On generalized symmetric Finsler spaces

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Abstract. In this article we study the geometric properties of generalized symmetric Finsler spaces. We first construct some examples which are generalized symmetric but not Berwald. Then we explore the relationship between weakly symmetric spaces and generalized symmetric spaces. In particular, we construct a series of examples of non-symmetric Riemannian manifolds which are weakly symmetric with a regular s-structure of order k, where $k \neq 2$.

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1 Introduction

In 1967, A.J. Ledger [15] initiated the study of generalized Riemannian symmetric spaces. These spaces are Riemannian manifolds (M, g) which admit at each point p in M an isometry s_p with p as an isolated fixed point. The definition of these spaces arises as a natural extension of symmetric spaces of É. Cartan. In fact, a generalized Riemannian symmetric space must be homogeneous [16]. Furthermore, if a regularity condition (trivially satisfied by globally symmetric spaces) is imposed on the isometries (s_p), then they can be chosen to have the same order n [10]. In this case, the spaces are said to be Riemannian regular n-symmetric spaces.

Symmetric Finsler spaces were first proposed and studied by Z.I. Szabó and the second author. A Finsler space (M, F) is called globally symmetric if any point of M is an isolated fixed point of an involutive isometry ([8], [12]). If we drop the involution property in the definition of symmetric Finsler spaces but keep the property that $s_x \circ s_y = s_z \circ s_x, z = s_x(y)$, we get a broader class of Finsler spaces called generalized symmetric spaces [14].

However, up to now very few geometric properties about generalized symmetric space have been studied. The purpose of this paper is to initiate a systematic study of such spaces.

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2 Preliminaries

In this section we present some fundamental definitions and facts in Finsler geometry.

Definition 2.1. Let V be an n-dimensional real vector space. A Minkowski norm on V is a real function F on V which is smooth on $V \setminus \{0\}$ and satisfies the following conditions:

1. $F(u) \geq 0$, $\forall u \in V$;
2. $F(\lambda u) = \lambda F(u), \forall \lambda > 0$, $u \in V$;
3. Given any basis $u_1, u_2, \ldots, u_n$ of V, write $F(y) = F(y^1, y^2, \ldots, y^n)$ for $y = y^1 u_1 + y^2 u_2 + \ldots + y^n u_n$. Then the Hessian matrix
$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$
is positive-definite at any point of $y \in V \setminus \{0\}$, where the subscript coordinates mean taking the partial differentials with respect to them.

Note that the condition 3 in the above definition combined with the non-negative condition 1 implies that a Minkowski norm must be positive definite in the sense that $F(u) > 0$, $\forall u \in V \setminus \{0\}$; see [1].

Definition 2.2. Let M be a connected smooth manifold. A Finsler metric on M is a function $F : TM \to [0, \infty)$ such that

1. F is C^∞ on the slit tangent bundle $TM \setminus \{0\}$;
2. The restriction of F to any $T_x M$, $x \in M$, is a Minkowski norm.

Let F be a Finsler metric on a smooth n-dimensional manifold M. On a standard local coordinate system of TM, the geodesic coefficients of F are defined by

$$G^i = \frac{1}{4} g^{ij} \{ [F^2]_{x^j y^k} y^k - [F^2]_{x^j} \}, \quad i = 1, 2, \ldots, n, \ x \in M, \ y \in T_x M.$$

Definition 2.3. A Finsler space (M, F) is called a Berwald space if on any standard local coordinate system of TM, the geodesic spray coefficients G^i are quadratic in $y \in TM_0$

The following results can be found in [4].

Proposition 2.1. *A connected Finsler space (M, F) is a Berwald space if and only if the parallel displacement along any piecewise smooth curve is a linear map, if and only if the holonomy group of (M, F) at any point of M consists of linear transformations.*

Affine and Riemannian s-manifold were fist defined in [13] following the introduction of generalized Riemannian symmetric spaces in [15]. They form a more generalized class than symmetric spaces studied by É. Cartan. Generalized symmetric Finsler spaces were first defined in [14]. This notion is a natural generalization of generalized Riemannian symmetric spaces.

Definition 2.4. Let (M, F) be a connected Finsler space, and $I(M, F)$ the full group of isometries of (M, F). An isometry of (M, F) with x as an isolated fixed point is called a symmetry at x, and will usually be denoted as s_x. A family $\{s_x | x \in M\}$ of symmetries on a connected Finsler manifold (M, F) is called an s-structure on (M, F).

