant mean curvature is totally umbilical, also they are spacelike slices, except in very exceptional cases. Recall that a spacetime is said to obey the *timelike (null) convergence condition* if the Ricci curvature is nonnegative on timelike (lightlike) direction. In [8] by those same authors and in [20] by S. Montiel, they all obtained uniqueness result for compact spacelike hypersurface with constant mean curvature in some spacetimes. More generally, in [5] L. J. Alías and A. G. Colares have studied the problem of uniqueness for compact spacelike hypersurface with constant higher order mean curvature immersed in generalized Robertson-Walker spacetimes, where the so-called generallized Robertson-Walker (GRW) spacetimes are Lorentzian warped products $-I \times_f M^n$ with 1-dimensional negative definite base I, warping function f and Riemannian fibre M^n, when the Riemannian fibre M^n has constant sectional curvature then $-I \times_f M^n$ is called a Robertson-Walker (RW) spacetime. In [6] the compact spacelike hypersurface was extended to complete spacelike hypersurface. Besides, there are many other

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authors have studied these problems in this branch, such as [10, 11], where M. Caballero and other authors have obtained some rigidity and uniqueness results for the spacelike hypersurface in GRW spacetimes.

In this paper, we will develop another method which follows the ideas of [9], using the operators L_k apply to the height function of the complete non-compact spacelike hypersurfaces in GRW spacetimes, according to impose a suitable restriction on the higher order mean curvature and norm of the gradient of the height function, we then obtain rigidity theorems in spacetimes, which without needing the restriction timelike convergence condition or null convergence condition.

Our approach is based on the use of second order linear differential operators L_k associate with the Newton transformations, combining with a consequence of a version of Stokes theorem on an n-dimensional, complete noncompact Riemannian manifold, which obtained by S. T. Yau in [24]. In secition 3, we obtain the following (Theorem 3.3)

> Let $\overline{M} = -I \times_f M^n$ be a RW spacetime, $\psi : \Sigma^n \to -I \times_f M^n$ (with $n \geq 2$) be a complete, connected spacelike hypersurface bounded away from the infinity of $\overline{M}$. Suppose that mean curvature H is bounded on Σ^n and $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$. Assume that either
> (i) when $k = 1$, $H_2 > 0$ and $0 < \frac{H_2}{H_1} \leq \frac{f'}{-f\langle N, \partial_t\rangle}$, or
> (ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map*, and $0 < \frac{H_{k+1}}{H_k} \leq \frac{f'}{-f\langle N, \partial_t\rangle}$.
> Then the hypersurface Σ^n is a slice.

In section 4, we establish nonparametric results of Theorem 3.1, 3.2 and 3.3. Taking into account the entire vertical graphs immersed in a RW spacetime, then we obtain the following nonparametric version of Theorem 3.3 (Corollary 4.1)

> Let $\overline{M} = -I \times_f M^n$ be a RW spacetime, $\Sigma^n(u)$ (with $n \geq 2$) be a entire spacelike vertical graph bounded away from the infinity of $\overline{M}$. Suppose that mean curvature H is bounded on Σ^n and $|Du|_{M^n} \in \mathcal{L}^1(M^n)$. For some constant $0 \leq \alpha < 1$, $|Du|^2_{M^n} \leq \alpha f^2(u)$, assume that either
> (i) when $k = 1$, $H_2 > 0$ and $0 < \frac{H_2}{H_1} \leq (1-\alpha)\frac{f'}{f}(h)$, or
> (ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map*, and $0 < \frac{H_{k+1}}{H_k} \leq (1-\alpha)\frac{f'}{f}(h)$.
> Then the hypersurface Σ^n is a slice.

In section 5, taking into account that the sign of derivation of the warping function f, we obtain our applications of the previous result. For instance, when the ambient spacetime is the steady state space, which is the model of half $\mathcal{H}^{n+1}$ of de Sitter space, then from Theorem 3.3 we have following consequence (Corollary 5.2).

> Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete, connected spacelike hypersurface bounded away from the infinity of $\mathcal{H}^{n+1}$. Suppose that the mean curvature H of Σ^n is bounded and $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$. Assume that either
> (i) when $k = 1$, $H_2 > 0$ and $\frac{H_1}{H_2} \geq \cosh\theta$, or
> (ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map* and $\frac{H_k}{H_{k+1}} \geq \cosh\theta$.
> Then the hypersurface Σ^n is a slice.

2 Preliminaries

In this section, we introduce some basic notations and facts that will appear along this paper.

Let M^n be a connected n-dimensional ($n \geq 2$) Riemannian manifold, I is a 1-dimensional manifold (either a circle or an open interval of $\mathbb{R}$), and $f : I \to \mathbb{R}$ is a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = -I \times_f M^n$, let π_I and π_M denote the projections onto the factors I and M, respectively.

The class of Lorentzian manifolds which will be our concern here is the one obtained by furnishing $\overline{M}^{n+1}$ with the metric

$$\langle v, w\rangle_p = -\langle (\pi_I)_* v, (\pi_I)_* w\rangle + (f \circ \pi_I)^2(p)\langle (\pi_M)_* v, (\pi_M)_* w\rangle,$$

for all $p \in \overline{M}^{n+1}$ and all $v, w \in T_p\overline{M}$. Following the terminology introduced in [7], such a space is called a *generalized Robertson-Walker* (GRW) Spacetimes, f is known as the warping function and we shall write $\overline{M}^{n+1} = -I \times_f M^n$ to denote it. When the Riemannian fibre M^n has constant sectional curvature, then $-I \times_f M^n$ is classically called a *Robertson-Walker* (RW) Spacetimes.

We remark (cf.[22]) that the GRW spacetime $-I \times_f M^n$ has constant sectional curvature $\overline{\kappa}$ if and only if it is an RW spacetime for which

$$\frac{f''}{f} = \overline{\kappa} = \frac{(f')^2 + \kappa}{f^2},$$

where κ is the (constant) value of the sectional curvature of M^n.

Consider a smooth immersion $\psi : \Sigma^n \to -I \times_f M^n$ of an n-dimensional connected manifold Σ^n is said to be a spacelike hypersurface if the induced metric via ψ is a Riemannian metric on Σ^n, which is also denoted by $\langle\ ,\ \rangle$. Since $\partial_t = (\frac{\partial}{\partial t})_{(t,x)}$, $(t, x) \in -I \times_f M^n$ is a unitary timelike vector field globally defined on the ambient spacetime, then there exists an unique timelike vector field N defined on the spacelike hypersurface Σ^n which is the same time-orientation as ∂_t. By using Cauchy-Schwarz inequality, we get

$$\langle N, \partial_t\rangle \leq -1 < 0 \quad on\ \Sigma^n. \tag{2.1}$$

We will refer to the normal vector field N as the *future-pointing Gauss map* of the spacelike hyperface Σ^n. The normal hyperbolic angle θ of Σ^n is the smooth function $\theta : \Sigma^n \to [0, +\infty]$ given by

$$\cosh\theta = -\langle N, \partial_t\rangle. \tag{2.2}$$

Denoting $\overline{\nabla}$ and ∇ are the Levi-Civita connection in $-I \times_f M^n$ and Σ^n, respectively. Then the Gauss and Weingarten formulas for the spacelike hypersurface $\psi : \Sigma^n \to -I \times_f M^n$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y\rangle N, \qquad \overline{\nabla}_X N = -AX,$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the shape operator of Σ^n with respect to the future-pointing Gauss map N.

Associate with the shape operator of Σ there are n algebraic invariants which are the elementary symmetric functions σ_k of its principal curvature $k_1, \cdots, k_n$, given by

$$S_k(p) = \sigma_k(k_1, k_2, \ldots, k_n) = \sum_{i_1 < \cdots < i_k} k_{i_1} \cdots k_{i_k},$$

where $k = 1, \ldots, n,\ S_0 = I$. The kth-mean curvature H_k of the hypersurface is then defined by

$$\binom{n}{k} H_k = (-1)^k S_k.$$

Thus $H_1 = -\frac{1}{n} tr(A) = H$ is the mean curvature, when k is even, it follows from the Gauss equation that H_k is a geometric quantity which is related to the intrinsic curvature of Σ.

