Tzitzeica geometry of soliton solutions for quartic interaction PDE

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Abstract. Geometric properties of graphs of solutions for the quartic interaction PDE are studied in the present work. Two classes of solutions are considered. One class is represented by soliton solutions, whereas the other class consists of solutions of a first order PDE system, which generates the quartic interaction PDE, in the sense of least squares type action. We prove that for both classes the graphs of solutions are Tzitzeica flat, i.e., the associated Tzitzeica curvature tensor vanishes. It is also shown how the quartic interaction PDE can be generated using a least squares type action.

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1 Recent topics about geometry of PDEs solutions

Geometric properties of solutions for partial differential equations (PDEs) provide surprising information for specific problems described by the equations. This topic has been the focus of numerous international conferences, workshops and research programs for a number of recent years. The directions of research include: *convexity of solutions, blow-up analysis, Sobolev type inequalities, hot spots, shapes of graphs of solutions* etc.

In this paper we are concerned with: (i) solutions of first order PDEs as submanifolds; (ii) Tzitzeica differential geometry for soliton solutions of the *quartic interaction PDE*; this is a fundamental equation of the Quantum Field Theory related to the famous *Klein-Gordon equation*; (iii) the geometry of least squares generators for quartic interaction PDE.

The paper is structured as follows. In Section 2, we lay out the theoretical considerations, introduce the basic notions, and recall some results from ([7], [8]) which will be used later.

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In Section 3, we state and prove our first result concerning the geometry of the Tzitzeica connection on graphs of soliton solutions of the quartic interaction PDE. More precisely, we prove that the graphs are Tzitzeica flat.

In Section 4, we prove that the quartic interaction PDE is an Euler-Lagrange prolongation of a first order PDE system. According to the Theorem 4.1, the solutions of the first order PDE system are also solutions of the quartic interaction PDE. The surprising fact is that this class of solutions has similar geometric properties with the class of soliton solutions.

2 Solutions of first order PDEs as sub-manifolds

All relevant mathematical ingredients used in the present work are supposed to be of class C^∞. The advantage of working with this class of objects consist in the fact that it is invariant under differentiation.

Let (T,k) and (M,g) be semi-Riemannian manifolds of dimension m, and n respectively, with $m<n$. The indexing of the components of the geometrical objects corresponding to the manifold T (manifold M) will be done using Greek (Latin) letters. Denote the local coordinates on the manifold T by $t=(t^\alpha)$, $\alpha=1,\dots,m$, and denote the local coordinates on the manifold M by $x=\left(x^i\right)$, $i=1,\dots,n$. The first order jet manifold $J^1(T,M)$, is endowed with the adapted coordinates $(t^\alpha,x^i,x^i_\alpha)$ (see for instance [3]).

A distinguished tensor field $X^i_\alpha(t,x(t))$, defined on $T\times M$ (with local coordinates (t^α,x^i)), defines a first order normal PDE system

$$\frac{\partial x^i}{\partial t^\alpha}(t)=X^i_\alpha(t,x(t)). \tag{2.1}$$

Suppose that the complete integrability conditions

$$\frac{\partial X^i_\alpha}{\partial t^\beta}+\frac{\partial X^i_\alpha}{\partial x^j}X^j_\beta=\frac{\partial X^i_\beta}{\partial t^\alpha}+\frac{\partial X^i_\beta}{\partial x^j}X^j_\alpha \tag{2.2}$$

are satisfied throughout. The solutions of the PDE system (2.1) correspond to $m-$dimensional leaves, which are sub-manifolds of co-dimension $n-m$, in M, diffeomorphic to T. These are the leaves of the foliation of the manifold M, induced by the integrable distribution $\mathcal{D}=\{X_\alpha\}$, $\alpha=1,\cdots,m$, where $X_\alpha=\left(X^1_\alpha,\cdots,X^n_\alpha\right)$. The geometric properties of the leaves, obtained as above, will be our main interest in what follows. The source of inspiration are the works [6], [8], [16].

