

A Hodge decomposition theorem on strongly pseudoconvex compact complex Finsler manifolds

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Abstract. In this paper, we prove that the Hodge-Laplace operator on strongly pseudoconvex compact complex Finsler manifolds is a self-adjoint elliptic operator. Then, from the decomposition theorem for self-adjoint elliptic operators, we obtain a Hodge decomposition theorem on strongly pseudoconvex compact complex Finsler manifolds.

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Key words: Complex Finsler manifold; Hodge-Laplace operator; Hodge decomposition theorem.

1 Introduction

It is well known that Hodge Laplacian plays an important role in the theory of harmonic integral and Bochner technique in differential geometry. For the harmonic integral theory in Kähler geometry one can ref. [12, 14, 17, 18, 19]. S. S. Chern had pointed out that "complex Finsler geometry plays an important role for researching the function theory of complex manifolds, since on every complex manifolds with or without boundary there exist a Carathéodory metric and a Kobayashi metric, and under proper condition they are $C^{(2)}$ metrics, and the most important fact is that naturally they are Finsler metrics, \dots , to extend harmonic integral to the case of Finslerian will be a new research region of differential geometry, and we expect the prospect is boundless" [10, 11]. In the past two decades, under the initiation of S. S. Chern, the global differential geometry of real and complex Finsler manifolds has gained a great development [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 15]. There are some results for Hodge Laplacian have been obtained on a complex Finsler manifold [20, 21, 22, 23, 24, 25]. In addition, there appear some papers about using complex connections of Finsler geometry to research the theory of integral representation of functions in several complex variables on complex Finsler manifolds [16].

D. Bao and B. Lackey [8] obtained a Hodge decomposition theorem for Finsler spaces. Later, C. P. Zhong and T. D. Zhong [25] studied the Hodge decomposition theorem on strongly Kähler Finsler manifolds. In [13], we obtained the Hodge theorem

for the natural projection of complex horizontal Laplacian on strongly pseudoconvex compact complex Finsler manifolds. In the present paper, we research the Hodge decomposition theorem for the Hodge-Laplace operator on strongly pseudoconvex compact complex Finsler manifolds. We first prove that the Hodge-Laplace operator on strongly pseudoconvex compact complex Finsler manifolds is a self-adjoint elliptic operator. And then, from the decomposition theorem for self-adjoint elliptic operators, we obtain a Hodge decomposition theorem on strongly pseudoconvex compact complex Finsler manifolds.

2 Complex Finsler manifolds

Let M be a compact complex manifold of dimension n and $\pi : T^{1,0}M \rightarrow M$ be the holomorphic tangent bundle of M . We denote by \tilde{M} the complement of zero section $\mathcal{O}(M)$ in $T^{1,0}M$, and $T^{1,0}\tilde{M}$ the holomorphic tangent bundle of \tilde{M} .

Let $\{\pi^{-1}(U), (z, v) = (z^1, \dots, z^n, v^1, \dots, v^n)\}$ be the complex coordinates on $T^{1,0}M$ induced by the covering of the system of complex coordinate neighborhoods $\{U, z = (z^1, \dots, z^n)\}$ on M , with $z^\alpha = x^\alpha + ix^{n+\alpha}$ and $v^\alpha = u^\alpha + iv^{n+\alpha}$, so that $\{x = (x^1, \dots, x^{2n})\}$ are the real coordinates on M and $\{(x, u) = (x^1, \dots, x^{2n}, u^1, \dots, u^{2n})\}$ are the coordinates on $T_{\mathbb{R}}M$ which is the real tangent bundle of M .

Set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, then the projectivized tangent bundle PTM of M is defined by $PTM = \tilde{M}/\mathbb{C}^*$ with projection $\tilde{\pi} : PTM \rightarrow M$. Thus the local coordinates (z, v) on $T^{1,0}M$ may be also considered as a local coordinate system for PTM as long as (v^1, \dots, v^n) is considered as a homogeneous coordinate system for fibres.