In what follows we will work based on the so-called *New transformations* $P_k : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which are defined from A, setting by $P_0 = I$ (the identity of $\mathfrak{X}(\Sigma)$) and for $1 \leq k \leq n$, $P_k = \binom{n}{k} H_k I + A \circ P_{k-1}$, then we have

$$tr(P_k) = c_k H_k,\ tr(A \circ P_k) = -c_k H_{k+1},$$

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$.

Associated with each *Newton transformations* P_k, we consider the second order linear differential operator $L_k : \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma)$, given by $L_r(f) = \mathrm{tr}(P_k \circ \nabla^2 f)$. Here $\nabla^2 f : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f, and it is given by $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle, \quad X, Y \in \mathfrak{X}(\Sigma)$.

Observe that

$$\begin{aligned} L_k f &= \mathrm{tr}(P_k \circ \nabla^2 f) = \sum_{i=1}^n \langle P_k \circ (\nabla_{E_i} \nabla f), E_i\rangle = \sum_{i=1}^n \langle \nabla_{E_i} \nabla f, P_k(E_i)\rangle \\ &= \sum_{i=1}^n \langle \nabla_{P_k(E_i)} \nabla f, E_i\rangle = \mathrm{tr}(\nabla^2 f \circ P_k), \end{aligned}$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on Σ. Moreover, we also have that

$$\mathrm{div}(P_k(\nabla f)) = \sum_{i=1}^n \langle (\nabla_{E_i} P_k)(\nabla f), E_i\rangle + \sum_{i=1}^n \langle (P_k(\nabla_{E_i} \nabla f), E_i\rangle \quad = \quad \langle \mathrm{div} P_k, \nabla f\rangle + L_k f,$$

where $\mathrm{div} P_k := \mathrm{tr}(\nabla P_k) = \sum_{i=1}^n (\nabla_{E_i} P_k)(E_i)$. For general k, and provided $\overline{M}^{n+1}$ has constant sectional curvature, it was shown by Rosenberg in [23] that

$$L_k f = \mathrm{div}(P_k(\nabla f)).$$

It follows from the formula above we have the operator L_k is elliptic if and only if P_k is positive definite. Observe that $L_0 = \Delta$, then the Laplacian of Σ which is always an elliptic operator in divergence form. We close this section by quoting two useful lemmas in which geometric conditions are given in order to guarantee the ellipticity of L_k when $k \geq 1$.

Lemma 2.1 ([5])**.** *If $H_2 > 0$ on Σ^n, then P_1 is positive definite for a appropriate choice of the* Gauss map *N.*

Lemma 2.2 ([5])**.** *Let Σ^n having an elliptic point with respect to an appropriate choice of the* Gauss map*. If $H_{r+1} > 0$ on Σ^n for some $2 \leq r \leq n-1$, then P_k is positive definite for all $1 \leq k \leq r$.*

3 Rigidity result in GRW spacetimes

Let $\psi : \Sigma \to -I \times_f M^n$ be a spacelike hypersurface with Gauss map N. The height function of Σ denoted by h, then we have h is the restriction of the projection $\pi_I(t,x) = t$ to Σ, that is $h = \pi_I \circ \psi : \Sigma \to I$, so that the gradient of h on Σ is $\nabla h = (\overline{\nabla}\pi_I)^T = -\partial_t^T$, where $\partial_t^T \in \mathfrak{X}(\Sigma)$ denotes the tangential component of ∂_t, i.e.,

$$\partial_t = \partial_t^T - \langle N, \partial_t \rangle N.$$

Denoting $|\ |$ as the norm of a vector field on Σ, we have

$$|\nabla h|^2 = \langle N, \partial_t \rangle^2 - 1.$$

We will need the following result of L.J. Alías and A.G. Colares.

