On some variations related to jerk motions

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Abstract. The purpose of this paper is to study some jerk motions considering some dynamic cases, other than the classical ones and extending the setting. We consider a special set of variations, finding a set of jerk motions, generalizing some cases known only in some particular forms till now. Finally we give a simple mechanical interpretation of these ideas. The subject is topical because we model a less studied type of movements that can lead to interesting new developments.

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1 Introduction

A lot of movements which occur in nature can be described as fast movements, which are modeled by the so called jerk movements.

These jerk movements involve the derivative of acceleration, i.e., the third order derivative of coordinates of the position vector. This kind of motions are present in various real situations and have a lot of applications, most of them in biomechanics.

We have to notice that most Lagrangians studied in mechanics are of first order. For example, the Newton dynamics is based on first order Lagrangian systems.

But there are a lot of other motions that have to be modeled using higher order Lagrangians. This is the case when the accelerations and some of the derivatives (higher order accelerations) are involved in the equations of motion, thus we have to consider some higher order Lagrangians.

In this order of ideas, the optimal curves of the action (2.2) on the Euclidean third order Lagrangian given by (2.1) are used in order to obtain some properties of natural movements of the human body (see [6, 14, 15]). According to [6], the motions performed by primates are based on the maximizing the smoothness; this is modeled by minimizing the Lagrangian (2.1). The same principle can be applied to other high speed movements, for example the movement of a hard disk drive system [10]. Some other aspects can be stressed in mechanics of robots [2] or of the vehicle suspensions [3]. Some cosmological aspects can be involved (see, for example the recent paper

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[8]). Some evolved mathematical tools are developed in [4, 7, 9], where one uses deep numerical methods or in [13], where one studies some control aspects.

A third order Euclidean quadratic Lagrangian, related to jerk motions (see [12]), is considered and used in this paper. For the sake of simplicity, in order to get simple formulas, we do not consider other exterior forces (as, for example, friction) like in [12].

Instead of an Euclidean Lagrangian, one may consider a third order Lagrangian on an arbitrary manifold, using for example a calculus like in [11]. Our goal is to stress the various influences of initial conditions on every component of the motion, since the general cases can be developed in a similar way.

Specifically, we consider in the paper some cases and we give examples when a curve-solution is uniquely determined giving explicitly its positions and/or velocities and/or accelerations at its ends. Other situations can be considered, involving more points and/or velocities and accelerations; these cases can be studied by following a similar way like in our work.

In this paper we achieve an original description of a mathematical background (not known in the present form in literature until now) of a jerk movement that is generated by a variation (3.1) that involves the velocities, the accelerations and the jerk (as the derivatives of accelerations), but not the positions. The idea can be used also for a general situation in higher order mechanics. For example, one can consider an accelerated motion (i.e., governed by a second order Lagrangian), where velocities and accelerations are involved, but the variations contain also velocities and accelerations.

A general variation, as considered in the paper, generalizes, in two different directions, the variation considered in [1], where for the k-th order Lagrangian, the variation involves the first k-1 accelerations.

One can also consider some other examples of jerk systems, involving the friction as in [11], obtaining in this way a more general setting.

In the same line, some other jerk systems, with a more complicated behavior, can be considered. A deep analysis of these equations, compared and correlated with experimental data, can produce new results and provide a better image of movements governed by a jerk Lagrangian.

The subject is topical because we model a less studied type of movements that can lead to interesting new developments.

2 Jerk movements with fixed ends

The first cases of jerk motions that are studied in literature (see, for example [6]), are governed by the quadratic Euclidean Lagrangian L:

$$T^3\mathbb{R}^2 = \mathbb{R}^8 \to \mathbb{R}, \quad L = \frac{1}{2}(\dddot{x}^2 + \dddot{y}^2) \tag{2.1}$$

and the motion is constrained to be performed on curves that minimize the usual action

$$I_L(\gamma) = \int_0^1 L dt, \tag{2.2}$$

such that $\gamma : I = [0,1] \to \mathbb{R}^2$ has the properties that the ends $\gamma(0)$ and $\gamma(1)$ are given, and the velocities and the accelerations at these points vanish. We can consider instead some general conditions:

$$\gamma(0) = \gamma_0, \gamma(1) = \gamma_1, \tag{2.3}$$

$$\gamma'(0) = \gamma_0', \gamma''(0) = \gamma_0'', \tag{2.4}$$

$$\gamma'(1) = \gamma_1', \gamma''(1) = \gamma_1''. \tag{2.5}$$

Thus, the third order derivatives of the coordinates are involved in the jerk formalism, through a Lagrangian form. The minimization curves of the action (2.2) that are subject to the conditions (2.3)-(2.5) are given by the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^i} - \frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot{x}^i} = 0, \tag{2.6}$$

where $x^1 := x$, $x^2 := y$.