Definition 2.1. [1] A strongly pseudoconvex complex Finsler metric on a complex manifold M is a continuous function $F : T^{1,0}M \rightarrow R^+$ satisfying

- (1) $G = F^2$ is smooth on $\tilde{M} = T^{1,0}M \setminus \mathcal{O}(M)$;
- (2) $F(v) > 0$ for all $v \in \tilde{M}$;
- (3) $F(\lambda v) = |\lambda|F(v)$ for all $v \in T^{1,0}M$ and $\lambda \in \mathbb{C}$;
- (4) The Hermitian matrix $(G_{\alpha\bar{\beta}})$ is positive definite on \tilde{M} , where

$$G_{\alpha\bar{\beta}} = \frac{\partial^2 G}{\partial v^\alpha \partial \bar{v}^\beta}.$$

A manifold M endowed with a strongly pseudoconvex complex Finsler metric will be called a strongly pseudoconvex complex Finsler manifold.

Condition (4) allows us to introduce a Hermitian structure on the vertical bundle \mathcal{V} . Indeed, if $v \in \tilde{M}$ and $W_1, W_2 \in \mathcal{V}_v$ with $W_j = W_j^\alpha \partial_\alpha$ ($j=1,2$), we get

$$\langle W_1, W_2 \rangle_v = G_{\alpha\bar{\beta}}(v) W_1^\alpha \bar{W}_2^\beta,$$

then there is a unique good complex vertical connection D making this Hermitian structure parallel. We call it the Chern-Finsler connection.

The connection matrix is given by

$$\omega_\beta^\alpha = G^{\bar{\tau}\alpha} \partial G_{\beta\bar{\tau}} = G^{\tau\alpha} (G_{\beta\tau;\mu} dz^\mu + G_{\beta\bar{\tau}\gamma} dv^\gamma),$$

in particular, the Christoffel symbols of the non-linear connection associated to F is given by

$$\Gamma_\mu^\alpha = G^{\bar{\tau}\alpha} G_{\bar{\tau};\mu}.$$

Denote by $\mathcal{H} \subset T^{1,0}\tilde{M}$ the complex horizontal bundle associated to the Chern-Finsler connection, the natural local frame for \mathcal{H} is $\{\delta_1, \dots, \delta_n\}$, where $\delta_\mu = \partial_\mu - \Gamma_\mu^\alpha \dot{\partial}_\alpha$. Now $\{\delta_\mu, \frac{\partial}{\partial v^\alpha}\}$ is a local frame for $T^{1,0}\tilde{M}$ with its dual coframe $\{dz^\mu, \delta v^\alpha\}$, where $\delta v^\alpha = dv^\alpha + \Gamma_\mu^\alpha dz^\mu$.

Writing

$$\omega_\beta^\alpha = \Gamma_{\beta;\mu}^\alpha dz^\mu + \Gamma_{\beta\gamma}^\alpha \delta v^\alpha,$$

we get

$$\begin{aligned} \Gamma_{\beta\gamma}^\alpha &= G^{\bar{\tau}\alpha} G_{\beta\bar{\tau}\gamma} = \Gamma_{\gamma\beta}^\alpha, \\ \Gamma_{\beta;\mu}^\alpha &= G^{\bar{\tau}\alpha} \delta_\mu(G_{\beta\bar{\tau}}), \end{aligned}$$

from which we have

$$\Gamma_{\beta;\mu}^\alpha(z, \lambda v) = \Gamma_{\beta;\mu}^\alpha(z, v), \quad \forall \lambda \in \mathbb{C} \setminus \{0\},$$

i.e. $\Gamma_{\beta;\mu}^\alpha(z, v)$ is zero homogenous on $T^{1,0}M$, thus it lives on PTM . Now we denote by $D_{\delta_\alpha}, D_{\delta_\beta}$ the covariant differentiation with respect to $\delta_\alpha, \delta_\beta$, respectively.