Suppose that the PDE system (2.1) has a solution $x(t)$. Differentiating along the solutions $x(t)$, and substituting $x^j_\beta=X^j_\beta$, yields the second order PDE system

$$\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}=\frac{\partial X^i_\alpha}{\partial x^j}X^j_\beta+\frac{\partial X^i_\alpha}{\partial t^\beta}. \tag{2.3}$$

On the other hand, a sub-manifold $S: x=x(t)$, $x(t_0)=x_0$, satisfies the *Gauss equations*

$$\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}(t)=\Lambda^\gamma_{\alpha\beta}(t)x^i_\gamma(t)+\Omega^a_{\alpha\beta}(t)N^i_a(x(t)), \tag{2.4}$$

where $N_a = N_a^i \partial_{x^i}$, $a = 1, \cdots, n-m$ are unit vector fields normal to S. Replacing the left hand side member by the right hand side member of (2.3) and using the *induced metric*

$$h_{\alpha\beta}(t) := \left(g_{ij} X_\alpha^i X_\beta^j\right)(t, x(t)), \tag{2.5}$$

on sub-manifold S, we obtain the components of the (non-metric) *Tzitzeica connection*

$$\Lambda^\gamma_{\alpha\beta}(t) = h^{\gamma\sigma}(t) g_{ik} X_\sigma^k \left[\frac{\partial X_\alpha^i}{\partial x^j} X_\beta^j + \frac{\partial X_\alpha^i}{\partial t^\beta}\right](t, x(t)), \tag{2.6}$$

and the *fundamental forms*

$$\Omega^a_{\alpha\beta}(t) = \delta^{ab} g_{ik} N_b^k \left[\frac{\partial X_\alpha^i}{\partial x^j} X_\beta^j + \frac{\partial X_\alpha^i}{\partial t^\beta}\right](t, x(t)). \tag{2.7}$$

The components of the *Tzitzeica mean curvature vector field*, with respect to the induced metric $h_{\alpha\beta}$, are given by

$$H^i(x(t)) = h^{\alpha\beta}(t) \Omega^a_{\alpha\beta}(t) N_a^i(x(t)). \tag{2.8}$$

Remark 2.1. Any PDE can be transformed into a (constrained or not) first order PDE system, and automatically, we can build the associated *Tzitzeica geometry.*

To a classical second order PDE, written in the explicit form (see for instance [1])

$$F(t, x(t), x_\alpha(t), x_{\alpha\beta}(t)) = 0, \tag{2.9}$$

where $t = (t^\alpha)$, $x = (x^i)$, $x_\alpha = (x^i_\alpha)$, $x_{\alpha\beta} = (x^i_{\alpha\beta})$, $\alpha = 1, \cdots, m$, $i = 1, \cdots, n$, one may associate a constrained first order PDE system

$$\begin{cases} \dfrac{\partial x^i}{\partial t^\alpha}(t) = \xi^i_\alpha(t), \\ \\ F(t, x(t), \xi_\alpha(t), \dfrac{\partial \xi_\alpha}{\partial t^\beta}(t)) = 0. \end{cases} \tag{2.10}$$

The associated "least squares type Lagrangian density" (see Section 4 for more details), with respect to the metric tensors (g_{ij}) and $(h_{\alpha\beta})$, is

$$L = \frac{1}{2} h^{\alpha\beta} g_{ij} \left(x^i_\alpha - \xi^i_\alpha\right)\left(x^j_\beta - \xi^j_\beta\right) + \frac{1}{2} F^2(t, x(t), \xi_\alpha(t), \xi_{\alpha\beta}(t)).$$

As an alternative approach, one might consider the Lagrangian density

$$L_\lambda = \frac{1}{2} h^{\alpha\beta} g_{ij} \left(x^i_\alpha - \xi^i_\alpha\right)\left(x^j_\beta - \xi^j_\beta\right) + \frac{1}{2} \lambda F(t, x(t), \xi_\alpha(t), \xi_{\alpha\beta}(t)),$$

where λ is a Lagrange multiplier (see, for instance, [2]).

