On some Kähler-Riemann type flows

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Abstract. The Ricci flow has a central position as one of the key tools of geometry. The Kähler-Riemann flow generalizes the notion of Riemann flow, along the line of Kähler-Ricci flow and Ricci flow. These extensions are natural, since resemble results of Ricci flow and Riemann flow, but also provide new methods of study and classification of Kähler spaces.

These techniques raised the problem of constructing families of related Kähler metrics obtained deforming the initial metric by certain Kähler-Riemann type flows, using properties of the h-projective, h-concircular and h-conharmonic curvature tensor fields on Kähler manifolds. Holomorphically projective, concircular and conharmonic mappings between Kähler manifolds, corresponding to these classes of evolution metrics, are characterized.

M.S.C. 2010: 53C44, 53C21.
Key words: Ricci flow; Kähler-Riemann flow; h-projective curvature tensor; h-concircular curvature tensor; h-conharmonic curvature tensor

1 Introduction

The Ricci flow, first introduced by Hamilton, is the equation

$$\frac{\partial g_{ij}}{\partial t} = -2S_{ij},$$

where S is the Ricci tensor field.

The Ricci flow, which evolves a Riemannian metric by its Ricci curvature, is a natural analogue of the heat equation for metrics. As a consequence, the curvature tensors evolve by a system of diffusion equations which tends to distribute the curvature uniformly over the manifold. Hence, one expects that the initial metric should be improved and evolve into a canonical metric, thereby leading to a better understanding of the topology of the underlying manifold.

Note that, due to the minus sign on the right hand side of the equation, a solution to this parabolic type equation, whose prototype was the heat equation, shrinks in the directions of the positive Ricci curvature and expands in directions of negative

Balkan Journal of Geometry and Its Applications, Vol.20, No.2, 2015, pp. 32-40.

Ricci curvature. Generally speaking, it compresses all the positive curvature parts of the manifold into nothingness, while expanding the negative curvature parts of the manifold until they become very homogeneous.

The original goal in studying Ricci flow was related to the classification problem of 3-manifolds, but generalizations in higher dimensions are still to be made.

Hamilton showed that on a compact three-manifold with an initial metric having positive Ricci curvature, the Ricci flow converges, after rescaling to keep constant volume, to a metric of positive constant sectional curvature, proving the manifold is diffeomorphic to the three-sphere S^3 or a quotient of the three-sphere by a linear group of isometries.

Hamilton [14] later introduced the notion of Ricci flow with surgery and laid out an ambitious program to prove the Poincaré and Geometrization conjectures. In a spectacular demonstration of the power of the Ricci flow, Perelman [23] developed new techniques which enabled him to complete Hamiltons program and settle these celebrated conjectures. More recently, the Ricci flow was used to prove the Brendle-Schoen Differentiable Sphere Theorem [2] and other geometric classification results [22].

In addition to these successes has been the development of the Kähler-Ricci flow. If the Ricci flow starts from a Kähler metric on a complex manifold, the evolving metrics will remain Kähler and the resulting PDE is called the Kähler-Ricci flow. Cao [6] used this flow, together with parabolic versions of the estimates of Yau and Aubin, to reprove the existence of Kähler-Einstein metrics on manifolds with negative and zero first Chern class.

Since then, the study of the Kähler-Ricci flow has developed into a vast field in its own right. There have been several different avenues of research involving this flow, including: existence of Kähler-Einstein metrics on manifolds with positive first Chern class and notions of algebraic stability [12]; the classification of Kähler manifolds with positive curvature in both the compact and non-compact cases [8]; extensions of the flow to non-Kähler settings [24].

The ideas of Riemann flow and Riemann wave have their origin in the work of Udrişte [25]. It turns out to be a very fruitful approach to consider the problem of curvature flow via the bialternate product Riemannian metrics on Riemann spaces [16], [26].

We extended these investigations in the complex case. In a similar fashion we consider the Kähler-Riemann flow as a major tool in Kähler geometry. The behavior of gradient Kähler-Riemann solitons with Bochner curvature tensor leads to a classification problem of Kähler manifolds [15]. In this framework it is possible to focus on describing certain families of Kähler metric using properties and techniques of some Kähler-Riemann type flows. In the third section we characterize the holomorphically projective, concircular and conharmonic mappings between Kähler manifolds, corresponding to these classes of evolution metrics, obtained by deforming the initial Kähler metric by some geometric Kähler-Riemann flows.

These are very recent developments in a field which we expect is only just beginning.

