Ricci and Riemann solitons

Iulia Elena Hirică, Constantin Udrişte

Abstract. Geometric flows (Ricci flow, Riemann flow, mean curvature flow etc), as a class of important geometric PDEs, have been highlighted in many fields of theoretical research and practical applications. They are very useful tool in understanding the topology of arbitrary Riemannian manifolds and have had a profound influence on modern geometric analysis.

The aim of this paper is quadruple: (i) to introduce in usage the extended name "moving scaled tensorial image Ricci soliton" and to define the "moving graph Ricci soliton" (like those in the context of PDEs), (ii) to introduce in usage the extended name "moving scaled tensorial image Riemann soliton" and to define the "moving graph Riemann soliton" (like those in the context of PDEs), (iii) to characterize the gradient Sasaki-Riemann solitons with harmonic potential function; (iv) to introduce the posynomial Ricci or Riemann flows.

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1 Introduction

During the last years, geometric evolution equations have been used to study geometric questions like isoperimetric inequalities, the Schönfliess conjecture, the Poincaré conjecture, Thurston's geometrization conjecture, the 1/4-pinching theorem, or Yau's uniformization conjecture. Particularly, the geometric flows is enjoying rapid growth providing new techniques of investigations in different directions of study in differential geometry, analysis and theoretical physics.

In differential geometry, the Ricci flow is an intrinsic geometric flow. It is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing out irregularities in the metric. The most important recent application is Perelman's proof of Thurston's conjecture [20].

Lately, increasingly more papers discuss Ricci solitons, but some authors hollowed this notion of content, losing the kinematic character of the soliton. Moreover, the

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concept of Ricci soliton seems to bypass the concept of soliton met in significant PDE (Korteweg-de Vries PDE, Wave PDE, reaction-diffusion PDE etc). We report a point of view that overcomes these impediments.

Firstly, let us explain our intention by considering: (i) the extended name "moving scaled tensorial image Ricci soliton" which reflects the kinematic sense of "Ricci soliton"; (ii) the new name "moving graph Ricci soliton" which is used to describe "another type of Ricci soliton" similar to solitons in the context of PDEs; (iii) the extended name "moving scaled tensorial image Riemann soliton" which is proper for "Riemann soliton"; (iv) the new name "moving graph Riemann soliton" which is used to introduce "another type of Riemann soliton", like solitons in PDEs.

Secondly, we study the Sasaki-Riemann flow and Sasaki-Riemann soliton.

2 Ricci flows. Ricci solitons

Let $(M, g(x))$ be a Riemannian manifold. The Riemannian metric $g(x) = (g_{ij}(x)))$ and its inverse $g^{-1}(x) = (g^{ij}(x)))$ determine: (i) the Christoffel symbols of the second kind

$$\Gamma^m{}_{ij} = \frac{1}{2} g^{mk} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right);$$

(ii) the Riemann tensor field $R(g)$ of components

$$R_{ijkl} = \frac{1}{2} \left(\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} \right)$$

$$+ g_{mn} \left(\Gamma^m{}_{jk} \Gamma^n{}_{il} - \Gamma^m{}_{jl} \Gamma^n{}_{ik} \right);$$

(iii) the Ricci tensor field $S(g)$ of components

$$S_{ij} = \frac{\partial \Gamma^l{}_{ij}}{\partial x^l} - \Gamma^m{}_{il} \Gamma^l{}_{jm} - \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial x^i} \left(\ln \sqrt{\det(g)} \right) \right).$$

Definition 2.1. A solution $g(x, t)$ of the nonlinear evolution PDE

$$\frac{\partial g}{\partial t}(x, t) = -2S(g)(x, t), \; t \in [0, T] \tag{2.1}$$

is called Ricci flow.

In harmonic local coordinates around a point p, the Ricci flow is a heat flow evolution since the Ricci tensor is reduced to $S_{ij}(p) = -\frac{1}{2}\Delta(g_{ij})(p)$. Similarly, we can introduce a complex evolutive PDE which mimics a Schrodinger equation, for the Riemannian metric.