The variations of a curve

$$\gamma(t) = (x^i(t)) \tag{2.7}$$

which provide (2.6) by means of the variational principle

$$\frac{dI_L(\gamma_\varepsilon)}{d\varepsilon}_{|\varepsilon=0} = 0, \tag{2.8}$$

have the form

$$\gamma_\varepsilon(t) = (x_\varepsilon^i(t) = x^i(t) + \varepsilon V^i(t)), \tag{2.9}$$

where

$$V^i(0) = V^i(1) = \frac{dV^i}{dt}(0) = \frac{dV^i}{dt}(1) = \cdots = \frac{d^3V^i}{dt^3}(0) = \frac{d^3V^i}{dt^3}(1) = 0. \tag{2.10}$$

It follows that the ends are fixed according to the conditions (2.3)-(2.5), together with their velocities and their accelerations.

The equation (2.6), used for the specific Lagrangian (2.1), has the form

$$x^{(6)}(t) = y^{(6)}(t) = 0$$

and its solutions are polynomials in t of degrees at most five:

$$x(t) = \sum_{j=1}^{5} X_j t^j, y(t) = \sum_{j=1}^{5} Y_j t^j. \tag{2.11}$$

We further describe some solutions of jerk movements with fixed ends.

If we denote

$$\gamma_u = (x_u, y_u), \gamma_u' = (x_u', y_u'), \gamma_u'' = (x_u'', y_u''), u = 0, 1, \tag{2.12}$$

then the solution (2.11) that fulfills the conditions (2.3)-(2.5) has as coefficients:

$X_0 = x_0$, $X_1 = x_0'$, $X_2 = \dfrac{1}{2}x_0''$, $X_3 = 10(x_1 - x_0) - 6x_0' - \frac{3}{2}x_0'' - 4x_1' + \dfrac{1}{2}x_1''$,

$X_4 = 15(x_0 - x_1) + 8x_0' + \frac{3}{2}x_0'' + 7x_1' - x_1''$, $X_5 = 6(x_1 - x_0) - 3x_0' - \dfrac{1}{2}x_0'' - 3 + \dfrac{1}{2}x_1''$
and similarly for Y_0, ...,Y_5. In particular, consider the conditions (2.3) and

$$\gamma'(0) = \gamma''(0) = 0, \tag{2.13}$$

$$\gamma\prime(1) = \gamma"(1) = 0. \tag{2.14}$$

This case generates the free jerk movements (see [6]). If $\gamma_0 = (x_0, y_0)$ and $\gamma_1 = (x_1, y_1)$, then there is a unique solution given by

$$\begin{cases} x(t) = x_0 + 10(x_1 - x_0)t^3 + 15(x_0 - x_1)t^4 + 6(x_1 - x_0)t^5, \\ y(t) = y_0 + 10(y_1 - y_0)t3 + 15(y_0 - y_1)t^4 + 6(y_1 - y_0)t^5. \end{cases}$$

The plot of $t \to x(t) = 6t^5 - 15t^4 + 10t^3$, obtained for $x_0 = 0$ and $x_1 = 1$, is depicted in Fig. 1a. A mechanical interpretation of this solution can be found in [6].

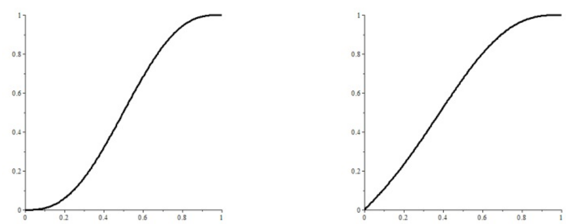

Figure 1: Plot of $(t, x(t))$, for a): (2.3)+(2.13) +(2.14), and b): (2.3)+(2.4)+(2.14).