Lemma 3.1 ([5])**.** *If $\psi : \Sigma^n \to -I \times_f M^n$ is a spacelike hypersurface immersed in a GRW spacetime with* Gauss map *N. Let $h = \pi_I \circ \psi$ denote the height function of Σ. Then for every $k = 0, \ldots, n-1$ we have*

$$L_k(h) = -(\log f)'(h)(c_k H_k + \langle P_k(\nabla h), \nabla h \rangle) - \langle N, \partial_t \rangle c_k H_{k+1} \tag{3.1}$$

Then from formula (3.1) we get that

$$L_k(h) = -\frac{1}{f(h)}[c_k(f'(h)H_k + \langle N, \partial_t \rangle f(h) H_{k+1}) + f'(h)\langle P_k(\nabla h), \nabla h \rangle] \tag{3.2}$$

In [24] S.T. Yau have the Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold, then in [15] A. Caminha et al. obtained a suitable consequence of S.T. Yau's result, we state it as following, where we denote $\mathcal{L}^1(\Sigma)$ be the space of Lebesgue integrable functions on Σ.

Lemma 3.2 ([15])**.** *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold Σ^n, such that* $\mathrm{div}X$ *does not change sign on Σ^n. If $|X| \in \mathcal{L}^1(\Sigma)$, then* $\mathrm{div}X = 0$.

Lemma 3.3 ([15])**.** *Let $\psi : \Sigma^n \to \overline{M}^{n+1}(\overline{\kappa})$ be a complete oriented hypersurface immersed in a space form $\overline{M}^{n+1}$ of constant sectional curvature $\overline{\kappa}$, with bounded second fundamental form. If $g : M \to \mathbb{R}$ is a smooth function such that $|\nabla g| \in \mathcal{L}(\Sigma)$ and $L_r g$ does not change sign on Σ, then $L_r g = 0$ on M.*

Along this work, we will assume the fibre M^n of the spacetime is complete and N will stand for the future-pointing *Gauss map* of the spacelike hypersurface Σ^n unless we change it.

According to [2] we say that a spacelike hypersurface $\psi : \Sigma^n \to -I \times_f M^n$ is *bounded away from the future infinity* of $-I \times_f M^n$ if there exists $\bar{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t,x) \in -I \times_f M^n; t \leq \bar{t}\}.$$

Analogously, we say that Σ^n is *bounded away from the past infinity* of $-I \times_f M^n$ if there exists $\underline{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t,x) \in -I \times_f M^n; t \geq \underline{t}\}.$$

Finally, Σ^n is said to be *bounded away from the infinity* if there exists $\underline{t} < \bar{t}$ such that $\psi(\Sigma)$ is contained in the slab bounded by the slices $\{\underline{t} \times M^n\}$ and $\{\bar{t} \times M^n\}$.

Theorem 3.4. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime,, $\psi : \Sigma^n \to -I \times_f M^n$ be a complete, connected spacelike hypersurface bounded away from the infinity of $\overline{M}$. Suppose*

$$\frac{f}{f'}(h)H_1 \leq -\frac{1}{\langle N, \partial_t \rangle}, \tag{3.3}$$

then if $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$, Σ^n is a slice.

Proof. Since N is the Gauss map such that $\langle N, \partial_t \rangle < 0$, then from (3.3) we have

$$1 + \langle N, \partial_t \rangle \frac{f}{f'}(h)H_1 \geq 0.$$

From formula (3.2) we have

$$\Delta h = -\frac{f'}{f}(h)[n(1 + \langle N, \partial_t \rangle \frac{f}{f'}(h)H_1) + |\nabla h|^2].$$

Thus we have $\mathrm{div}(\nabla h) = \Delta h$ does not change sign on Σ^n under the condition that $f' \neq 0$. Now, taking into account $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$, we can apply Lemma 3.2 to get that $\mathrm{div}(\nabla h)$ vanishes on Σ^n. Therefore, we can conclude that $|\nabla h|$ is identically zero on Σ^n, that is the hypersurface Σ^n is a slice. □

Theorem 3.5. *Let $\overline{M} = -I \times_f M^n$ be a RW spacetime, $\psi : \Sigma^n \to -I \times_f M^n$ (with $n \geq 2$) be a complete, connected spacelike hypersurface bounded away from the infinity of $\overline{M}$. Suppose that mean curvature H_1 is bounded on Σ^n and $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$. Assume that either*

(i) when $k = 1$, $H_2 > 0$ and

$$\frac{f'}{-f\langle N, \partial_t \rangle} \leq \frac{H_2}{H_1} < 0, \ \textit{or} \tag{3.4}$$

(ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map*, and

$$\frac{f'}{-f\langle N, \partial_t \rangle} \leq \frac{H_{k+1}}{H_k} < 0. \tag{3.5}$$

Then the hypersurface Σ^n is a slice.