Definition 2.2. (Moving scaled tensorial image Ricci soliton) Let $\varphi_t(x) : M \to M, t \in [0, T], \varphi_0(x) = id$, be a family of diffeomorphisms and $\sigma(t)$, $\sigma(0) = 1$, be a positive scale. Let $g(x, 0)$ be a fixed Riemannian metric. The Ricci flow $g(x, t) = \sigma(t)\varphi_t^*(x)g(x, 0)$ is called moving scaled tensorial image Ricci soliton. The initial Riemannian metric $g(x, 0)$ is called the profile of the soliton.

Theorem 2.1. *Let us suppose that the family of diffeomorphisms $\varphi_t(x)$ is generated by the vector field $X(x)$. The evolutive metric $g(x,t) = \sigma(t)\varphi_t^*(x)g(x,0)$ is a Ricci soliton iff the profile metric $g(x,0) = g(x)$ is a solution of the nonlinear stationary PDE*

$$S(g) + \frac{1}{2}\mathcal{L}_X\, g + \lambda g = 0.$$

The profile metrics of Ricci solitons are natural generalizations of Einstein metrics. A Ricci soliton is called: (i) shrinking, when the profile is associated to $\lambda < 0$; (ii) steady, when $\lambda = 0$ and (iii) expanding, when $\lambda > 0$.

In case $X = \nabla f$, the PDE whose unknown is the profile $g(x)$ can also be written as $S(g) + Hess_f + \lambda g = 0$ and the profile g determines a *gradient Ricci soliton*. The function f is called *potential function* of the Ricci soliton.

Remark 2.3. Some authors define the Ricci soliton only by the above PDEs. One should mention that the kinematic character is missing, in this way.

Remark 2.4. Perelman [20] proved that a Ricci soliton on a compact n-dimensional manifold is a gradient Ricci soliton.

On a compact manifold M^n, a gradient steady or expanding Ricci soliton is generated by an Einstein metric profile ([10], [16]).

In dimension three or less all compact shrinking solitons are produced by positive constant curvature profiles ([11],[16],[20]).

Definition 2.5. (Moving graph Ricci soliton) A Ricci flow $g(x,t) = \phi(f(x) - \omega t)$, $g_{ij}(x,t) = \phi_{ij}(f(x) - \omega t)$, is called moving graph Ricci soliton. The Riemannian metric $\phi(f(x)) = (\phi_{ij}(f(x)))$ is called the profile of the soliton.

Having in mind the soliton solutions in PDEs, the function f should be geodesic affine, i.e., the co-vector field df must be parallel.

To find a moving graph Ricci soliton we follow the following steps: (i) we give the function f; (ii) we compute $df = (f_i)$ and

$$\frac{\partial g_{ij}}{\partial t} = -\phi'_{ij}\ \omega,\ \frac{\partial g_{ij}}{\partial x^l} = \phi'_{ij} f_l,\ \Gamma^m{}_{ij} = \frac{1}{2}\phi^{mk}\left(\phi'_{ki} f_j + \phi'_{kj} f_i - \phi'_{ij} f_k\right); \tag{2.2}$$

(iii) replacing in the PDE (2.1), it follows the profile ODEs as a second order Riccati ODEs system in the unknown profile matrix function $\phi(f(x)) = (\phi_{ij}(f(x)))$.

3 Riemann flows. Riemann solitons

Problem ([21], [22]) *Extend in a natural way the concept of Ricci flow to a nonlinear PDE which involve the Riemann curvature tensor and interpret the metric $g(x,t)$ as solution of previous PDE. The notion of Ricci soliton is replaced by Riemann soliton as a kinematic solution of Riemann flow, whose profile generalizes the space of constant sectional curvature.*

Let $G_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}$ and R_{ijkl} be the components of the curvature tensor field.