An s-structure $\{s_x | x \in M\}$ is called of order k $(k \geq 2)$ if $(s_x)^k = \mathrm{id}$ for all $x \in M$ and k is the least integer of satisfying the above property. Obviously a Finsler space is symmetric if and only if it admits an s-structure of order 2. An s-structure $\{s_x\}$ on (M, F) is called regular if for every pair of points $x, y \in M$,

$$s_x \circ s_y = s_z \circ s_x, \ z = s_x(y).$$

Definition 2.5 ([14])**.** A generalized symmetric Finsler space is a connected Finsler manifold (M, F) admitting a regular s-structure. A Finsler space (M, F) is said to be k-symmetric $(k \geq 2)$ if it admits a regular s-structure of order k.

Theorem 2.2 ([5], [8])**.** *Let (M, F) be a globally symmetric Finsler space. Then (M, F) is a Berwald space. Furthermore, the connection of F coincides with the Levi-Civita connection of a Riemannian metric Q such that (M, Q) is a Riemannian globally symmetric space.*

3 Generalized symmetric Randers metrics

Let (M, F) be a connected Finsler space. Then the group $I(M, F)$ of isometries of (M, F) is a Lie transformation group of M ([7]). If $I(M, F)$ acts transitively on M, then (M, F) is called a homogeneous Finsler space. In this case the homogeneous Finsler manifold M can be written as the form $M = G/H$, where G is a Lie group acting isometrically and transitively on M, and H is the isotropy subgroup of G at a point in M. Moreover, if the Lie algebra $\mathfrak{g}$ of G has a decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \qquad \text{(direct sum of subspaces)}$$

where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ such that

$$\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}, \quad \text{for all } h \in H,$$

then the homogeneous Finsler manifold $(G/H, F)$ is called reductive. In this case, the tangent space $T_o(G/H)$, where $o = eH$ is the origin, can be canonically identified with $\mathfrak{m}$. Note that the isotropy subgroup $I_x(M, F)$ of $I(M, F)$ at a point $x \in M$ is compact ([7]), and M can be written as

$$M = I(M, F)/I_x(M, F).$$

Then $M = I(M, F)/I_x(M, F)$ is a reductive homogeneous manifold. Thus in the paper, we only consider reductive homogeneous Finsler spaces.

An n-dimensional Finsler space (M, F) is said to have almost isotropic $\mathbf{S}$-curvature if there exists a smooth function $c(x)$ on M and a closed 1-form η such that

$$\mathbf{S}(x, y) = (n + 1)(c(x)F(y) + \eta(y)), x \in M, y \in T_xM.$$

For the definition and fundamental properties of $\mathbf{S}$-curvature we refer to [4]. Now we prove

Theorem 3.1. *Let (M, F) be a generalized symmetric Finsler space. Then (M, F) has almost isotropic* **S***-curvature if and only if it has vanishing* **S***-curvature.*

Proof. Since (M, F) is generalized symmetric, for any point $x \in M$, there is an symmetry s_x with x as an isolated fix point. Suppose (M, F) has almost isotropic **S** curvature. Then

$$\mathbf{S}(x, y) = (n+1)(c(x)F(y) + \eta(y)),\ x \in M, y \in T_xM.$$

Since ds_x is a linear isometry, we have $\mathbf{S}(x, y) = \mathbf{S}(s_x(x), ds_x(y))$. Thus

$$(n+1)(c(x)F(y) + \eta(y)) = (n+1)(c(s_x(x))F(ds_x(y)) + \eta(ds_x(y)).$$

Notice that $c(x) = c(s_x(x))$ and $F(y) = F(ds_x(y))$. Thus we have $\eta(ds_x(y)) = \eta((y))$. On the other hand, we also have $\eta((ds_x - \mathrm{id})(y)) = 0$, where id is the identity transformation on T_xM. Now select a basis $y_1, y_2, \ldots y_n$ of T_xM. Then we have $\eta((ds_x - \mathrm{id})(y_i)) = 0$, $\forall i$. Since s_x is a symmetry with x as an isolated fixed point, ds_x is an isometry without fixed vector. Thus $(ds_x - \mathrm{id})$ is also a nonsingular transformation on T_xM and $(ds_x - \mathrm{id})(y_1), (ds_x - \mathrm{id})(y_2), \ldots, (ds_x - \mathrm{id})(y_n)$ is a basis of T_xM. This implies that $\eta = 0$. Hence the **S**-curvature of (M, F) vanishes at x. Since (M, F) is homogeneous, (M, F) has vanishing **S**-curvature everywhere.