Remark 2.2. If the initial second order PDE is given in the normal form

$$x_{\alpha\beta} = F_{\alpha\beta}(t, x(t), x_\alpha(t)), \quad \alpha \leq \beta, \tag{2.11}$$

then the corresponding first order PDE system is

$$\text{(2.12)} \qquad \begin{cases} \dfrac{\partial x^i}{\partial t^\alpha}(t) = \xi^i_\alpha(t) \\[2ex] \dfrac{\partial \xi^i_\alpha}{\partial t^\beta} = F^i_{\alpha\beta}(t, x(t), \xi_\alpha(t)). \end{cases}$$

In this case, the corresponding "least squares type Lagrangian density", becomes

$$L = \frac{1}{2} h^{\alpha\beta} g_{ij} \left(x^i_\alpha - \xi^i_\alpha\right)\left(x^j_\beta - \xi^j_\beta\right) + \frac{1}{2} h^{\alpha\mu} h^{\beta\lambda} g_{ij} \left(\xi^i_{\alpha\beta} - F^i_{\alpha\beta}\right)\left(\xi^j_{\mu\lambda} - F^i_{\mu\lambda}\right).$$

We also should keep in mind the complete integrability conditions

$$\frac{\partial^2 x}{\partial t^\gamma \partial t^\beta} = \frac{\partial^2 x}{\partial t^\beta \partial t^\gamma}, \quad \frac{\partial^2 \xi_\alpha}{\partial t^\gamma \partial t^\beta} = \frac{\partial^2 \xi_\alpha}{\partial t^\beta \partial t^\gamma}, \quad \alpha, \beta, \gamma = 1, \cdots, m.$$

Remark 2.3. When the initial second order PDE system is pseudo-linear, i. e.,

$$\text{(2.13)} \qquad h^{\alpha\beta}(t) x^i_{\alpha\beta}(t) + F^i(t, x(t), x_\alpha(t)) = 0,$$

one may associate the first order PDE system

$$\text{(2.14)} \qquad \begin{cases} \dfrac{\partial x^i}{\partial t^\alpha}(t) = \xi^i_\alpha(t) \\[2ex] \dfrac{\partial \xi^i_\alpha}{\partial t^\beta} = \dfrac{\partial \xi^i_\beta}{\partial t^\alpha} \\[2ex] h^{\alpha\beta}(t) \dfrac{\partial \xi^i_\alpha}{\partial t^\beta}(t) + F^i(t, x(t), \xi_\alpha(t) = 0, \end{cases}$$

together with the "least squares type Lagrangian density"

$$\begin{aligned} L &= \frac{1}{2} h^{\alpha\beta} g_{ij} \left(x^i_\alpha - \xi^i_\alpha\right)\left(x^j_\beta - \xi^j_\beta\right) + \frac{1}{2} h^{\alpha\mu} h^{\beta\lambda} g_{ij} \left(\frac{\partial \xi^i_\alpha}{\partial t^\beta} - \frac{\partial \xi^i_\beta}{\partial t^\alpha}\right)\left(\frac{\partial \xi^j_\alpha}{\partial t^\beta} - \frac{\partial \xi^j_\beta}{\partial t^\alpha}\right) \\ &+ \frac{1}{2}\, g_{ij} \left(h^{\alpha\beta} \frac{\partial \xi^i_\alpha}{\partial t^\beta} + F^i\right)\left(h^{\alpha\beta} \frac{\partial \xi^j_\alpha}{\partial t^\beta} + F^j\right). \end{aligned}$$

3 Tzitzeica geometry of solitons associated to quartic interaction PDE

In the context of Quantum Field Theory, a classical free scalar field is a solution of the Klein-Gordon equation, which is a relativistic analogue of the Schrödinger equation.

The Klein-Gordon equation can be altered in such a manner that the solutions of the modified version, which is called *quartic interaction PDE*, are fields with quartic interaction in Quantum Field Theory (see for example [5]).

In this section, we are interested in the geometric properties of graphs of soliton solutions for quartic interaction PDE, as four-dimensional submanifolds immersed in the product manifold of (T,k) and $(\mathbb{R},\delta=1)$, where by (T,k) we denote the four-dimensional Minkowski space-time, with the metric signature $(-+++)$.