Being D a good vertical connection, it extends to a complex linear connection on \tilde{M} (still called the Chern-Finsler connection in this article). Using the complex horizontal map $\Theta: \mathcal{V} \rightarrow \mathcal{H}$, we can transfer the Hermitian structure \langle, \rangle on \mathcal{H} just by setting

$$\forall H, K \in \mathcal{H}_v, \langle H, K \rangle_v = \langle \Theta^{-1}(H), \Theta^{-1}(K) \rangle_v,$$

and then we can define a Hermitian structure on $T^{1,0}\tilde{M}$ by requiring \mathcal{H} to be orthogonal to \mathcal{V} . It is easy to check that these definitions are compatible enough so to get

$$X\langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle X, D_{\bar{X}} Z \rangle,$$

for all $X \in T^{1,0}\tilde{M}$, and $Y, Z \in \mathcal{X}(T^{1,0}\tilde{M})$.

3 Laplace operators on complex Finsler manifolds

Firstly, we introduce the definition of a complex horizontal Laplacian on the projectivized tangent bundle PTM .

Let (M, F) be an n -dimensional strongly pseudoconvex compact complex Finsler manifold with a complex Finsler metric F , then F induces a natural Hermitian metric on $T^{1,0}\tilde{M}$,

$$\tilde{G} = G_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + G_{\alpha\bar{\beta}} \delta v^\alpha \otimes \delta \bar{v}^\beta.$$

It descends to a non-degenerated metric (still denote it by \tilde{G})

$$\tilde{G} = G_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + (\ln G)_{\alpha\bar{\beta}} \delta v^\alpha \otimes \delta \bar{v}^\beta,$$

on the total space PTM . We further denote:

$$\omega_{\mathcal{V}} = \sqrt{-1}(\ln G)_{\alpha\bar{\beta}} \delta v^\alpha \wedge \delta \bar{v}^\beta, \omega_{\mathcal{H}} = \sqrt{-1}G_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

Then the invariant volume form of PTM is given by

$$dv = \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} \wedge \frac{\omega_{\mathcal{H}}^n}{n!}.$$

Since $\omega_{\mathcal{H}}^n$ is a horizontal (n, n) -form, the above expression is invariant by replacing δv^α and $\delta \bar{v}^\beta$ by dv^α and $d\bar{v}^\beta$, respectively.

If we denote by $d\sigma$ the pure vertical form of the volume form of PTM , then

$$d\sigma = \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!}.$$

So we have

$$dv = d\sigma \wedge \frac{\omega_{\mathcal{H}}^n}{n!} = g d\sigma \wedge d\chi,$$

where

$$g = \det(G_{\alpha\bar{\beta}}), d\chi = \frac{\tau^n}{n!}, \tau = \sqrt{-1} \sum dz^i \wedge d\bar{z}^i.$$

Let $\mathcal{A}^{p,q}$ be the space of smooth horizontal (p, q) -forms on PTM , i.e., those coefficients of every $\varphi \in \mathcal{A}^{p,q}$ are zero homogeneous with respect to fibre coordinates v .

Let $\dim M = n$, we denote

$$A_p = (\alpha_1, \dots, \alpha_p), \alpha_1 < \alpha_2 < \dots < \alpha_p, 1 \leq \alpha_i \leq n, \\ A_{n-p} = (\alpha_{p+1}, \dots, \alpha_n), \alpha_{p+1} < \dots < \alpha_n, 1 \leq \alpha_i \leq n,$$

where $(\alpha_1, \dots, \alpha_p, \alpha_{p+1}, \dots, \alpha_n)$ is a permutation of $(1, 2, \dots, n)$. Similarly, denote $B_q = (\beta_1, \dots, \beta_q), B_{n-q} = (\beta_{q+1}, \dots, \beta_n), C_p = (c_1, \dots, c_p), C_{n-p} = (c_{p+1}, \dots, c_n), D_q = (d_1, \dots, d_q), D_{n-q} = (d_{q+1}, \dots, d_n)$, then with these notations, the elements of $\mathcal{A}^{p,q}$ in local coordinates are

$$\varphi = \sum_{A_p, B_q} \varphi_{A_p \bar{B}_q} dz^{A_p} \wedge d\bar{z}^{\bar{B}_q}, \\ \psi = \sum_{C_p, D_q} \psi_{C_p \bar{D}_q} dz^{C_p} \wedge d\bar{z}^{\bar{D}_q},$$