Another case that includes the above one, is when one considers the conditions (2.3), (2.4) and (2.14), that give coefficients in the forms:

$X_0 = x_0$, $X_1 = x_0'$, $X2 = \dfrac{1}{2}x_0''$, $X_3 = 10(x_1 - x_0) - 6x_0' - \frac{3}{2}x_0''$,

$X_4 = 15(x_0 - x_1) + 8x_0' + \frac{3}{2}x_0''$, $X_5 = 6(x_1 - x_0) - 3x_0' - \dfrac{1}{2}x_0''$

and similarly for Y_0,... ,Y_5. The plot of $t \to x(t) = t + \dfrac{1}{2}t^2 + \dfrac{5}{2}t^3 - \dfrac{11}{2}t^4 + \dfrac{5}{2}t^5$, obtained for $x_0 = 0$, $x_1' = x_1'' = 0$, $x_1 = x_0' = x_0'' = 1$ is depicted in Fig. 1b.

3 Jerk movements with no necessarily fixed ends

Let us consider the Lagrangian (2.1), the action (2.2) on curves the form (2.7), but we reconsider the variation (2.9). The curve γ_ε given by (2.9) lifts to a curve $\gamma_\varepsilon^{(3)} : I \to T^3\mathbb{R}^2 = \left(\mathbb{R}^2\right)^4$,

$$\gamma_\varepsilon^{(3)}(t) = \left(x^i\left(t\right) + \varepsilon V^i(t), \frac{dx^i}{dt} + \varepsilon\frac{dV^i}{dt}, \dots, \frac{d^3x^i}{dt^3} + \varepsilon\frac{d^3V^i}{dt^3}\right).$$

We consider a new set of variations $\Gamma_\varepsilon^{(3)} : I \to T^3\mathbb{R}^2$, having the form

$$\Gamma_\varepsilon^{(3)}(t) = \left(x^i\left(t\right), \frac{dx^i}{dt}, \dots, \frac{d^3x^i}{dt^3} + \varepsilon\frac{d^3V^i}{dt^3}\right). \tag{3.1}$$

A variation in this form extends a variation considered in [1]: for a k-th order Lagrangian $L : T^k M \to \mathbb{R}$ one considers an action (2.2) and a set of variations $\Gamma_\varepsilon^{(k)} : I \to T^k M$ having the form

$$\Gamma_\varepsilon^{(k)}(t) = \left(x^i(t), \frac{dx^i}{dt}, \ldots, \frac{d^{k-1}x^i}{dt^3}(t) + \varepsilon V^i, \frac{d^k x^i}{dt^k} + \varepsilon \frac{dV^i}{dt}\right), \tag{3.2}$$

where the curve has the local form (2.7) and second we consider a more general form of variations

$$V^i(0) = V^i(1) = \frac{dV^i}{dt}(0) = \frac{dV^i}{dt}(1) = \cdots = \frac{d^{k-1}V^i}{dt^{k-1}}(0) = \frac{d^{k-1}V^i}{dt^{k-1}}(1) = 0. \tag{3.3}$$

Here M is a differentiable manifold. The critical curves of the actions, according to this set of variations, are solutions of the equations (see [1, eqn. (5)]):

$$\frac{\partial L}{\partial x^{(k-1)i}} - \frac{d}{dt}\frac{\partial L}{\partial x^{(k)i}} = 0.$$

In a similar way one can prove the following result.

Theorem 3.1. *Let $L : T^3 M \to \mathbb{R}$ be a third order Lagrangian on a differentiable manifold M. Let us consider the variational principle (2.8) applied to the action (2.2) on curves, according to the set of variations having the form (3.2). Then the critical curves are solutions of the equation*

$$\frac{\partial L}{\partial \dot{x}^i} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \dddot{x}^i} = 0.$$

The equation (3.3), used for the specific Lagrangian (2.1), has the form $x^{(5)}(t) = y^{(5)}(t) = 0$, and their solutions are polynomials in t of degrees at most four:

$$x(t) = \sum_{j=1}^{4} X_j t^j,\ y(t) = \sum_{j=1}^{4} Y_j t^j. \tag{3.4}$$

In the sequel we consider again the notations (2.12).

3.1 Solutions of jerk movements generated by variations on velocities of curves, when the second end point is not given

First we construct the solutions (3.7) that fulfill the conditions

$$\gamma(0) = \gamma_0 \tag{3.5}$$

together with the conditions (2.4) and (2.5). In this case one obtains a curve of the form (3.4), whose coefficients are
$X_0 = x_0, X_1 = x_0', X_2 = \frac{1}{2}x_0'', X_3 = -x_0' - \frac{2}{3}x_0'' + x_1' - \frac{1}{3}x_1''$,
$X_4 = \frac{1}{2}x_0' + \frac{1}{4}x_0'' - \frac{1}{2}x_1' + \frac{1}{4}x_1''$ and similarly for Y_0,... ,Y_4.