Definition 3.1. A solution $g(x,t)$ of the nonlinear evolution PDE

$$\frac{\partial G_{ijkl}}{\partial t}(x,t) = -2R_{ijkl}(g)(x,t),\ t \in [0,T] \tag{3.1}$$

is called Riemann flow.

Some results in the Riemann flow resemble the case of Ricci flow (for details, see [22]).

Theorem 3.1. (Short time existence and uniqueness) *Let (M,g) be a Riemannian n-dimensional manifold. Then there exists $\epsilon > 0$ such that the initial value problem*

$$\frac{\partial G_{ijkl}}{\partial t}(x,t) = -2R_{ijkl}(g)(x,t),\ g(x,0) = g_0(x)$$

has unique solution $g(x,t)$ on $M \times [0,\epsilon]$.

Definition 3.2. (Moving scaled tensorial image Riemann soliton) Let $\varphi_t(x) : M \to M, t \in [0,T], \varphi_0(x) = id$, be a family of diffeomorphisms and $\sigma(t)$, $\sigma(0) = 1$, be a positive scale. Let $g(x,0)$ be a fixed Riemannian metric. The Riemann flow $g(x,t) = \sigma(t)\varphi_t^*(x)g(x,0)$ is called moving scaled tensorial image Riemann soliton. The Riemannian metric $g(x,0)$ is called the profile of the Riemann soliton.

Theorem 3.2. *Suppose the family of diffeomorphisms $\varphi_t(x)$ is generated by the vector field $X(x)$. The evolutive metric $g(x,t) = \sigma(t)\varphi_t^*(x)g(x,0)$ is a Riemann soliton iff the profile metric $g(x,0) = g(x)$ is a solution of the nonlinear stationary PDE*

$$R(g) + \lambda G + \frac{1}{2} g \wedge \mathcal{L}_X g = 0,$$

where λ is a constant and $\wedge$ is the Kulkarni-Nomizu product.

Riemann profile metrics g generalize the class of metrics of constant sectional curvature. A Riemann soliton is called shrinking when the profile corresponds to $\lambda < 0$, steady when $\lambda = 0$ and expanding when $\lambda > 0$.

If X is a gradient, i.e., $X = \nabla f$, then we get the notion of gradient Riemann solitons, whose profile $g(x)$ satisfies the PDE

$$R(g) + \lambda G + g \wedge \nabla^2 f = 0,$$

for some smooth potential function f on M.

Remark 3.3. Some authors define the Riemann soliton only by the above PDEs. In this way, the kinematic character is missing.

Proposition 3.3. *A Riemann soliton on a compact Riemannian manifold is a gradient Riemann soliton.*

Definition 3.4. (Moving graph Riemann soliton) A Riemann flow $g(x,t) = \phi(f(x) - \omega t)$, $g_{ij}(x,t) = \phi_{ij}(f(x) - \omega t)$, is called moving graph Riemann soliton. The Riemannian metric $\phi(f(x)) = (\phi_{ij}(f(x)))$ is called the profile of the soliton.

To determine a moving graph Riemann soliton, we fix the function f and we replace the expressions (2.2) in the PDE (3.1). It follows the profile ODEs as a second order Riccati ODEs system in the unknown profile matrix function $\phi(f(x)) = (\phi_{ij}(f(x)))$.

4 The Sasakian case

The Sasakian geometry can be viewed as an odd-dimensional counterpart of the Kähler geometry. Indeed, just as Kähler geometry is the natural intersection of complex, symplectic, and Riemannian geometry, so Sasakian geometry is the natural intersection of CR, contact, and Riemannian geometry.

We consider that the underlying manifold $(M^{2m+1}, g, \Phi, \xi, \eta)$ is a Sasakian manifold, where ξ is a unit Killing vector field (called the Reeb vector field), η is the dual one-form of ξ, Φ is a (1, 1)-tensor defined by $\Phi(Y) = \nabla_Y \xi$ and $R(Y, \xi)Z = -g(Y, Z)\xi + g(Z, \xi)Y$, for any vector fields $Y, Z \in TM$.