The "only if" part is obvious. □

Now we consider generalized symmetric Randers metrics. For the definitions and fundamental properties of Randers metrics, see [4]. Note that a Randers metric can be written as $F(x, y) = \alpha(x, y) + \langle U, y\rangle_x$, $x \in M$, $y \in T_x(M)$, where α is a Riemannian metric, U is a smooth vector field whose length with respect to α is less than 1 everywhere and $\langle\cdot,\cdot\rangle_x$ is the inner product on the tangent space $T_x(M)$ induced by α.

Theorem 3.2. *A generalized symmetric Randers space must be Riemannian.*

We need the following Lemma.

Lemma 3.3. *Let (M, F) be a generalized symmetric Randers space with F defined by the Riemannian metric α and the vector field U. Then the regular s-structure $\{s_x\}$ of (M, F) is also a regular s-structure of the Riemannian manifold (M, α).*

Proof. Let s_x be a symmetry of (M, F) at x. For $p \in M$, set $q = s_x(p)$. Then for any $y \in T_p(M)$ we have

$$\begin{aligned} F(p, y) &= \alpha(p, y) + \langle U|_p, y\rangle_p = F(q, ds_x(y)) \\ &= \alpha(q, ds_x(y)) + \langle U|_q, ds_x(y)\rangle_q. \end{aligned}$$

Replacing y with $-y$ in the above equation, we get

$$\alpha(p, y) - \langle U|_p, y\rangle_q = \alpha(q, ds_x(y)) - \langle U|_q, ds_x(y)\rangle_q.$$

Therefore we have

$$\alpha(p, y) = \alpha(q, ds_x(y)).$$

Thus s_x is a symmetry with respect to the underlying Riemannian metric α. □

Proof of Theorem 3.2. Suppose $F(x,y)=\sqrt{\langle y,y\rangle_x}+\langle U,y\rangle_x$. Since (M,F) is a generalized symmetric space, it has a regular s-structure and for any $x\in M$ there exists a symmetry s_x with x as an isolated fixed point. Since (M,F) is a homogeneous, by Lemma 3.3, s_x is also a symmetry of (M,α). Thus we have

$$\begin{aligned} F(x,ds_x(y)) &= \sqrt{\langle ds_x(y),ds_x(y)\rangle_x}+\langle U|_x,ds_x(y)\rangle_x \\ &= \sqrt{\langle y,y\rangle_x}+\langle U|_x,ds_x(y)\rangle_x \\ &= F(x,y). \end{aligned}$$

Therefore $\langle U|_x,ds_x(y)\rangle_x=\langle U|_x,y\rangle_x,\ \forall y\in T_xM$. Since a regular s-structure induces a tensor field S of type (1,1) defined by $S_x=(ds_x)_x$ and it is an orthogonal transformation on T_xM without any nonzero fixed vectors, we have $\langle U|_x,(S-\mathrm{id})|_x(y)\rangle_x=0,\ \forall y\in T_xM$. Since $(S-\mathrm{id})|_x$ is an invertible linear transformation, we have $U|_x=0,\forall x\in M$. Hence F is Riemannian.

4 A rigidity Theorem

In this section we prove a rigidity theorem that a locally projective flat generalized symmetric Finsler space with almost isotropic **S**-curvature is either Riemannian or locally Minkowskian. We first recall some definitions.

Definition 4.1. Let F be a Finsler metric on an n-dimensional manifold M. F is said to be of scalar (flag) curvature if $\mathbf{K}(P,y)=\mathbf{K}(x,y)$ is a scalar function on $TM\setminus\{0\}$; It is said to have isotropic flag curvature if $\mathbf{K}(P,y)=\mathbf{K}(x)$ is a scalar function on M; It is said to have constant flag curvature if $\mathbf{K}(P,y)$ is a constant.

Clearly, a Finsler metric is of scalar curvature $\mathbf{K}(y)$ if and only if for any $y\in TM\setminus\{0\}$ the flag curvature $\mathbf{K}(P,y)$ is independent of the tangent planes P containing y. In particular, $R_y=0$ if and only if $\mathbf{K}(P,y)=0$, that is, a Finsler metric is of zero curvature if and only if it is R-flat.