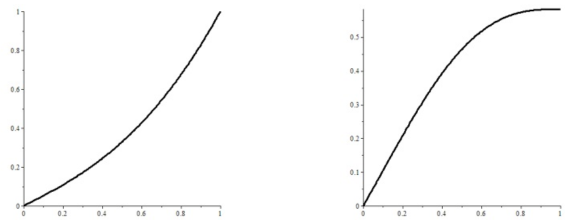

Figure 2: Plot of $(t, x(t))$, for a): (3.5)+(2.4)+(2.5) and b): (3.5)+(2.4)+(2.14).

In this case, the expected end point $\gamma(1) = \gamma_1$ is

$$\gamma 1 = (X_0 + ... + X_4, Y_0 + ... + Y_4) =$$
$$\left(x_0 + \tfrac{1}{2}x_0' + \tfrac{1}{12}x_0'' + \tfrac{1}{2}x_1' - \tfrac{1}{12}x_1'', y_0 + \tfrac{1}{2}y_0' + \tfrac{1}{12}y_0'' + \tfrac{1}{2}y_1' - \tfrac{1}{12}y_1''\right).$$

A graphical representation of $t \to x(t) = \frac{1}{2}t + \frac{1}{12}t^2 + \frac{1}{2}t^3 - \frac{1}{12}t^4$, obtained for $x_0' = x_0'' = x_1' = x_1'' = 1$, $x_0 = 0$ is depicted in Figure 2a. Using the above case, one can consider the conditions (3.5), (2.4) and (2.14) giving together as solution a curve (3.4) that has as coefficients

$X_0 = x_0$, $X_1 = x_0'$, $X_2 = x_0''$, $X_3 = -x_0' + \frac{2}{3}x_0''$, $X_4 = \frac{1}{2}x_0' + \frac{1}{4}x_0''$

and similarly for Y_0,... ,Y_4. In this case, the expected end point $\gamma(1) = \gamma_1$ is

$\gamma_1 = (X_0 + ... + X_4, Y_0 + ... + Y_4) = (x_0 + \frac{1}{2}x_0' + \frac{1}{12}x_0'', y_0 + \frac{1}{2}y_0' + \frac{1}{12}y_0'')$.

A graphical representation of $t \to x(t) = t + \frac{1}{2}t^2 - \frac{5}{3}t^3 + \frac{3}{4}t^4$, obtained for $x_0' = x_1' = x_1'' = 0$, $x_0' = x_0'' = 1$ is depicted in Figure 2b.

Another case is when one considers the conditions (3.5), (2.13) and (2.5), that give together as solution a curve (3.4) that has as coefficients

$$X_0 = x_0, X_1 = 0, X_2 = 0, X_3 = x_1' - \frac{1}{3}x_1'', X_4 = -\frac{1}{2}x_1' + \frac{1}{4}x_1''$$

and similarly for Y_0,... ,Y_4. In this case, the expected end point $\gamma(1) = \gamma_1$ is

$$\gamma_1 = (X_0 + ... + X_4, Y_0 + ... + Y_4) = (x_0 + \frac{1}{2}x_1' - \frac{1}{12}x_1'', y_0 + \frac{1}{2}y_1' - \frac{1}{12}y_1'').$$

A graphical representation of $t \to x(t) = \frac{2}{3}t^3 - \frac{1}{4}t^4$, obtained for $x_0 = x_0' = x_1' = x_1'' = 0$, $x_1 = 1$ is depicted in Figure 3a.

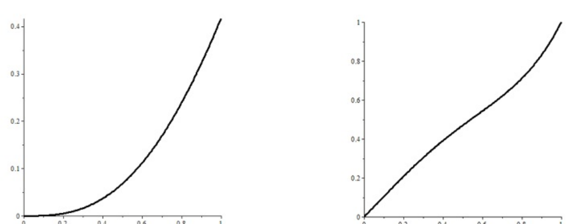

Figure 3: Plot of $(t, x(t))$, for a): (3.5)+(2.13)+(2.5) and b): (2.3)+(2.4)+(3.6).