Remark 4.1. A compact Riemannian manifold (M, g) is Sasakian if and only if its metric cone $(C(M) = \mathbf{R}_+ \times M, \bar{g} = dr^2 + r^2 g)$ is Kähler.

This condition is equivalent with one of the followings statements:

1) There exists a unit Killing vector field ξ and the tensor field $\Phi(X) = \nabla_X \xi$ satisfies

$$(\nabla_X \phi)(Y) = g(\xi, Y)X - g(X, Y)\xi,$$

$\forall X, Y$ vector fields on M.

2) There exists a unit Killing vector field ξ so that the Riemann curvature satisfies

$$R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi,$$

$\forall X, Y$ vector fields on M.

A Sasakian manifold is said to be η-Sasaki-Einstein if there are two constants λ and ν such that $S(g) = \lambda g + \nu \eta \otimes \eta$.

η-Sasaki-Einstein condition is equivalent to the transverse Kähler metric g^T (associated to the Reeb foliation $\mathcal{F}_\xi$) being Einstein, where

$$g^T(X, Y) = \frac{1}{2} d\eta(X, \Phi(Y)).$$

Remark 4.2. If a Sasakian manifold satisfies the gradient Sasaki-Ricci soliton PDE, then f is a constant function (i.e. trivial gradient Ricci soliton) and (M, g) is an Einstein manifold ([12]).

Since all compact Ricci solitons are gradient ones, the previous result implies that there is no compact non-Einstein Ricci soliton in Sasakian manifolds.

On the other hand, there exist non-gradient expanding Ricci solitons on noncompact Sasakian manifolds which are not Einstein. They are left invariant metrics on some solvable Lie groups and appear as type III singularities models of Ricci flow [1], [17]. The one with the lowest dimension is the three dimensional Heisenberg group with a left invariant metric.

On a Sasakian manifold $(M^{2n+1}, g, \Phi, \xi, \eta)$, the concept of *Riemann flow* becomes the *Sasaki-Riemann flow*, i.e., a solution $g(x, t)$ of the nonlinear evolution PDE

$$\frac{\partial G_{ijkl}}{\partial t}(x, t) = -2R_{ijkl}(g(x, t)).$$

The evolutive metric $g(x,t)$ is a Sasaki-Riemann soliton iff the profile metric $g(x,0) = g(x)$ is a solution of the nonlinear stationary PDE

$$R(g) + \lambda G + \frac{1}{2} g \wedge \mathcal{L}_X g = 0,$$

where λ is a constant.

If X is a gradient ($X = \nabla f$), then we get the notion of gradient Sasaki-Riemann solitons, whose profile metric $g(x)$ satisfies the nonlinear stationary PDE

$$R(g) + \lambda G + g \wedge \nabla^2 f = 0,$$

for some smooth potential function f on M.

5 The Sasaki-Riemann soliton

Theorem 5.1. *Let $(M^{2n+1}, g, \Phi, \xi, \eta)$ be a Sasakian manifold, whose metric g satisfies the gradient Sasaki-Riemann soliton PDE*

$$R(g) + \lambda G + g \wedge \nabla^2 f = 0.$$

Suppose that the potential function f is harmonic, then $(M^{2n+1}, g, \Phi, \xi, \eta)$ is a Sasaki space form.

Moreover the Kähler cone $(C(M), \bar{g})$ is Ricci-flat and also the transverse Kähler structure g^T associated to the Reeb foliation $\mathcal{F}_\xi$ is Kähler-Einstein.

Proof. Let $\mathcal{D} \subset TM$ be the distribution defined by $\eta(Y) = g(Y,\xi) = 0$. Then $\mathcal{D}$ is nowhere integrable as η is a contact 1-form.