Proof. (i) when $k = 1$, from (3.4) we have the mean curvature $H_1 < 0$ is bounded and $H_2 > 0$ on Σ. From the Cauchy-Schwarz inequality we have

$$H_1^2 \geq H_2,$$

then we get that there exists a constant c such that

$$|A|^2 = n^2H^2 - n(n-1)H_2 < c.$$

Thus $|A|$ is bounded on Σ^n.

Now we change N to the past-pointing Gauss map, then we obtain that the sign both H_1 and $\langle N, \partial_t\rangle$ are changed, so from Lemma 2.1 we have P_1 is positive definite. Under this condition, from (3.4) we have $f'(h)H + f(h)\langle N, \partial_t\rangle H_2 \leq 0$. Besides, we also get $f' < 0$, thus $f'(h)\langle P_1\nabla h, \nabla h\rangle \leq 0$. From formula (3.2) we have

$$L_1(h) = -\frac{1}{f(h)}[c_1(f'(h)H_1 + \langle N, \partial_t\rangle f(h)H_2) + f'(h)\langle P_1(\nabla h), \nabla h\rangle].$$

Thus we get $L_1(h)$ does not change sign on Σ^n. Now we apply Lemma 3.3 to conclude that $L_1(h)$ vanishes on Σ^n. Therefore, returning to the expression of $L_1(h)$ we get that ∇h is identically zero on Σ^n, then Σ^n is a slice $\{t\} \times M^n$.

(ii) When $2 \leq k \leq n-1$, from (3.5) we have $H_kH_{k+1} < 0$ and $f' < 0$, thus from the hypothesis we can change the N to the past-pointing Gauss map such that there exists an elliptic point. Under this condition, we have both H_k and H_{k+1} are positive definite, then we apply Lemma 2.2 get that P_j are positive definite and H_j are positive for every $1 \leq j \leq k$. Then in a similar way as (i), we can get $|A|$ is bounded,

$$f'(h)H_k + f(h)\langle N, \partial_t\rangle H_{k+1} \leq 0$$

and

$$f'(h)\langle P_k(\nabla h), \nabla h\rangle \leq 0.$$

Thus L_kh does not change sign on Σ.

Now we can use Lemma 3.3 to conclude that L_kh vanishes on Σ^n. Considering the expression of L_kh in (3.2) and the hypothesis (3.5), we conclude that ∇h is identically zero on Σ^n, then Σ^n is a slice $\{t\} \times M^n$. □

Theorem 3.6. *Let $\overline{M} = -I \times_f M^n$ be a RW spacetime, $\psi : \Sigma^n \to -I \times_f M^n$ (with $n \geq 2$) be a complete, connected spacelike hypersurface bounded away from the infinity of $\overline{M}$. Suppose that mean curvature H_1 is bounded on Σ^n and $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$. Assume that either*
(i) when $k = 1$, $H_2 > 0$ and

$$0 < \frac{H_2}{H_1} \leq \frac{f'}{-f\langle N, \partial_t\rangle}, \text{ or} \tag{3.6}$$

(ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map*, and

$$0 < \frac{H_{k+1}}{H_k} \leq \frac{f'}{-f\langle N, \partial_t\rangle}. \tag{3.7}$$

Then the hypersurface Σ^n is a slice.