Let

$$u : T \longrightarrow \mathbb{R},\quad t=\left(t^1,t^2,t^3,t^4\right)\longmapsto u\left(t^1,t^2,t^3,t^4\right).$$

The quartic interaction PDE (see for example [5]) is

$$u_{11}-u_{22}-u_{33}-u_{44}=\mu^2 u-\lambda u^3, \tag{3.1}$$

where μ is the *mass term*, λ is the (strictly positive) *coupling constant*, indices means partial derivatives and whose solutions are real valued functions (scalar fields) u.

Let us consider the soliton solutions, in implicit form, given by (see for example [4])

$$\int\left[C_1+\frac{2}{k_1^2-(k_2^2+k_3^2+k_4^2)}\int f(u)du\right]^{-1/2}du=k_\alpha t^\alpha+C_2, \tag{3.2}$$

where $f(u)=\mu^2u-\lambda u^3$ and $C_1,C_2,k_\alpha,\ \alpha=1,2,3,4,$ are some constants such that $k_1^2\neq k_2^2+k_3^2+k_4^2$.

Denoting by $F(u)$ the left hand side member of the equation (3.2), and taking the partial derivatives, we obtain

$$\frac{\partial}{\partial t^\alpha}F(u)=\frac{\partial F}{\partial u}u_\alpha=k_\alpha \implies u_\alpha=k_\alpha\frac{1}{\frac{\partial F}{\partial u}}=k_\alpha Y(u),\quad \alpha=1,2,3,4, \tag{3.3}$$

where $Y(u)=\dfrac{1}{\frac{\partial F}{\partial u}}$.

The product manifold $(M,g)=(T\times\mathbb{R},\ k+\delta)$, with coordinates (t^1,t^2,t^3,t^4,u), is a Lorentzian manifold. The graphs of solutions are integral manifolds of the smooth integrable distribution $\mathcal{D}=\{Y_1,Y_2,Y_3,Y_4\}$, spanned by

$$\begin{aligned}Y_1&=(1,0,0,0,Y_1^5),\quad Y_2=(0,1,0,0,Y_2^5),\\ Y_3&=(0,0,1,0,Y_3^5),\quad Y_4=(0,0,0,1,Y_4^5),\end{aligned}$$

defined on the manifold M, where

$$Y_\alpha^5(x(t))=k_\alpha Y(u(t)). \tag{3.4}$$

Computing the induced pseudo-metric, on an integral sub-manifold of the distribution $\mathcal{D}$, using (2.5), yields

$$(h_{\alpha\beta})=\begin{pmatrix}-1+k_1^2Y^2 & k_1k_2Y^2 & k_1k_3Y^2 & k_1k_4Y^2\\ k_2k_1Y^2 & 1+k_2^2Y^2 & k_2k_3Y^2 & k_2k_4Y^2\\ k_3k_1Y^2 & k_3k_2Y^2 & 1+k_3^2Y^2 & k_3k_4Y^2\\ k_4k_1Y^2 & k_4k_2Y^2 & k_4k_3Y^2 & 1+k_4^2Y^2\end{pmatrix}. \tag{3.5}$$

The contravariant components are

$$\left(h^{\alpha\beta}\right)=\frac{1}{\Delta_h}\begin{pmatrix}1+c_1Y^2 & -k_1k_2Y^2 & -k_1k_3Y^2 & -k_1k_4Y^2\\ -k_2k_1Y^2 & -1+c_2Y^2 & k_2k_3Y^2 & k_2k_4Y^2\\ -k_3k_1Y^2 & k_3k_2Y^2 & -1+c_3Y^2 & k_3k_4Y^2\\ -k_4k_1Y^2 & k_4k_2Y^2 & k_4k_3Y^2 & -1+c_4Y^2\end{pmatrix}, \tag{3.6}$$

where
$$c_1 = \left(k_2^2 + k_3^2 + k_4^2\right), \quad c_2 = \left(k_1^2 - k_3^2 - k_4^2\right),$$
$$c_3 = \left(k_1^2 - k_2^2 - k_4^2\right), \quad c_4 = \left(k_1^2 - k_2^2 - k_3^2\right).$$

The determinant associated to the previous pseudo-metric is

$$\Delta_h = -1 - \left(k_4^2 + k_3^2 + k_2^2 - k_1^2\right) Y^2.$$

Hence, we shall impose $k_4^2 + k_3^2 + k_2^2 > k_1^2$, in order for $\Delta_h < 0$ to hold.