where $dz^{A_p} = dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p}, d\bar{z}^{\bar{B}_q} = d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}, dz^{C_p} = dz^{c_1} \wedge \dots \wedge dz^{c_p}, d\bar{z}^{\bar{D}_q} = d\bar{z}^{d_1} \wedge \dots \wedge d\bar{z}^{d_q}$. Thus at each point $(z, v) \in PTM$, we define

$$\langle \varphi, \psi \rangle_{PTM} = \varphi_{A_p \bar{B}_q} \overline{\psi^{\bar{A}_p B_q}},$$

where

$$\psi^{\bar{A}_p B_q} = \sum_{\lambda, \mu} G^{\bar{\alpha}_1 \lambda_1} \dots G^{\bar{\alpha}_p \lambda_p} G^{\beta_1 \bar{\mu}_1} \dots G^{\beta_q \bar{\mu}_q} \psi_{\lambda_1 \dots \lambda_p \bar{\mu}_1 \dots \bar{\mu}_q}.$$

Notice that there is a natural Hermitian inner product in $\mathcal{A}^{p,q}$ which is induced by the complex Finsler metric F , i.e.,

$$(\varphi, \psi)_{PTM} = \int_{PTM} \langle \varphi, \psi \rangle_{PTM} dv.$$

Let $\varphi \in \mathcal{A}^{p,q}$, $\varphi = \sum_{A_p, B_q} \varphi_{A_p \bar{B}_q} dz^{A_p} \wedge dz^{\bar{B}_q}$, then an operator $\bar{\partial}_H : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}$ is defined by

$$\bar{\partial}_H \varphi = \sum_{A_p, B_{q+1}} (\bar{\partial}_H \varphi)_{A_p \bar{B}_{q+1}} dz^{A_p} \wedge dz^{\bar{B}_{q+1}},$$

where

$$(\bar{\partial}_H \varphi)_{A_p \bar{B}_{q+1}} = \sum_{j=1}^{q+1} (-1)^{p+j+1} \delta_{\bar{\beta}_j} \varphi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_1, \dots, \widehat{\bar{\beta}_j}, \dots, \bar{\beta}_{q+1}}.$$

Let $\bar{\partial}_H^*$ be the adjoint operator of $\bar{\partial}_H$ with respect to the inner product $(\cdot, \cdot)_{PTM}$ on the complex projectivized tangent bundle PTM . That is,

$$(\bar{\partial}_H \varphi, \psi)_{PTM} = (\varphi, \bar{\partial}_H^* \psi)_{PTM}, \text{ for all } \varphi \in \mathcal{A}^{p,q}, \psi \in \mathcal{A}^{p,q+1}.$$

Lemma 3.1. [25] *Assume that (M, F) is an n -dimensional strongly pseudoconvex compact complex Finsler manifold, then for any $\varphi \in \mathcal{A}^{p,q}$, we have*

$$(\bar{\partial}_H^* \varphi)_{A_p \bar{B}_{q-1}} = -(-1)^p \sum_{\beta, \mu} G^{\beta \bar{\mu}} \delta_{\beta} \varphi_{\alpha_1 \dots \alpha_p \bar{\mu} \bar{\beta}_1 \dots \bar{\beta}_{q-1}} + \text{lower order terms}.$$

Definition 3.1. [25] $\square_H = \bar{\partial}_H \bar{\partial}_H^* + \bar{\partial}_H^* \bar{\partial}_H$ is called the complex horizontal Laplacian on PTM , and \square_H maps $\mathcal{A}^{p,q}$ into $\mathcal{A}^{p,q}$.

Lemma 3.2. [25] *Assume that (M, F) is an n -dimensional strongly pseudoconvex compact complex Finsler manifold, then for any $\varphi \in \mathcal{A}^{p,q}$, we have*

$$(\square_H \varphi)_{A_p \bar{B}_q} = - \sum_{\nu, \mu} G^{\nu \bar{\mu}} \delta_{\nu} \delta_{\bar{\mu}} \varphi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} + \text{lower order terms}.$$

Now, we consider differential forms on M , let $\mathcal{E}^{p,q}$ be the space of smooth (p, q) -forms on M . Since every elements of $\mathcal{E}^{p,q}$ can be regarded in a natural manner as elements of $\mathcal{A}^{p,q}$ via the canonical projection $\tilde{\pi} : PTM \rightarrow M$, it follows that $\mathcal{E}^{p,q} \subset \mathcal{A}^{p,q}$.