Proof. (i) when $k=1$, from (3.6) we have the mean curvature $H_1 > 0$ is bounded and $H_2 > 0$ on Σ. From the Cauchy-Schwarz inequality we have $H_1^2 \geq H_2$. Similarly, we get $|A|$ is bounded and P_1 is positive definite respect to the future-pointing Gauss map N on Σ^n. Therefore, we get both $f'(h)H_1 + f(h)\langle N, \partial_t\rangle H_2$ and $\langle P_1(\nabla h), \nabla h\rangle$ are non-negative. Besides, from (3.6) we also get f' is positive, then it's easy to get that $L_1(h) \leq 0$ on Σ. As a application of Lemma 3.3 we have that $L_1(h)$ vanishes on Σ^n. Thus, returning consider the expression of $L_1(h)$ and the hypothesis we get that ∇h is identically zero on Σ^n, then we have Σ^n is a slice $\{t\} \times M^n$.

(ii) when $2 \leq k \leq n-1$, from (3.7) we have $H_k H_{k+1} > 0$ and $f' > 0$, from the hypothesis we have that there exists an elliptic point respect to the future-pointing Gauss map, making that both H_{k+1} and H_k are positive. Now we apply Lemma 2.2 to conclude that P_j are positive definite and H_j are positive for every $1 \leq j \leq k$. Then from H_1 is bounded and $H_2 > 0$ we have $|A|$ is bounded. Under the condition above we have that

$$f'(h)H_k + f(h)\langle N, \partial_t\rangle H_{k+1} \geq 0$$

and

$$f'(h)\langle P_k(\nabla h), \nabla h\rangle \geq 0.$$

Hence, from formula (3.2) we have that $L_k(h) \leq 0$ on Σ^n. From Lemma 3.3 we have that $L_k(h)$ vanishes on Σ^n, then in a similar way, we have ∇h is identically zero. Thus Σ^n is a slice. □

4 Entire vertical graphs in a GRW spacetimes $-I \times_f M^n$

Let $\Omega \subseteq M^n$ be a connected domain of M^n. A vertical graph over Ω is determined by a smooth function $u \in \mathcal{C}^\infty(\Omega)$ and it is given by

$$\Sigma^n(u) = \{(u(x), x) : x \in \Omega\} \subset -I \times_f M^n.$$

The metric induced on Ω from the Lorentzian metric on the ambient space via $\Sigma^n(u)$ is

$$\langle\ ,\ \rangle = -du^2 + f^2(u)\langle\ ,\ \rangle_{M^n}.$$

The graph is said to be entire if $\Omega = M^n$. It can be easily seen that a graph $\Sigma^n(u)$ is a spacelike hypersurface if and only if $|Du|^2_{M^n} < f^2(u)$, Du being the gradient of u in Ω and $|Du|_{M^n}$ is its norm, both with respect to the metric $\langle\ ,\ \rangle_{M^n}$ in Ω. In this context, we obtain the following non-parametric version of Theorem 3.3, where the fibre M^n of the RW spacetime $-I \times_f M^n$ is complete.

Corollary 4.1. *Let $\overline{M} = -I \times_f M^n$ be a RW spacetime, $\Sigma^n(u)$ (with $n \geq 2$) be an entire spacelike vertical graph bounded away from the infinity of $\overline{M}$. Suppose that mean curvature H_1 is bounded on Σ^n and $|Du|_{M^n} \in \mathcal{L}^1(M^n)$. For some constant $0 \leq \alpha < 1$, $|Du|^2_{M^n} \leq \alpha f^2(u)$, assume that either*
(i) when $k=1$, $H_2 > 0$ and

$$0 < \frac{H_2}{H_1} \leq (1-\alpha)\frac{f'}{f}(h),\ \textit{or}$$

(ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map*, and

$$0 < \frac{H_{k+1}}{H_k} \leq (1-\alpha)\frac{f'}{f}(h).$$

Then the hypersurface Σ^n is a slice.

Proof. We can obtain the proof in [9] about $\nabla h \in \mathcal{L}^1(M^n)$. Furthermore, following the proof of Theorem 3.3 we can obtain the result. ☐

Following the same ideas of Corollary 4.1, we also obtain non-parametric versions of Theorem 3.1 and Theorem 3.2.