We have the following result concerning the Tzitzeica curvature of an integral manifold of the distribution $\mathcal{D}$.

Theorem 3.1. *Let S be an integral manifold of the distribution $\mathcal{D}$. Then*

i) the components of the Tzitzeica connection Λ are

$$\Lambda^{\gamma}_{\alpha\beta} = h^{\gamma\sigma} Y^5_{\sigma} \frac{\partial Y^5_{\alpha}}{\partial u} Y^5_{\beta} = \left(h^{\gamma\sigma} k_{\sigma}\right) k_{\alpha} k_{\beta} \left(Y\right)^2 \frac{\partial Y}{\partial u}, \tag{3.7}$$

ii) the curvature tensor of the manifold (S, Λ) is identically zero.

Proof. i) Noticing that the vector fields depend only on the last coordinate, and making the substitution $Y^5_{\alpha} = k_{\alpha} Y$, from (2.6), one obtains (3.7).

ii) The curvature tensor of the Tzitzeica connection is

$$R^{\gamma}_{\sigma\alpha\beta} = \frac{\partial \Lambda^{\gamma}_{\sigma\beta}}{\partial t^{\alpha}} - \frac{\partial \Lambda^{\gamma}_{\sigma\alpha}}{\partial t^{\beta}} + \Lambda^{\eta}_{\sigma\beta}\Lambda^{\gamma}_{\eta\alpha} - \Lambda^{\eta}_{\sigma\alpha}\Lambda^{\gamma}_{\eta\beta}, \quad \alpha, \beta, \sigma, \eta = 1, 2, 3, 4.$$

Substituting (3.7), we have

$$\begin{aligned} R^{\gamma}_{\sigma\alpha\beta} = {} & \frac{\partial}{\partial t^{\alpha}} \left[\left(h^{\gamma\eta} k_{\eta}\right) k_{\sigma} k_{\beta} \left(Y\right)^2 \frac{\partial Y}{\partial u} \right] - \frac{\partial}{\partial t^{\beta}} \left[\left(h^{\gamma\nu} k_{\nu}\right) k_{\sigma} k_{\alpha} \left(Y\right)^2 \frac{\partial Y}{\partial u} \right] \\ & + \left[\left(h^{\eta\nu} k_{\nu}\right) k_{\sigma} k_{\beta} \left(Y\right)^2 \frac{\partial Y}{\partial u} \right] \left[\left(h^{\gamma\mu} k_{\mu}\right) k_{\eta} k_{\alpha} \left(Y\right)^2 \frac{\partial Y}{\partial u} \right] \\ & - \left[\left(h^{\eta\nu} k_{\nu}\right) k_{\sigma} k_{\alpha} \left(Y\right)^2 \frac{\partial Y}{\partial u} \right] \left[\left(h^{\gamma\mu} k_{\mu}\right) k_{\eta} k_{\beta} \left(Y\right)^2 \frac{\partial Y}{\partial u} \right]. \end{aligned}$$

Since the last two terms cancel each other, and using the fact that $\dfrac{\partial}{\partial t_{\alpha}} = k_{\alpha} Y \dfrac{\partial}{\partial u}$, one obtains

$$\begin{aligned} R^{\gamma}_{\sigma\alpha\beta} & = k_{\sigma} k_{\beta} k_{\alpha} Y \frac{\partial}{\partial u} \left[\left(h^{\gamma\eta} k_{\eta}\right) \left(Y\right)^2 \frac{\partial Y}{\partial u} \right] \\ & - k_{\sigma} k_{\alpha} k_{\beta} Y \frac{\partial}{\partial u} \left[\left(h^{\gamma\nu} k_{\nu}\right) \left(Y\right)^2 \frac{\partial Y}{\partial u} \right] = 0. \end{aligned}$$

□

Remark 3.1. It is worth noticing that the above results do not depend on the function $Y(u(t))$ but only on the fact that $Y^5_{\alpha}(x(t)) = k_{\alpha} Y(u(t))$. Thus, it should hold in other cases too.