We denote

$$G^{A_p \bar{B}_q C_p \bar{D}_q} = G^{\bar{c}_1 \alpha_1} \dots G^{\bar{c}_p \alpha_p} G^{\bar{\beta}_1 d_1} \dots G^{\bar{\beta}_q d_q},$$

we have

$$\overline{G^{A_p \bar{B}_q C_p \bar{D}_q}} = G^{C_p \bar{D}_q A_p \bar{B}_q},$$

where the overline denotes the conjunction, and the matrix $(G^{A_p \bar{B}_q C_p \bar{D}_q})$ is positive definite. Set

$$\begin{aligned} N(z) &= \int_{PT_z M} g d\sigma, \\ N^{A_p \bar{B}_q C_p \bar{D}_q} &= \frac{1}{N(z)} \left(\int_{PT_z M} G^{A_p \bar{B}_q C_p \bar{D}_q} g d\sigma \right), \\ dM &= N(z) d\chi = N(z) \frac{\tau^n}{n!}. \end{aligned}$$

Proposition 3.3. [24] $N^{A_p \bar{B}_q C_p \bar{D}_q}$ is a contravariant Hermitian tensor of rank $2(p+q)$ on M .

Proposition 3.4. [24] dM is a real invariant (n, n) -form on M .

$\mathcal{E}^{p,q}$ is viewed as a subspace of $\mathcal{A}^{p,q}$, its inner product (\cdot, \cdot) is induced from $\mathcal{A}^{p,q}$, that is, for any $\varphi, \psi \in \mathcal{E}^{p,q}$, we have

$$(\varphi, \psi) = (\varphi, \psi)_{PTM}.$$

Let $\varphi, \psi \in \mathcal{E}^{p,q}$, we define

$$(3.1) \quad \langle \varphi, \psi \rangle = \varphi_{A_p \bar{B}_q} N_{A_p \bar{B}_q C_p \bar{D}_q} \overline{\psi_{C_p \bar{D}_q}},$$

then

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle dM.$$

Let $\bar{\partial}^* : \mathcal{E}^{p,q+1} \rightarrow \mathcal{E}^{p,q}$ be the formal adjoint operator of the $\bar{\partial}$ defined by

$$(\bar{\partial}\varphi, \psi) = (\varphi, \bar{\partial}^*\psi), \text{ for any } \varphi \in \mathcal{E}^{p,q}, \psi \in \mathcal{E}^{p,q+1}.$$

Lemma 3.5. [24] Let (M, F) be a strongly pseudoconvex compact complex Finsler manifold, then for any $\psi \in \mathcal{E}^{p,q+1}$, we have

$$(\bar{\partial}^*\psi)_{A_p \bar{B}_q}(z) = N_{C_p \bar{D}_q A_p \bar{B}_q}(z) N_{E_p \bar{F}_q C_p \bar{D}_q} \partial_\nu \bar{\partial}_{\bar{\mu}} \psi_{E_p \bar{F}_q} + \text{lower order terms}.$$

Using the inner product $(\cdot, \cdot)_{PTM}$ for smooth horizontal (p, q) -forms on PTM , we take the \mathcal{L}_2 closure of $\mathcal{A}^{p,q}$ and denote that as $L^{p,q}(PTM)$. The weak extensions of $\bar{\partial}_H$ and $\bar{\partial}_H^*$ still preserve $L^{p,q}(PTM)$, and remain formal adjoint of each other. So $\square_H = \bar{\partial}_H \bar{\partial}_H^* + \bar{\partial}_H^* \bar{\partial}_H$ is well defined on $L^{p,q}(PTM)$, we also call it complex horizontal Laplacian.

Using the inner product (\cdot, \cdot) for smooth (p, q) -forms on M , we take the \mathcal{L}_2 closure of $\mathcal{E}^{p,q}$ and denote that as $L^{p,q}$. The weak extension of $\bar{\partial}$ and $\bar{\partial}^*$ still preserve $L^{p,q}$, and remain formal adjoint of each other, the weak extensions of $\bar{\partial}$ and $\bar{\partial}_H$ still agree, when acting on $L^{p,q}$.