For any $Y \in \mathcal{D}$ and $Z \in TM$, we have

$$R(Y,\xi,Z,Y) = -g(Y,Z)g(\xi,Y) + g(Z,\xi)g(Y,Y) = g(Z,\xi)|Y|^2.$$

Therefore $S(\xi,\xi) = 2n$. Since ξ is a Killing vector field, from the Sasaki-Riemann soliton equation one has

$$\mathcal{L}_\xi(\mathcal{L}_X g)(Y,Z) = R(\xi,Y,X,Z) + g(\nabla_Y \nabla_\xi X, Z) + R(\xi,Z,X,Y) +$$

$$+ g(\nabla_Z \nabla_\xi X, Y) + g(\nabla_Y \xi, \nabla_Z X) + g(\nabla_Y X, \nabla_Z \xi) = 0.$$

The gradient Sasaki-Riemann soliton equation leads to

$$\frac{2}{2n-1}\{[(2n+1)\lambda + \alpha]g - S\} = \mathcal{L}_X g$$

and therefore

$$g(\nabla_\xi X, \xi) = \frac{-1}{2n-1}[(2n+1)\lambda + \alpha + 2n],$$

where $\alpha = \nabla_i X^i = div(X)$.

If $Y \in \mathcal{D}$ and $Z = \xi$, then

$$R(X,\xi,\xi,Y) + \nabla_Y g(\nabla_\xi X, \xi) + g(\nabla_\xi \nabla_\xi X, Y) = 0.$$

Hence
$$-\frac{1}{2n-1}Y(\alpha)+g(X,Y)+g(\nabla_\xi\nabla_\xi X,Y)=0.$$
The gradient Sasaki-Riemann soliton equation leads to
$$Hess_f(\xi,W)=\frac{-1}{2n-1}[(2n+1)\lambda+\alpha+2n]g(\xi,W).$$
From the last two formulas, we get
$$\nabla_\xi\nabla f=\frac{-1}{2n-1}[(2n+1)\lambda+\alpha+2n]\xi.$$
Thus we have
$$g(\nabla f,Y)-\frac{1}{2n-1}Y(\triangle(f))=0,\forall Y\in\mathcal{D}.$$

If f is harmonic and M is compact, then f is constant and $R=-\lambda G$. If M is noncompact, then ∇f is parallel to ξ; hence $\nabla f=0$ as $\mathcal{D}$ is nowhere integrable, i.e., f is a constant function. Consequently, the profile of the gradient Sasaki-Riemann soliton satisfies the equation $R=-\lambda G$.

□

Theorem 5.2. *Let $(M^{2n+1},g,\Phi,\xi,\eta)$, be a Sasakian manifold, whose metric g satisfies the Sasaki-Riemann soliton PDE*
$$R+\lambda G+\frac{1}{2}g\wedge\mathcal{L}_X g=0.$$
If X is pointwise collinear to ξ, then the manifold is a Sasaki space form.

Proof. Let $X=u\xi$, where u is a smooth function in M. The Sasaki-Riemann soliton equation becomes
$$(2n-1)Z(u)\eta(Y)+(2n-1)Y(u)\eta(Z)+2S(Z,Y)+2[(2n+1)\lambda+\alpha]g(Z,Y)=0,$$
where $\alpha=\nabla_i X^i$. If one considers $Y=Z=\xi$ the equation leads to
$$(2n-1)\xi(u)=-2n-(2n+1)\lambda-\alpha.$$
From this two formulas, we obtain
$$Y(u)=-\frac{(2n+1)\lambda+\alpha+2n}{2n-1}\ \eta(Y).$$
This is equivalent with
$$du=-\frac{(2n+1)\lambda+\alpha+2n}{2n-1}\ \eta$$
and, using $d^2u=0$ and the fact that $d\eta$ is nowhere vanishing, we get $du=0$. Hence $u=$ constant.