Proposition 4.1 ([3])**.** *Let (M,F) be an n-dimensional Finsler manifold of scalar flag curvature with flag curvature $\mathbf{K}=K(x,y)$. Suppose that the $\mathbf{S}$-curvature is almost isotropic, i.e.,*

$$\mathbf{S}=(n+1)\{cF+\eta\},$$

where $c=c(x)$ is a scalar function and $\eta=\eta_i(x)y^i$ is a closed 1-form on M. Then there is a scalar function $\sigma=\sigma(x)$ on M such that the flag curvature has the form

$$\mathbf{K}=3\frac{c_{x^m}y^m}{F}+\sigma$$

Now we prove the following result:

Theorem 4.2. *Let (M,F) be an n-dimensional ($n\geq 3$) generalized symmetric Finsler space of scalar flag curvature with flag curvature $K=K(x,y)$. If the $\mathbf{S}$-curvature is almost isotropic, then $\mathbf{K}$ is a constant.*

Proof. By Theorem 3.1 we have $\mathbf{S}=0$. By Proposition 4.1, there is a scalar function $\sigma=\sigma(x)$ on M such that the flag curvature $\mathbf{K}=\sigma(x)$. Then by Schur's lemma (see [1]) $\mathbf{K}$ must be a constant. □

Let $F = F(x, y)$ be a Finsler metric on an open domain $U \subset \mathbb{R}^n$. Then the geodesics of F satisfy the following system of ordinary different equations

$$\frac{d^2x^i}{dt^2} + G^i(x, \frac{dx}{dt}) = 0.$$

A Finsler metric F is said to be projectively flat on U if all geodesics are straight lines. This is equivalent to saying that the geodesic coefficients G^i of F have the following form

$$G^i = p(x, y)y^i.$$

A Finsler metric F on a manifold M is said to be locally projectively flat if at any point, there is a local coordinate system (x^i) in which F is projectively flat ([4]).

Lemma 4.3 ([11])**.** *Any locally projectively flat Finsler metric is of scalar flag curvature.*

Proposition 4.4 ([4])**.** *Let $F = F(x, y)$ be a projectively flat Finsler metric on an open subset $U \subset R^n$. Suppose that F has almost isotropic S-curvature. Then F is determined as follows.*

1. *If $\mathbf{K} \neq -c(x)^2 + \frac{c_x m(x) y^m}{F(x,y)}$ at every point $x \in U$, then $F = \alpha + \beta$ is a Randers metric on U*
2. *If $\mathbf{K} \equiv -c(x)^2 + \frac{c_x m(x) y^m}{F(x,y)}$, then $c(x) = c$ is a constant, and either F is locally Minkowskian $(c = 0)$ or up to a scaling, F can be expressed as*

$$F = \begin{cases} \Theta(x, y) + \frac{<a,y>}{1+<a,x>}, & \text{if } c = \frac{1}{2}, \\ \Theta(x, -y) - \frac{<a,y>}{1+<a,x>}, & \text{if } c = -\frac{1}{2}. \end{cases} \tag{4.1}$$

We further obtain the following

Theorem 4.5. *Let (M, F) be a locally projective flat Finsler space. If F is generalized symmetric and has almost isotropic S-curvature, then F is either Riemannian or locally Minkowskian.*

Proof. Since F is locally projectively flat, for any point p, there is a local coordinate system (x^i) on an open neighborhood of p in which F is projective flat. Since M is homogeneous, we can write M as the form $M = G/H$, where G is a Lie group acting isometrically and transitively on M and H is the isotropy subgroup at a point of M. Thus we just need to consider the point $o = eH$ where e is the identity element in G. There is a local coordinate system (x^i) around $o = eH$, such that $(x_1, x_2, \ldots, x_n) = (x^i) : U \to R^n$ is a local coordinate on an open subset $U \subset M$ around $o = eH \in U$ and such that the spray coefficients are given by $G^i = Py^i$, where $P = \frac{F_{x^k} y^k}{2F}$. By Proposition 4.3 and Theorem 3.1, (M, F) has vanishing **S**-curvature on U, hence (M, F) has vanishing **S**-curvature everywhere. Then by Lemma 4.5 and Theorem 4.4, $\mathbf{K}$ is a constant.