$L^{p,q}(PTM)$ and $L^{p,q}$ are Hilbert spaces, and $L^{p,q}$ is the closed subspace of $L^{p,q}(PTM)$, then we have a \mathcal{L}_2 orthogonal projection $P : L^{p,q}(PTM) \rightarrow L^{p,q}$. Thus $P \circ \square_H$ maps $L^{p,q}$ into itself. So we have the following definition.

Definition 3.2. [13] $\tilde{\square}_H = P \circ \square_H : L^{p,q} \rightarrow L^{p,q}$ is called the natural projection of complex horizontal Laplacian \square_H on M .

Lemma 3.6. Let $\varphi \in L^{p,q}$. Then $\bar{\partial}\varphi = \bar{\partial}_H\varphi$ and $\bar{\partial}^*\varphi = P\bar{\partial}_H^*\varphi$.

Definition 3.3. $\square = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^* : L^{p,q} \rightarrow L^{p,q}$ is called the Hodge-Laplacian on M .

It is easy to see that the Hodge-Laplacian \square is a self-adjoint operator.

Definition 3.4. [13, 24] $\varphi \in L^{p,q}$ is called a horizontal harmonic form on M , if $\tilde{\square}_H\varphi=0$, or if $\square_H\varphi=0$. And $\varphi \in L^{p,q}$ is called a harmonic form on M , if $\square\varphi=0$.

From Lemma 3.6, we know that the space of horizontal harmonic forms is the subset of the space of harmonic forms. In [13], we obtain the Hodge theorem for the natural projection of complex horizontal Laplacian on strongly pseudoconvex compact complex Finsler manifolds. Moreover, we prove that the space of horizontal harmonic forms is a subgroup of the Dolbeault cohomology group.

4 Hodge decomposition for the Hodge-Laplacian on strongly pseudoconvex compact complex Finsler manifolds

Let M be a complex manifold of dimension n and $\{U, z = (z^1, \dots, z^n)\}$ be the complex local coordinates on M and $z^\mu = x^\mu + ix^{\mu+n}$. Let L be a linear partial differential operator

$$L : \mathcal{E}^{p,q} \rightarrow \mathcal{E}^{p,q},$$

where $\mathcal{E}^{p,q} = \Gamma(M, \wedge^{p,q} T^*M)$ is the space of the smooth differential forms of degree (p, q) on M . Then in a real local coordinate $\{U, x = (x^1, \dots, x^{2n})\}$ on M , L can be written as

$$L = \sum_{|\alpha| \leq k} (-i)^{|\alpha|} A_\alpha D^\alpha,$$

where $\{A_\alpha\}$ are matrices of C^∞ functions on U , $D_i = \frac{\partial}{\partial x^i}$ and $D^\alpha = D_1^{\alpha_1} \dots D_{2n}^{\alpha_{2n}}$. And then the k -symbol $\sigma_k(L)$ of L is given by [18]

$$\sigma_k(L)(x, v) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha,$$

where $v = \xi_1 dx_1 + \dots + \xi_{2n} dx_{2n}$ is a real cotangent vector. For each fixed (x, v) , $\sigma_k(L)(x, v)$ is a linear mapping from $\wedge^{p,q} T_x^* M \rightarrow \wedge^{p,q} T_x^* M$, given by the usual multiplication of a vector in $\wedge^{p,q} T_x^* M$ by the matrix

$$\sum_{|\alpha|=k} A_\alpha(x) \xi^\alpha.$$

Definition 4.1. [18] A linear partial differential operator L

$$L : \mathcal{E}^{p,q} \rightarrow \mathcal{E}^{p,q}$$

is called elliptic if for any $v \neq 0$, the linear map

$$\sigma_k(L)(x, v) : \wedge^{p,q} T_x^* M \rightarrow \wedge^{p,q} T_x^* M$$

is injective (or isomorphism).