Remark 3.2. The unit vector field $N(t) = \frac{1}{l}(k_1 Y, -k_2 Y, -k_3 Y, -k_4 Y, 1)$, where $l = \sqrt{-\Delta_h}$, is orthogonal to any integral manifold of the distribution $\mathcal{D}$. Hence, by direct computation, from (2), we obtain the components of the *second fundamental form*

$$\Omega_{\alpha\beta} = k_\alpha k_\beta \frac{1}{2l} \frac{\partial \left(Y^2\right)}{\partial u}. \tag{3.8}$$

Thus, the components of the *Tzitzeica mean curvature vector field*, by direct computation, are

$$H^i\left((t)\right) = \frac{1}{2l}\frac{\partial \left(Y^2\right)}{\partial u}\left[Y^2 \frac{\left(k_1^2+k_2^2+k_3^2+k_4^2\right)}{\Delta_h} + \left(k_2^2+k_3^2+k_4^2-k_1^2\right)\right] N^i(t). \tag{3.9}$$

4 The geometry of least squares generators for quartic interaction PDE

In the present section we shall study the geometry of another class of solutions of the quartic interaction PDE, which have a close connection to the soliton solutions. More precisely, we shall study the Tzitzeica geometry corresponding to a first order PDE system which is a generator, in the sense of geometric dynamics, of the quartic interaction PDE. The classical approach of geometric dynamics ([10]-[15]) consists in extending normal first order PDE systems to second order Euler - Lagrange type PDE systems; the solutions of the first order systems are included in the set of extremal points of least squares type Lagrangians (see [8]). In this respect, we recall the following key result ([7] Theorem 2.3, [10]).

Theorem 4.1. *With the above notations, we have that each solution of the PDE system (2.1) is an extremal for the Lagrangian*

$$L = \frac{1}{2} h^{\alpha\beta} g_{ij} (x^i_\alpha - X^i_\alpha)(x^j_\beta - X^j_\beta)\sqrt{|h|}. \tag{4.1}$$

Note that the converse assertion is not true.

Let, as above, (T, k) be the Minkowski space and $(M, g) = (T \times \mathbb{R},\ s + \delta)$, and consider the immersion (graph map)

$$x : T \to M, \quad \left(t^1, t^2, t^3, t^4\right) \longmapsto \left(t^1, t^2, t^3, t^4, u\left(t\right)\right), \tag{4.2}$$

i.e. $x^i\left(t\right) = t^i,\ i = 1, 2, 3, 4,\ x^5(t) = u(t)$.

Let $X^5_\alpha(x(t)) = k_\alpha X(u(t)),\ \alpha = 1, 2, 3, 4$, where

$$X(x(t)) = \frac{1}{\sqrt{2\lambda(k_2^2+k_3^2+k_4^2-k_1^2)}}\left(\lambda\left(u(t)\right)^2 - \mu^2\right). \tag{4.3}$$

and k_α are nonzero real numbers such that $k_1^2 < k_2^2 + k_3^2 + k_4^2$.

We introduce the first order normal PDE system

$$(4.4) \qquad \begin{cases} \dfrac{\partial x^i}{\partial t^\alpha} = \delta^i_\alpha = X^i_\alpha(x(t)), \quad i,\alpha = 1,2,3,4, \\ \\ \dfrac{\partial x^5}{\partial t^\alpha} = X^5_\alpha(x(t)). \end{cases}$$

The associated "least squares type Lagrangian" (4.1), corresponding to the metric tensors (g_{ij}) and $\left(h^{\alpha\beta}\right)$, respectively, becomes

$$(4.5) \qquad \begin{aligned} L \ &= \frac{1}{2}\Big[h^{\alpha\beta} g_{ij}\left(x^i_\alpha - X^i_\alpha\right)\left(x^j_\beta - X^j_\beta\right) \\ &- \ \left(x^5_1 - X^5_1\right)^2 + \left(x^5_2 - X^5_2\right)^2 + \left(x^5_3 - X^5_3\right)^2 + \left(x^5_4 - X^5_4\right)^2\Big] \\ &\alpha,\beta,i,j = 1,\cdots,4, \end{aligned}$$

(recall that $k_{11} = g_{11} = -1,\ k_{22} = g_{22} = k_{33} = g_{33} = k_{44} = g_{44} = 1,\ g_{55} = 1$, and non-diagonal terms are zero).