If $\mathbf{K} \neq 0$ at T_oM, then by Theorem 3.2, F is a locally flat Riemannian metric on M. Now the Beltrami's theorem and Cartan's local classification theorem in Riemannian geometry state that every locally projectively flat Riemannian metric is, up to scaling, locally isometric to α_μ for some constant μ where

$$\alpha_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \quad y \in T_xB^n(r_n) \cong R^n,$$

where $r_\mu = \frac{1}{\sqrt{-\mu}}$ if $\mu < 0$, and $r_\mu = +\infty$ if $\mu \geq 0$.

If $\mathbf{K} = 0$, then $c(x) = c = 0$. In this case F is locally Minkowskian. □

5 Non-Berwald generalized symmetric Finsler spaces

Recall that a generalized symmetric Finsler space (M, F) must be a k-symmetric space for some $k \geq 0$ (see [16]). If $k = 2$, then (M, F) is a symmetric Finsler space and by Theorem2.2, (M, F) is a Berwald space.

It is natural to ask whether a generalized symmetric Finsler space (of order $k > 2$) is a Berwald space? The answer is negative. Next we will construct a counter example.

For generalized symmetric Riemannian space we have the following theorem

Theorem 5.1. *Let (M, g) be a connected generalized k-symmetric Riemannian space, with $k \geq 3$. Then M can be written as a coset space G/H which admits a G-invariant non-Riemannian Finsler metric such that (M, F) is a connected generalized symmetric Finsler space.*

Proof. Let (M, g) be a connected generalized k-symmetric Riemannian space, with $k \geq 3$. Then the full group of isometries of (M, g), denoted as G, is a Lie transformation group on M. Note that G must be transitive on M, hence M can be written as a coset space $M = G/H$, where H is the isotropic subgroup of G. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the corresponding reductive decomposition of the coset space G/H. We assert that H cannot be transitive on the unit sphere of $\mathfrak{m}$ defined by the Riemannian metric g. In fact, otherwise the Riemannian manifold (M, g) must be isotropic (see [9]) and (M, g) must be a Riemannian globally symmetric space of rank 1. This implies that $k = 2$, which is a contradiction. Now by the main theorem of §4.2 of [5], there exists a G-invariant non-Riemannian Finsler metric F on M. Then it is easily seen that (M, F) is a generalized symmetric Finsler space. □

To construct a counter example, we start with the definition of generalized Heisenberg groups of H-type.

Definition 5.1. Let V and Z be two real vector spaces of dimension n and m, $m \geq 1$, both equipped with an inner product which we shall denote by the same symbol $\langle , \rangle$. Let $j : Z \to \mathrm{End}(V)$ be a linear map such that

- $|j(a)x| = |x||a|, \ x \in V, \ a \in Z,$
- $j(a)^2 = -|a|^2 I, \ a \in Z.$

we define the Lie algebra $\mathfrak{n}$ as the direct sum of V and Z together with the brackets defined by

- $[a + x, b + y] = [x, y] \in Z,$

- $\langle [x,y],a\rangle = \langle j(a)x,y\rangle$.

where $a,b \in Z$ and $x,y \in V$. Then $\mathfrak{n}$ is said to be a Lie algebra of H-type. It is a 2-step nilpotent Lie algebra with center Z. The simply connected, connected Lie group N whose Lie algebra is $\mathfrak{n}$ is called a Lie group of H-type or a generalized Heisenberg group.

For more details we refer to [17] and [2]

Example 5.2. Let $(N,\langle,\rangle)$ be a six dimensional group of H-type, with an orthonormal basis x_1,x_2,x_3,x_4,a_1,a_2, and the only nonzero Lie brackets

$$\begin{cases} [x_1,x_2]=a_1, \quad [x_1,x_3]=a_2, \\ [x_2,x_4]=-a_2, \quad [x_3,x_4]=a_1. \end{cases}$$