We consider the Dolbeault complex on a complex manifold M ,

$$(4.1) \quad \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{E}^{p,n} \longrightarrow 0.$$

Then this has an associated symbol sequence

$$(4.2) \quad \longrightarrow \wedge^{p,q-1} T_x^* M \xrightarrow{\sigma_1(\bar{\partial})(x,v)} \wedge^{p,q} T_x^* M \xrightarrow{\sigma_1(\bar{\partial})(x,v)} \wedge^{p,q+1} T_x^* M \longrightarrow,$$

where $\sigma_1(\bar{\partial})(x, v) : \wedge^{p,q} T_x^* M \rightarrow \wedge^{p,q+1} T_x^* M$ is a linear map for each (x, v) . And the above symbol sequence is exact. Let $\varphi = (\varphi_{A_p \bar{B}_q}) \in \wedge^{p,q} T_x^* M$ be a non zero vector. Then $\sigma_1(\bar{\partial})(x, v)\varphi = ((\sigma_1(\bar{\partial})(x, v)\varphi)_{A_p \bar{B}_{q+1}}) \in \wedge^{p,q+1} T_x^* M$, which is given by

$$(4.3) \quad (\sigma_1(\bar{\partial})(x, v)\varphi)_{A_p \bar{B}_{q+1}} = \sum_{j=1}^{q+1} (-1)^{p+j+1} \bar{\eta}_{\beta_j} \varphi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_1, \dots, \widehat{\bar{\beta}_j}, \dots, \bar{\beta}_{q+1}},$$

where $v = \xi_1 dx^1 + \cdots + \xi_{2n} dx^{2n} = \eta + \bar{\eta}$, $\eta = \eta_\mu dz^\mu$ and $\eta_\mu = \frac{1}{2}(\xi_\mu - i\xi_{\mu+n})$.

Let $\bar{\partial}^*$ be the adjoint operator of $\bar{\partial}$. Then its associated symbol sequence is

$$(4.4) \quad \longleftarrow \wedge^{p,q-1} T_x^* M \xrightarrow{\sigma_1(\bar{\partial}^*)(x,v)} \wedge^{p,q} T_x^* M \xrightarrow{\sigma_1(\bar{\partial}^*)(x,v)} \wedge^{p,q+1} T_x^* M \longleftarrow,$$

where $\sigma_1(\bar{\partial}^*)(x, v) : \wedge^{p,q} T_x^* M \rightarrow \wedge^{p,q-1} T_x^* M$ is a linear map. Then from Lemma 3.5, $\sigma_1(\bar{\partial}^*)(x, v)\varphi = ((\sigma_1(\bar{\partial}^*)(x, v)\varphi)_{A_p \bar{B}_{q-1}}) \in \wedge^{p,q-1} T_x^* M$, which is given by

$$(4.5) \quad (\sigma_1(\bar{\partial}^*)(x, v)\varphi)_{A_p \bar{B}_{q-1}} = N_{C_p \bar{D}_{q-1} A_p \bar{B}_{q-1}} N^{E_p \bar{\mu} \bar{F}_{q-1} \overline{C_p \bar{\nu} \bar{D}_{q-1}}} \eta_{\nu} \varphi_{E_p \bar{\mu} \bar{F}_{q-1}}.$$

In any linear space $\wedge^{p,q} T_x^* M$, there is an inner product \langle, \rangle , which is given by (3.1). Let $(\sigma_1(\bar{\partial}))^*$ be the adjoint of $\sigma_1(\bar{\partial})$ with respect to the inner product (3.1). We have the following lemma.

Lemma 4.1. $(\sigma_1(\bar{\partial}))^* = \sigma_1(\bar{\partial}^*)$.