As ξ is Killing, the Sasaki-Riemann soliton is based on a profile satisfying the equation $R=-\lambda G$. Consequently, the manifold is a Sasaki space form. □

Open problem. Classification of Sasaki-Riemann gradient soliton for arbitrary potential function.

6 Posynomial Ricci or Riemann flows

Let $x^1, ..., x^n$ denote n real positive variables, and $x = (x^1, ..., x^n)$ the associated vector. A real valued function of the form

$$f(x) = c\ (x^1)^{a_1}...(x^n)^{a_n},$$

where $c > 0$ and $a_i \in \mathbb{R}$, is called a *monomial function.* A sum of one or more monomials, i.e., a function of the form

$$F(x) = \sum_{k=1}^{K} c_k\ (x^1)^{a_{1k}}...(x^n)^{a_{nk}},$$

where $c_k > 0$, is called a *posynomial function.* This kind of functions are basically in *geometric programming* [4].

Our belief is that what is good programming is good also in differential geometry. This is why we impose special forms for Ricci and Riemann flows:

(i) the monomial form

$$g_{kl}(x,t) = c_{kl}(t)\ (x^1)^{a_{1(kl)}}...(x^n)^{a_{n(kl)}},\ \text{ no sum;}$$

(ii) the posynomial form

$$g_{ij}(x,t) = c_{kij}(t)\ (x^1)^{a_{1k}}...(x^n)^{a_{nk}},\ \text{sum after k.}$$

7 Examples

Example 7.1. Let us find some geometrical entities produced by a particular monomial Riemannian metric.

On $(1,\infty) \times [0,\infty) \times [0,\infty)$, we consider a Riemannian monomial metric g whose non-zero components are given by $g_{11} = c_1(t)$, $g_{22} = xc_2(t), g_{33} = yc_3(t)$, $g_{12} = c_4(t)$, with $c_1(t) > 0$, $c_2(t) > 0$, $c_3(t) > 0$, $c_1(t)c_2(t) - c_4(t)^2 \geq 0$.

The non-zero components of Christoffel symbols are

$$\Gamma^1_{12} = -\frac{0.5c_2(t)c_4(t)}{xc_1(t)c_2(t) - c_4(t)^2},\ \Gamma^1_{22} = -\frac{0.5xc_2(t)^2}{xc_1(t)c_2(t) - c_4(t)^2},\ \Gamma^1_{33} = \frac{0.5c_3(t)c_4(t)}{xc_1(t)c_2(t) - c_4(t)^2},$$

$$\Gamma^2_{12} = \frac{0.5c_1(t)c_2(t)}{xc_1(t)c_2(t) - c_4(t)^2},\ \Gamma^2_{22} = \frac{0.5c_2(t)c_4(t)}{xc_1(t)c_2(t) - c_4(t)^2},\ \Gamma^2_{33} = -\frac{0.5c_1(t)c_3(t)}{xc_1(t)c_2(t) - c_4(t)^2}.$$

It follows the equations of geodesics

$$\frac{d^2z}{ds^2} + \frac{1}{y}\frac{dy}{ds}\frac{dz}{ds} = 0,$$

$$\frac{d^2x}{ds^2} - \frac{c_2(t)c_4(t)}{xc_1(t)c_2(t) - c_4(t)^2}\frac{dx}{ds}\frac{dy}{ds}$$

$$-\frac{1}{2}\frac{xc_2(t)^2}{xc_1(t)c_2(t) - c_4(t)^2}\left(\frac{dy}{ds}\right)^2 + \frac{1}{2}\frac{c_3(t)c_4(t)}{xc_1(t)c_2(t) - c_4(t)^2}\left(\frac{dz}{ds}\right)^2 = 0,$$

$$\frac{d^2y}{ds^2}+\frac{c_1(t)c_2(t)}{xc_1(t)c_2(t)-c_4(t)^2}\frac{dx}{ds}\frac{dy}{ds}$$
$$+\frac{1}{2}\frac{c_2(t)c_4(t)}{xc_1(t)c_2(t)-c_4(t)^2}\left(\frac{dy}{ds}\right)^2-\frac{1}{2}\frac{c_1(t)c_3(t)}{xc_1(t)c_2(t)-c_4(t)^2}\left(\frac{dz}{ds}\right)^2=0.$$

The straight line $x(t)=as+b, y(s)=c, z(s)=d$ is a geodesic.