Set

$$\begin{cases} U_1 = x_1+ix_4, \\ U_2 = x_2+ix_3, \\ U_3 = -a_1+ia_2, \end{cases}$$

and define a linear map $\mathcal{S}$ of $\mathfrak{n}$ by

$$\mathcal{S}(U_j) = e^{\frac{2\pi i}{3}}U_j, \ j=1,2,3.$$

Then $\mathcal{S}$ is an isometric automorphism of the Lie algebra $(\mathfrak{n},\langle,\rangle)$ and $\mathcal{S}^3=\mathrm{id}$. Hence N is a 3-symmetric space. Now consider the linear map $\mathcal{S}'$ defined by

$$\mathcal{S}'(U_1)=iU_1, \ S'(U_2)=iU_2, \ \mathcal{S}'(U_3)=-U_3.$$

It is easily seen that $\mathcal{S}'$ is also an isometric automorphism, hence $(\mathfrak{n},\langle,\rangle)$ is also a 4-symmetric space. Thus the six-dimensional group of type H is both 3- and 4-symmetric. By Theorem 5.1, $(N,\langle,\rangle)$ admits a left invariant non-Riemannian Finsler metric. It is easily seen that N is a connected and simply connected indecomposable two-step nilpotent Lie group. Then by Proposition 6.7 of [5], any left invariant non-Riemannian Finsler metric F on N must be non-Berwald.

6 Generalized symmetric spaces with weakly symmetric structure

In the literature, there is still another notion generalizing the notion of Riemannian symmetric spaces, which is the weak symmetry. A Riemannian manifold (M,g) is called weakly symmetric if for any $m\in M$ and $u\in T_m(M)$ there exists an isometry σ of (M,g) such that $\sigma(m)=m$ and $d\sigma(u)=-u$. The set $\{(\sigma_x,u)|x\in M, u\in T_xM\}$ on a connected Riemannian manifold (M,g) is called a weak symmetric structure if for any point x in M and $u\in T_xM$ there exists an isometry σ_x of (M,g) such that $\sigma_x(x)=x$ and $d\sigma_x(u)=-u$.

Recently the second author defines the concept of k-fold symmetric spaces: Let (M,Q) be an n-dimensional connected Riemannian manifold and $1\le k\le n$. Then

(M, Q) is called k-fold symmetric if given any tangent vector $\xi_1, \xi_2, \ldots, \xi_k$ at a point $x \in M$, there exists an isometry σ such that $\sigma(x) = x$ and $d\sigma(\xi_i) = -\xi_i, i = 1, 2, \ldots, k$. Obviously, if $k = 1$, then a k-fold symmetric Riemannian manifold is weakly symmetric.

For 2-fold symmetric Riemannian manifolds we have the following theorem:

Theorem 6.1 ([6]). *A connected simply connected* 2*-fold symmetric Riemannian manifold must be globally symmetric.*

It is an interesting question to ask whether a connected simply connected Riemannian manifold admitting a regular s-structure (with order $k > 2$) as well as a weak symmetric structure is a k-fold symmetric Riemannian manifold? More precisely, whether a connected simply connected Riemannian manifold admitting a regular s-structure (with order $k > 2$) as well as a weak symmetric structure is globally symmetric? The answer is negative. We now construct an example.

Example 6.1. The five-dimensional Heisenberg group N can be realized as a matrix group

$$N = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ u & v & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let (N, g) be the space $\mathbb{R}^5(x, y, z, u, v)$ endowed with the Riemannian metric

$$g = dx^2 + dy^2 + du^2 + dv^2 + \lambda^2(xdu - ydv + dz)^2, \qquad \lambda > 0.$$

The typical symmetry of order 4 at the point $(0, 0, 0, 0, 0)$ is the transformation $\theta : N \to N$, $\theta(x') = -y$, $\theta(y') = x$, $\theta(z') = -z$, $\theta(u') = -v$, $\theta(v') = u$. Thus N is a 4-symmetric space with Riemannian metric g defined above.

Let $\mathfrak{n}$ be the Lie algebra of N, and fix a basis of $\mathfrak{n}$ as the following:

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the Lie brackets are

$$[x_i, y_j] = \delta_{ij} z, [x_i, x_j] = [y_i, y_j] = 0, [x_i, z] = [y_i, z] = 0, i, j = 1, 2, \ldots, n.$$