The connection between the first order PDE system (4.4) and the quartic interaction PDE (3.1) becomes clear by the following

Proposition 4.2. *i) The quartic interaction PDE (3.1) is an Euler-Lagrange prolongation of the system (4.4), with respect to the manifolds* (T,k) *and* $(T\times\mathbb{R}, k+\delta)$*, respectively.*

ii) There exist infinitely many suitable geometric structures and infinitely many vector fields which realize the above prolongation.

Proof. Writing the Euler - Lagrange equation

$$\frac{\partial L}{\partial x^5} - \frac{\partial}{\partial t^\alpha}\frac{\partial L}{\partial x^5_\alpha} = 0,$$

with respect to the last coordinate function $x^5(t) = u(t)$, one obtains

$$(4.6) \qquad X^5_1\frac{\partial X^5_1}{\partial x^5} - \sum_{\alpha=2}^{4} X^5_\alpha\frac{\partial X^5_\alpha}{\partial x^5} = \frac{\partial^2 x^5}{\partial t^1\partial t^1} - \sum_{\alpha=2}^{4}\frac{\partial^2 x^5}{\partial t^\alpha\partial t^\alpha}.$$

Making the corresponding substitutions yields

$$\mu^2 u - \lambda u^3 = u_{11} - u_{22} - u_{33} - u_{44},$$

which is precisely the quartic interaction PDE (3.1).

In order to prove ii), it is enough to choose as the component g_{55} any strictly positive constant. □

Let $\mathcal{D}' = \{X_1, X_2, X_3, X_4\}$ be the distribution generated by the smooth vector fields

$$X_1 = (1,0,0,0,X^5_1), \quad X_2 = (0,1,0,0,X^5_2)$$
$$X_3 = (0,0,1,0,X^5_3), \quad X_4 = (0,0,0,1,X^5_4),$$

defined on the manifold M. The Frobenius integrability conditions reduce to

$$\frac{\partial X_\alpha^5}{\partial x^5} X_\beta^5 = \frac{\partial X_\beta^5}{\partial x^5} X_\alpha^5, \quad \alpha, \beta = 1, 2, 3, 4, \tag{4.7}$$

which means that the corresponding distribution is integrable.

The geometry of the integral sub-manifolds of the integrable distribution $\mathcal{D}'$ can be studied precisely as in the previous section. The induced metric tensor on integral manifolds of the distribution $\mathcal{D}'$, as well as its inverse, are precisely as in (3.5) and (3.6), respectively, with X instead of Y.

Using the Remark 3.1, one obtains the following, similar to (3.1),

Theorem 4.3. *Let S' be an integral manifold of the distribution $\mathcal{D}'$. Then*

i) the components of the Tzitzeica connection Λ) are

$$\Lambda_{\alpha\beta}^\gamma = h^{\gamma\sigma} X_\sigma^5 \frac{\partial X_\alpha^5}{\partial u} X_\beta^5 = (h^{\gamma\sigma} k_\sigma)\, k_\alpha k_\beta\, (X)^2\, \frac{\partial X}{\partial u}, \tag{4.8}$$

ii) the curvature tensor of (S', Λ) is identically zero.

5 Conclusions

In the present paper we considered geometric objects which correspond in a natural way to PDEs and their solutions. These objects provide, very often, insights for the qualitative study of equations. One such object is the Tzitzeica connection together with its associated curvature tensor, which vanishes on graphs of the two classes of solutions we had considered.

The present work also provides an example of how the tools of geometric dynamics can be useful when we deal with geometric properties of graphs of solutions of PDEs.

A starting point for further research is the Remark 3.1, according to which, the result of Theorem 3.1 is valid for a larger class of equations.

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