2 Riemann flow. Kähler-Riemann flow.

The idea of Ricci flow was generalized [25], [26] to the concept of Riemann flow, which is a PDE that evolves the metric tensor G:

$$\frac{\partial G_{ijkl}}{\partial t} = -2R_{ijkl},$$

where $G = \frac{1}{2} g \wedge g$, R is the Riemann curvature tensor associated to the metric g and " $\wedge$ " is the Kulkarni-Nomizu product. For $(0, 2)$-tensors a and b, their *Kulkarni-Nomizu product* [17] $a \wedge b$ is given by

$$(a \wedge b)(X_1, X_2; X, Y) = a(X_1, X)b(X_2, Y) + a(X_2, Y)b(X_1, X)$$

$$-a(X_1, Y)b(X_2, X) - a(X_2, X)b(X_1, Y).$$

These extensions are natural, since some results in the Riemann flow resemble the case of Ricci flow. For instance, the Riemann flow satisfies the short time existence and the uniqueness [16]. Also [25]:

Theorem A. *If (M, g_0) is a Riemann manifold $(n \geq 2)$ of constant negative sectional curvature, then an evolution metric of the Riemann flow is given by $g_t = (1 + (n-1)t)\, g_0$. The manifold expands homothetically for all time.*

Theorem B. *For the round unit sphere (S^n, g_0), $n \geq 2$, an evolution metric of the Riemann flow is $g_t = (1 - (n-1)t)\, g_0$ and the sphere collapses to a point in finite time.*

The following natural question arises:

How much of the above can be developed for the complex case?

In particular, if M is a Kähler manifold with an initial metric g_0, what sort of information can we obtain by deforming the metric by certain geometric flow?

The previous approach can be adapted to the complex case. We consider (M, J, g) a Kähler manifold. The Kähler Ricci flow is:

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -2S_{i\bar{j}}.$$

These equations become strictly parabolic and it is easy to prove the short time existence.

We extend the notion of Riemann flow for a Riemann space and the notion of Kähler-Ricci flow on complex case to the concept of Kähler-Riemann flow on a Kähler manifold (M, J, g)[15] :

$$\frac{\partial G_{i\bar{j}k\bar{l}}}{\partial t} = -2R_{i\bar{j}k\bar{l}},$$

where R is the Riemann curvature tensor field and $G = \frac{1}{2} g \wedge g$.

Conjecture (Short time existence and uniqueness)

Let (M, J, g_0) be a complex n-dimensional $(n \geq 2)$ Kähler manifold. Then there exists $\epsilon > 0$ such that the initial value problem

$$\frac{\partial G_{i\bar{j}k\bar{l}}(x,t)}{\partial t} = -2R_{i\bar{j}k\bar{l}}(x,t), G(x,0) = G_0$$

has unique solution $G(x,t)$ on $M \times [0,\epsilon]$.

We generalize the notion of Ricci soliton, according to the Kähler-Riemann flow, in the following manner:

A solution of the Kähler-Riemann flow is said to be a Kähler-Riemann soliton if it moves along under a one parameter family of automorphisms of M generated by some holomorphic vector field X i.e.

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{l}}(\nabla_{\bar{j}}X_k + \nabla_{\bar{k}}X_j) + g_{k\bar{j}}(\nabla_{\bar{l}}X_i + \nabla_i X_{\bar{l}}).$$

In the case that X is gradient of some potential function$f$, the metric $g_{i\bar{j}}$ is said to be a gradient Kähler-Riemann soliton and one has

$$R_{i\bar{j}k\bar{l}} + \lambda G_{i\bar{j}k\bar{l}} = g_{i\bar{l}}\nabla_{\bar{j}}\nabla_k f + g_{k\bar{j}}\nabla_{\bar{l}}\nabla_i f;$$

$$\nabla_i \nabla_j f = 0.$$

A solution is said to be expanding (shrinking, respectively steady) gradient Kähler-Riemann soliton if there exists a potential function f and a constant λ such that $\lambda < 0$ ($\lambda > 0$, respectively $\lambda = 0$).

Proposition 2.1 *Let $(\mathbf{R}^2, g_\Sigma)$ be a manifold with*

$$g_\Sigma = \frac{dzd\bar{z}}{1+|z|^2}.$$

Letting function $f = \frac{1}{2}\log(1+|z|^2)$, then the metric is Kähler on $\mathbf{C}$ and is a gradient steady Kähler-Riemann soliton, with potential function f.

Proof. Indeed, $R_{1\bar{1}1\bar{1}} = 2g_{1\bar{1}}f_{,1\bar{1}} = \frac{1}{(1+|z|^2)^3}$. □

In complex local coordinates, the Bochner curvature tensor on a complex n-dimensional Kähler manifold (M, J, g) is given by

$$B_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \frac{1}{n+2}(g_{i\bar{j}}S_{k\bar{l}} + g_{i\bar{l}}S_{k\bar{j}} + g_{k\bar{l}}S_{i\bar{j}} +$$

$$+ g_{k\bar{j}}S_{i\bar{l}} + \frac{\rho}{2(n+1)(n+2)}(g_{k\bar{l}}g_{i\bar{j}} - g_{k\bar{j}}g_{i\bar{l}}),$$

where ρ is the scalar curvature. One should mention that this curvature tensor is invariant under the h-projective transformations on Kähler manifolds.