Thus $\mathfrak{n}$ is a 2-step nilpotent Lie algebra. Now let $\mathfrak{g} = \mathfrak{u}(2) + \mathfrak{n}$ (direct sum of subspaces) and define the brackets as follows. The brackets among the elements in $\mathfrak{u}(2)$ are the usual operations. For $A \in \mathfrak{u}(2)$ we define $[A, z] = 0$, and for the element

$$w = \sum_{i=1}^{n} (a_i x_i + b_i y_i), \quad a_i . b_i \in R,$$

we set

$$z_i = a_i + \sqrt{-1}b_i,\ \ i = 1, 2.$$

Let

$$(z'_1, z'_2) = (z_1, z_2)A$$

and write $z'_i = a'_i + \sqrt{-1}b'_i,\ \ a'_i, b'_i \in R,\ \ i = 1, 2$. Then we define

$$[A, w] = w' = \sum_{i=1}^{n}(a'_i x_i + b'_i y_i).$$

It is easy to check that the Jacobian identities hold among these brackets. Therefore these brackets together with the brackets of $\mathfrak{n}$ define a Lie algebra structure on $\mathfrak{g}$. By the definition, we have $[\mathfrak{u}(2), \mathfrak{n}] \subset \mathfrak{n}$. Now we define an endomorphism τ of $\mathfrak{g}$ by

$$\tau(A) = \overline{A},\ \ \tau(x_i) = -x_i,\ \ \tau(y_i) = y_i,\ \ \tau(z) = -z,\ \ i = 1, 2.$$

where $A \in \mathfrak{u}(2)$ and $\overline{A}$ is the complex conjugate matrix of A. It is easy to check that τ is a real automorphism of the real Lie algebra $\mathfrak{g}$ and $\tau^2 = \mathrm{id}$. Now $(\mathfrak{g}, \mathfrak{u}(2))$ is a weakly symmetric Lie algebra with respect to $\{\mathrm{id}, \tau\}$ (for details, see [5], pp. 147). Thus $\mathfrak{n} \cong \mathfrak{g}/\mathfrak{u}(2)$ is a weakly symmetric algebra and there exists a Riemannian metric Q on N such that (N, Q) is a weakly symmetric space.

Using this example we can construct infinitely many examples which are both k-symmetric ($k > 2$) and weakly symmetric, but not globally symmetric.

Let $\mathfrak{n}$ be a $(4n + 1)$-dimensional Heisenberg Lie algebra with a basis

$$x_1, x_2, \dots, x_{2n-1}, x_{2n}, y_1, y_2, \dots, y_{2n-1}, y_{2n}, z.$$

Define an automorphism $\overline{\theta}$ on $\mathfrak{n}$ by

$$\overline{\theta}(x_{2k-1}) = x_{2k}, \overline{\theta}(x_{2k}) = -x_{2k-1}, \overline{\theta}(y_{2k-1}) = y_{2k}, \overline{\theta}(y_{2k}) = -y_{2k-1}\ \ k = 1, 2 \dots, n,$$

and $\overline{\theta}(z) = -z$. Then $\overline{\theta}$ induces an automorphism of N such that $\theta^4 = \mathrm{id}$. Since $\overline{\theta}$ has no fixed vector, identifying N with $R(u_1, \dots, u_{2n}, v_1, \dots, v_{2n}, a)$ gives the Riemannian metric

$$g = \sum_{i=1}^{2n} du_i^2 + \sum_{i=1}^{2n} dv_i^2 + \lambda^2 (\sum_{i=1}^{n} u_{2i-1}dv_{2i-1} - u_{2i}dv_{2i} + da)^2$$

on N, where $\lambda > 0$. It is easy to check that $g(X, Y) = g(\overline{\theta}(X), \overline{\theta}(Y))$. Thus (N, g) is a 4-symmetric space. Let $\mathfrak{g} = \mathfrak{u}(2n) + \mathfrak{n}$. Then using a similar method as above we can prove that $(\mathfrak{g}, \mathfrak{u}(2n))$ is a weakly symmetric Lie algebra with respect to $\{\mathrm{id}, \tau\}$, where τ is the endomorphism of $\mathfrak{g}$ defined by

$$\tau(A) = \overline{A},\ \ \tau(x_i) = -x_i,\ \ \tau(y_i) = y_i,\ \ \tau(z) = -z,\ \ i = 1, 2, \dots, 2n.$$

Then $\mathfrak{n} \cong \mathfrak{g}/\mathfrak{u}(2n)$ is a weakly symmetric Lie algebra. Hence there exists a Riemannian metric Q on N such that (N, Q) is a weakly symmetric space. This gives an infinite series of examples which are both 4-symmetric and weakly symmetric but not globally symmetric.

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