Proof. Let $\varphi = (\varphi_{A_p \bar{B}_q}) \in \wedge^{p,q} T_x^* M$, $\psi = (\psi_{A_p \bar{B}_{q+1}}) \in \wedge^{p,q+1} T_x^* M$. For each (x, v) , from (3.1), (4.3) and (4.5), we have

$$\langle \sigma_1(\bar{\partial})(x, v)\varphi, \psi \rangle = \langle \varphi, \sigma_1(\bar{\partial}^*)(x, v)\psi \rangle.$$

That is, the lemma is proved. \square

Let $\sigma_2(\square)$ be the 2-symbol of Hodge-Laplacian \square . Then from Lemma 4.1, we have

$$\begin{aligned} \sigma_2(\square) &= \sigma_2(\bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*) = \sigma_1(\bar{\partial}^*) \sigma_1(\bar{\partial}) + \sigma_1(\bar{\partial}) \sigma_1(\bar{\partial}^*) \\ &= \sigma_1(\bar{\partial})^* \sigma_1(\bar{\partial}) + \sigma_1(\bar{\partial}) \sigma_1(\bar{\partial})^*. \end{aligned}$$

Now we consider a diagram of finite dimensional Hilbert spaces and linear mappings,

$$\begin{array}{ccccc} \wedge^{p,q-1} T_x^* M & \xrightarrow{\sigma_1(\bar{\partial})(x,v)} & \wedge^{p,q} T_x^* M & \xrightarrow{\sigma_1(\bar{\partial})(x,v)} & \wedge^{p,q+1} T_x^* M \\ \updownarrow & & \updownarrow & & \updownarrow \\ \wedge^{p,q-1} T_x^* M & \xrightarrow{\sigma_1(\bar{\partial})^*(x,v)} & \wedge^{p,q} T_x^* M & \xrightarrow{\sigma_1(\bar{\partial})^*(x,v)} & \wedge^{p,q+1} T_x^* M, \end{array}$$

which is exact at $\wedge^{p,q} T_x^* M$. Then $\wedge^{p,q} T_x^* M = \text{Im} \sigma_1(\bar{\partial})(x, v) + \text{Ker}(\bar{\partial})^*(x, v)$, moreover, $\sigma_1(\bar{\partial})^*(x, v)\sigma_1(\bar{\partial})(x, v) + \sigma_1(\bar{\partial})(x, v)\sigma_1(\bar{\partial})^*(x, v)$ is an isomorphism map on $\wedge^{p,q} T_x^* M$. So the Hodge-Laplacian \square is an elliptic operator.

Theorem 4.2. *The Hodge-Laplacian \square is a self-adjoint elliptic operator.*

From the decomposition theorem for self-adjoint elliptic operators [18], we have the following Hodge decomposition theorem.

Theorem 4.3. *Assume that (M, F) is an n -dimensional strongly pseudoconvex compact complex Finsler manifold. Then*

(1) *The space of all harmonic forms on M , $\mathcal{H}^{p,q} = \{\varphi \in L^{p,q} | \square\varphi = 0\} \subseteq \mathcal{E}^{p,q}$, and $\dim \mathcal{H}^{p,q} < \infty$.*

(2) *There exists an operator $\mathcal{G} : L^{p,q} \rightarrow L^{p,q}$, such that*

$$I = P_H + \square \mathcal{G},$$

where P_H is the projective operator from $L^{p,q}$ onto $H = \mathcal{H}^{p,q}$, and $\mathcal{G}(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p,q}$.
 (3) $\mathcal{E}^{p,q} = \mathcal{H}^{p,q} \oplus \square(\mathcal{E}^{p,q}) = \mathcal{H}^{p,q} \oplus \bar{\partial}(\mathcal{E}^{p,q-1}) \oplus \bar{\partial}^*(\mathcal{E}^{p,q+1})$.

From the above theorem, we know that the space of harmonic forms is isomorphism to the Dolbeault cohomology group.

Remark 4.2. From [13],[23] and [24], one can easy to know that the operator \square is more complex than the operators \square_H and $\tilde{\square}_H$. Therefore, it is difficult to discuss the Bochner technique about the Hodge Laplace operator \square , but we can obtain the Bochner technique about the horizontal Laplace operator \square_H in [20].

Remark 4.3. If the complex Finsler metric F is a Kähler metric, then the operator \square is the usual Hodge Laplacian. Thus Theorem 4.3 reduces to the Hodge decomposition theorem on a compact Kähler manifold.

Remark 4.4. If the complex Finsler manifold is strongly Kähler or Kähler [9], Theorem 4.3 reduces to the Hodge decomposition theorem on a compact strongly Kähler Finsler manifold [25].

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