Ultimately, the main goal is to give a classification theorem [15]:

Theorem C. *Let (M, g, f) be a complex n-dimensional gradient Kähler-Riemann soliton with vanishing Bochner curvature tensor. Then the manifold has constant holomorphic sectional curvature. Consequently, M is $\mathbf{C}^n$ (steady case), $\mathbf{CP}^n$ (shrinking case), B^n (expanding case), or their quotients.*

3 Classes of related metrics produced by certain geometric Kähler-Riemann flows

Conformal and h-projective mappings represent a successful technique used to convert one mathematical problem and solution into another. In order to obtain families of equivalent Kähler metrics, verifying geometric properties, we are interested in a specific method, using some Kähler-Riemann flows.

a) h-projective equivalent metrics

Let M be a $m = 2n$ real dimensional complex manifold.

If Kähler metrics g and $\bar{g}$ are projective equivalent (i.e. if their unparametrized geodesic coincide), then the associated Levi-Civita connections coincide i.e. $\nabla = \overline{\nabla}$ and there are only trivial examples of projective Kähler metrics.

Otsuki and Tashiro introduced another notion in the complex case. Therefore, in h-projective geometry, the unparametrized geodesics are replaced by the "generalized complex geodesics", known as h-planar curves.

Let (M, J, g) be a Kähler manifold and ∇ the Levi-Civita connection.

A regular curve $\gamma : I \mapsto M$ is called h-planar with respect to g if satisfies $\nabla_{\gamma'}\gamma' = \alpha\gamma' + \beta J(\gamma')$, for some functions $\alpha, \beta : I \mapsto \mathbf{R}$.

Let g and $\bar{g}$ be two Kähler metrics on the complex manifold M. We call g and $\bar{g}$ h-projectively equivalent if each h-planar curve of g is h-planar with respect to $\bar{g}$ and viceversa.

∇ and $\overline{\nabla}$ are h-projectively equivalent iff there exists a (real) 1-form θ such that

$$\overline{\nabla}_X Y - \nabla_X Y = \theta(X)Y + \theta(Y)X - \theta(JX)JY - \theta(JY)JX.$$

A bi-holomorphic mapping $f : M \mapsto M$ is called h-projective transformation if f^*g is h-projectively equivalent to g. Equivalently, we can require that f preserves the set of h-planar curves.

The h-projective curvature tensor ${}^H P$, given by:

$${}^H P(X,Y)Z = R(X,Y)Z - \frac{1}{2n+2}[S(Y,Z)X - S(X,Z)Y -$$

$$-S(Y,JZ)JX + S(X,JZ)JY + 2S(X,JY)JZ],$$

is invariant under h-projective transformations.

One can produce a family of h-projectively equivalent metrics on Kähler manifolds by deforming the initial metric in the following manner:

Theorem 3.1. *Let (M, J, g_0) be a Kähler manifold. The class $\overset{1}{g}_t$ of h-projectively equivalent Kähler metrics, given by the h-projective-Kähler-Riemann flow*

$$\frac{\partial \overset{1}{G}_{i\bar{j}k\bar{l}}}{\partial t} = -2^H P_{i\bar{j}k\bar{l}}, \overset{1}{G}(x,0) = G_0$$

verifies

$$\overset{1}{G}(x,t) = -2\,{}^H P(g_0(x))t + G_0(x)$$

on $M \times [0, \epsilon]$.

Proof. Implicit solution of a Cauchy problem associated to the ${}^{H}P$-Kähler-Riemann flow . □

As a consequence of the previous theorem and using properties of holomorphically projective mappings obtained by Chuda and Mikes [11] one has:

Proposition 3.1. *Let* $t \in (0, \epsilon), \epsilon > 0,\ f_t : (M, J, g_0) \mapsto (M, J, \overset{1}{g}_t)$ *be a holomorphically projective mapping,* $x_0 \in M$ *and* $\bar{x}_0 = f(x_0)$.

If $\overset{1}{g}_t\ (\bar{x}_0) = k(t) g_0(x_0)$ *and* ${}^{H}P_0(x_0) \neq 0$, *then* f *is a homothety and one has*

$$\frac{\partial^2 k(t)}{\partial t^2} + k(t)(\frac{\partial k(t)}{\partial t})^2 = 0, k(0) = 1.$$

b) concircular equivalent metrics

Let $g \mapsto \bar{g} = e^{2u} g$ be a conformal transformation of the metric g on the Riemann space (M, g), where u is a nowhere zero function on M.