Corollary 4.2. *Let $\overline{M} = -I \times_f M^n$ be a GRW spacetime, $\Sigma^n(u)$ (with $n \geq 2$) be an entire spacelike vertical graph bounded away from the infinity of $\overline{M}$. Suppose that $|Du|_{M^n} \in \mathcal{L}^1(M^n)$. Then for some constant $0 \leq \alpha < 1$, $|Du|_{M^n} \leq \alpha f(u)$, assume that*

$$H_1\frac{f}{f'}(u) \leq 1-\alpha,$$

then the hypersurface Σ^n is a slice.

Corollary 4.3. *Let $\overline{M} = -I \times_f M^n$ be a RW spacetime, $\Sigma^n(u)$ (with $n \geq 2$) be a entire spacelike vertical graph bounded away from the infinity of $\overline{M}$. Suppose that mean curvature H is bounded on Σ^n and $|Du|_{M^n} \in \mathcal{L}^1(M^n)$. Then for some constant $0 \leq \alpha < 1$, $|Du|_{M^n} \leq \alpha f(u)$, assume that either*
(i) when $k=1$, $H_2 > 0$ and

$$(1-\alpha)\frac{f'}{f}(u) \leq \frac{H_2}{H_1} < 0, \text{ or}$$

(ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map*, and

$$(1-\alpha)\frac{f'}{f}(u) \leq \frac{H_{k+1}}{H_k} < 0.$$

Then the hypersurface Σ^n is a slice.

5 Applications

In this section, we'll give some examples which are the applications of Theorem 3.2 and 3.3. The de Sitter space $\mathbb{S}_1^{n+1}$ are spacetimes obtained into Minkowski space $\mathbb{R}_1^{n+2}$ as the hyperquadrics

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{R}_1^{n+2} \mid |p|^2 = 1\}.$$

If we choose a unit timelike vector $a \in \mathbb{R}_1^{n+2}$, then we may consider the vector field

$$X(p) = a - \langle p, a\rangle p \quad p \in \mathbb{S}_1^{n+1} \quad |a|^2 = -1,$$

which is the conformal timelike field. Consequently, we can consider $\mathbb{S}_1^{n+1}$ as the warped product $-\mathbb{R}^- \times_{\cosh t} \mathbb{S}^n$, where $\mathbb{S}^n$ means the Riemannian unit sphere.

In what follows, we will consider a spacelike hypersurface Σ^n immersed in $\mathbb{S}_1^{n+1}$, then from theorem 3.2 and formula (2.2), we obtain the following

Corollary 5.1. *Let $\psi : \Sigma^n \to \mathbb{S}_1^{n+1}$ be a complete, connected spacelike hypersurface bounded away from the past infinity. Suppose that the mean curvature H of Σ^n is bounded and $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$. Assume that either*
(i) when $k = 1$, $H_2 > 0$ and $-\tanh t \leq \frac{H_2}{H_1} \cosh\theta < 0$, *or*
(ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map*, and $-\tanh t \leq \frac{H_{k+1}}{H_k} \cosh\theta < 0$,
Then the hypersurface Σ^n is a slice.

Now we consider the half $\mathcal{H}^{n+1}$ of the de Sitter space $\mathbb{S}_1^{n+1}$, which models are the so-called *steady state space*([21]). From [2] we have that the steady state space admits the following RW spacetime model:

$$\mathcal{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n.$$

Then from Theorem 3.3 and (2.2) we have the following

Corollary 5.2. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete, connected spacelike hypersurface bounded away from the infinity of $\mathcal{H}^{n+1}$. Suppose that the mean curvature H of Σ^n is bounded and $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$. Assume that either*
(i) when $k = 1$, $H_2 > 0$ and $\frac{H_1}{H_2} \geq \cosh\theta$, *or*
(ii) when $2 \leq k \leq n-1$, there exists an elliptic point on Σ with respect to an appropriate choice of the *Gauss map*, and $\frac{H_k}{H_{k+1}} \geq \cosh\theta$. Then the hypersurface Σ^n is a slice.

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Authors' address:

Junhong Dong and Ximin Liu

School of Mathematical Sciences,
Dalian University of Technology,
Dalian 116024, China.
E-mail: dongjunhongrun@163.com , ximinliu@dlut.edu.cn