The foregoing metric is not a Ricci flow. Indeed, the non-zero components of the Ricci tensor,

$$S_{11}=-\frac{1}{4}\frac{c_1(t)^2c_2(t)^2}{(xc_1(t)c_2(t)-c_4(t)^2)^2},$$
$$S_{12}=-S_{33}=-\frac{1}{4}\frac{c_1(t)c_2(t)(xc_1(t)c_2(t)+yc_2(t)c_4(t)-c_4(t)^2)}{y(xc_1(t)c_2(t)-c_4(t)^2)^2},$$
$$S_{22}=-\frac{1}{4}\frac{xy^2c_1(t)c_2(t)^3+xyc_1(t)c_2(t)^2c_4(t)+x^2c_1(t)^2c_2(t)^2}{(xc_1(t)c_2(t)-c_4(t)^2)^2y^2}$$
$$-\frac{1}{4}\frac{-yc_2(t)c_4(t)^3-2xc_1(t)c_2(t)c_4(t)^2+c_4(t)^4}{(xc_1(t)c_2(t)-c_4(t)^2)^2y^2},$$

do not lead to PDEs (2.1) independent with respect to x, y.

Example 7.2. Let us give a Ricci flat monomial pseudo-Riemannian metric. On $\mathbb{R}^3$, we consider the family of monomial pseudo-Riemannian metrics g with non-zero components $g_{11}=xc_1(t)$, $g_{33}=zc_2(t)$, $g_{12}=2c_3(t)$. These produce a Ricci flat manifold, i.e., $S_{ij}(g)=0$.

Example 7.3. Let us give a Ricci flow using a fundamental non-degenerate tensor g whose non-zero components are $g_{11}=f(x,t)$, $g_{22}=k_2f(x,t)$, $g_{12}=k_1f(x,t)$, $g_{33}=h(z,t)$. Since the non-zero components of Ricci tensor $S(g)$ are

$$S_{11}=-\frac{1}{2}\frac{(f_{xx}-f_x^2)k_2}{f^2(k_1^2-k_2)},\ S_{22}=-\frac{1}{2}\frac{(f_{xx}-f_x^2)k_2^2}{f^2(k_1^2-k_2)},\ S_{12}=-\frac{1}{2}\frac{(f_{xx}-f_x^2)k_1k_2}{f^2(k_1^2-k_2)},$$

the PDEs of the Ricci flow are reduced to $f_t=\frac{(f_{xx}-f_x^2)k_2}{f^2(k_1^2-k_2)}$, $h_t=0$. The Ricci flow is generated by five families of functions (if and only if $k_1^2=2k_2$, $k>0$)

$$f(x,t)=\frac{2c_2(t+c_1)e^{\pm kx}}{(1-c_2e^{\pm kx})^2};f(x,t)=\frac{2(t+c_1)}{(x+c_2)^2};f(x,t)=k(t+c_1)(2k\pm\tan(kx+c_2)).$$

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Authors' addresses:

Iulia Elena Hirică
Univ. of Bucharest, Faculty of Mathematics and Computer Science,
14 Academiei Str., RO-010014, Bucharest 1, Romania.
E-mail: ihirica@fmi.unibuc.ro

Constantin Udrişte
Univ. Politehnica of Bucharest, Faculty of Applied Sciences,
Dept. of Mathematics-Informatics,
Splaiul Independentei 313, Bucharest 060042, Romania.
E-mails: anet.udri@yahoo.com , udriste@mathem.pub.ro