The Levi-Civita connections are related by

$$\overline{\nabla}_X Y = \nabla_X Y + X(u)Y + Y(u)X - g(X, Y) grad \quad (u).$$

The *tensor field of the conformal change* $B \in \mathcal{T}^{0,2}(M)$ has the components

$$B_{ij} = u_{i,j} - u_i u_j,\ u_i = \frac{\partial u}{\partial x^i}, i, j = \overline{1, n}.$$

If $B = \frac{1}{n}\ Tr(B) g$, then the conformal change is called *concircular transformation.*

A concircular transformation carries all the circles of the manifold into circles (a curve in a Riemannian manifold is called *circle* when the first curvature is constant and all the other curvatures are identically zero).

On a Kähler manifold one consider the h-concircular curvature tensor

$${}^{H}Z(X, Y)W = R(X, Y)W -$$

$$-\frac{\rho}{2n(2n-1)}[g(X, W)Y - g(Y, W)X + g(JX, W)JY - g(JY, W)JX],$$

invariant under concircular transformations, where ρ is the scalar curvature.

A new approach of finding families of concircular related Kähler metrics is to use the h-concircular Kähler-Riemann flow:

Theorem 3.2 *Let* (M, J, g_0) *be a Kähler manifold.*

The class $\overset{2}{g}_t = e^{2u_t} g_0$ *of concircular related Kähler metrics, with* g_0 *given by the* ${}^{H}Z$*-Kähler-Riemann type flow,*

$$\frac{\partial \overset{2}{G}_{i\bar{j}k\bar{l}}}{\partial t} = -2\ {}^{H}Z_{i\bar{j}k\bar{l}},\ \overset{2}{G}\ (x, 0) = G_0$$

satisfies

$$\overset{2}{G}\ (x, t) = -2\ {}^{H}Z(g_0(x))\, t + G_0(x)$$

on $M \times [0, \epsilon]$.

Proof. Implicit solution of a Cauchy problem associated to a ${}^{H}Z$-Kähler-Riemann type flow. □

Properties of concircular geometry proved by Yano lead to the following characterization of Einstein Kähler manifolds:

Proposition 3.2.

Let $t \in (0, \epsilon), \epsilon > 0,\ f_t : (M, J, g_0) \mapsto (M, J, \overset{2}{g}_t)$ *be a concircular mapping.*

If (M, J, g_0) *is an Einstein Kähler manifold, then* $(M, J, \overset{2}{g}_t)$ *is an Einstein Kähler manifold.*

c) conharmonic equivalent metrics

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In general, a harmonic function does not transform into a harmonic function, by a conformal change of metrics.

The condition under which the harmonic functions remain invariant have been studied by Ishii, who introduced the *conharmonic transformation* as a subgroup of the conformal transformations satisfying the condition

$$u^k_k = g^{ij} u_{ij} = 0, u_{ij} = u_{i,j} - u_i u_j + \frac{1}{2} u_k u^k g_{ij},\ i, j, k = \overline{1, n}.$$

On a Kähler manifold the h-conharmonic curvature tensor

$${}^{H}C(X,Y)Z = R(X,Y)Z + \frac{1}{2n+4}[S(X,Y)Z - S(Y,Z)X +$$

$$+ S(JX,Z)JY - S(JY,Z)JX + 2S(JX,Y)JZ + g(Z,X)s(Y) -$$

$$- g(Y,Z)s(X) + g(JX,Z)s(JY) - g(JY,Z)s(JX) + 2g(JX,Y)s(JZ)],$$

where $S(X,Y) = g(s(X),Y)$, is invariant under conharmonic transformations.

Starting with a h-conharmonic Kähler-Riemann flow, we outline the relation to certain conharmonic related metrics:

Theorem 3.3 *Let* (M, J, g_0) *be a Kähler manifold.*

The class $\overset{3}{g}_t = e^{2u_t} g_0$ *of conharmonically related Kähler metrics with* g_0*, given by the* ${}^{H}C$*-Kähler-Riemann type flow,*

$$\frac{\partial \overset{3}{G}_{i\bar{j}k\bar{l}}}{\partial t} = -2\ {}^{H}C_{i\bar{j}k\bar{l}}, \overset{3}{G}(x, 0) = G_0$$

satisfies

$$\overset{3}{G}(x,t) = -2\ {}^{H}C(g_0(x))\, t + G_0(x)$$

on $M \times [0, \epsilon]$.

Proof. Implicit solution of a Cauchy problem associated to a ${}^{H}C$-Kähler Riemann type flow. □

Using certain results concerning conharmonic transformations considered by B.H. Kim, I.B. Kim, S.M. Lee [18], the previous family of metrics satisfies:

Proposition 3.3. *Let* $t \in (0, \epsilon), \epsilon > 0,\ f_t : (M, J, g_0) \mapsto (M, J, \overset{3}{g}_t)$ *be a conharmonic mapping. If* M *is compact, then* f *is a homothety.*

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