3.2 Solutions of jerk movements for which the second end point is an initial point

First we construct the solutions (3.4) that fulfill the conditions (2.3), (2.4) and also the condition

$$\gamma'(1) = \gamma'_1. \tag{3.6}$$

In this case one obtains a curve (3.4) that has as coefficients

$$X_0 = x_0, X_1 = x'_0, X_2 = \frac{1}{2}x''_0, X_3 = 4(x_1 - x_0) - 4x'_1 - x''_0 - x'_1,$$
$$X_4 = 3(x_1 - x_0) + 3x'_0 + \frac{1}{2}x''_0 + x'_1$$

and similarly for Y_0,... ,Y_4. A graphical representation of $t \to x(t) = t + \frac{1}{2}t^2 - 2t^3 + \frac{3}{2}t^4$, obtained for $x_0 = 0$, $x'_0 = x''_0 = x_1 = x'_1 = 1$ is depicted in Figure 3b.

Another case is when we consider the conditions (2.3), (3.7) with $\gamma'_0 = 0$ and (2.14). One obtains a curve (3.4) that has as coefficients

$$X_0 = x_0, X_1 = 0, X_2 = 0, X_3 = 4(x_1 - x_0), X_4 = 3(x_0 - x_1)$$

and similarly for Y_0,... ,Y_4. A graphic of $t \to x(t) = 4t^3 - 3t^4$, for $x'_0 = x_1 = x'_1 = x''_1 = 0$, $x_1 = 1$ is depicted in Figure 4a.

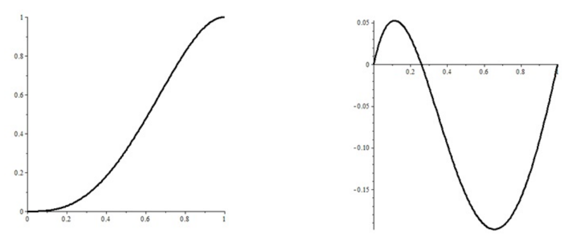

Figure 4: Path $(t, x(t))$, for *a*): (2.3)+(3.7,$\gamma'_0 = 0$)+(2.14) and *b*): (2.3)+(3.7)+(2.5).

In the case of solutions (3.4) that fulfill the conditions (2.3), (2.5) and

$$\gamma'(0) = \gamma'_0, \tag{3.7}$$

one obtains a curve (3.4) having as coefficients:

$$X_0 = x_0, X_1 = x'_0, X_2 = 6(x_1 - x_0) - 3(x'_0 + x'_1) + \frac{1}{2}x''_0,$$
$$X_3 = 8(x_0 - x_1) + 3x'_0 + 5x'_1 - x''_1, X_4 = 3(x_1 - x_0) - x'_0 - 2x'_1 + \frac{1}{2}x''_1$$

and similarly for Y_0,... ,Y_4. A graphical representation of $t \to x(t) = t - \frac{11}{2}t^2 + 7t^3 - \frac{5}{2}t^4$, obtained for $x_0 = 0$, $x_1 = x'_0 = x'_1 = x''_1 = 1$ is depicted in Figure 4b. A special case of this is when the conditions (2.3), (3.7) and (2.14) give together as a solution a curve (3.4), the case represented in Fig. 4a.

4 Conclusions

Jerk motions are involved in various fast movements, where the derivative of the acceleration, i.e., the third order derivative of the position vector coordinates are involved. This kind of motions are involved in various real situations, according to various bibliographical sources.

The original jerk case, as studied in [6], starts from a critical situation (with null velocity and acceleration) and ends also at a critical point. We depict here some cases when the end points are not necessarily both critical, but also some cases when none of them is critical. Thus, our approach can be used to control the free jerk motions.

The above new and original description considered in our paper contains a mathematical background (not known in the present form in literature until now) of a jerk movement that is generated by a variation (3.1) that involves the velocities, the accelerations and the jerk (as the derivatives of accelerations), but not positions. The idea can be used in some general settings from the higher order mechanics.

A mechanical interpretation of the variation (2.10) studied in the paper can be performed using two joint (united) moving bodies such that the motion is initially (for $t < 0$) generated only by one of the moving bodies; then, at a given time ($t = 0$), the second moving body (initially inactive) has an accelerated motion during a given period of time (for example $t \in [0, 1]$), such that at the end point ($t = 1$) the second moving body stops its action or, at most, its acceleration remains a constant, while the first moving body continues its programmed action.

The settings of our paper open new directions of investigation, that can be studied in other subsequent works. The mathematical models presented in the paper can be related to specific mechanical models, in various situations, in order to validate the theoretical assumptions. Using the settings of this paper and [12], one can also construct a mathematical model in order to involve other exterior forces, such as frictions. Moreover, the study can involve a set of constraints (as in [5]) or a control (as in [13]).

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