On para-Norden metric connections

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Dedicated to Professor Constantin Udrişte at his 75-th anniversary

Abstract. The aim of this paper is the construction of some para-Norden metric connections of the first type and of the second type on almost para-Norden manifolds. These are metric connections with respect to the associated twin metric, have nonvanishing torsion and in some aspects they are similar with the para-Bismut connection from para-Hermitian geometry.

M.S.C. 2010: 53C15, 53C25, 53C05, 53C07, 53C12.
Key words: almost para-complex manifold; Norden metric; connection.

1 Introduction

The systematic development of the theory of Riemannian almost product manifolds was initiated in [20]. Also, in [13] is given a classification of these manifolds with respect to the covariant differentiation of the almost product structure, and in [19] is given a classification of the Riemannian almost product manifolds with zero trace of the almost product structure.

On the other hand, the para-complex structures (a particular type of almost product structures) on smooth manifolds were introduced in [8], and a survey of further results in para-complex geometry (including para-Hermitian and para-Kähler geometry) can be found for instance in [5, 6]. Their are also useful in the study of para-Kenmotsu manifolds [3]. Also, other further significant developments are due in some recent surveys [1, 4], where some aspects concerning the geometry of para-complex manifolds are presented systematically by analogy with the geometry of complex manifolds using certain para-holomorphic coordinate systems. This type of even dimensional geometry is now in the mainstream of research as the surveys [1] and [5] and their several citations prove. From the other point of view, the analyticity of tensor fields on (almost) para-complex manifolds, endowed eventualy with para-Norden metrics, is intensively studied in [15, 16].

In contrast with the case of the almost para-Hermitian manifolds, where the action of the almost para-complex structure on the tangent space at each point of the manifold is an anti-isometry regarding a pseudo-Riemannian metric, in the almost

Balkan Journal of Geometry and Its Applications, Vol.21, No.2, 2016, pp. 45-54.

para-complex manifolds endowed with para-Norden metrics the action of the almost para-complex structure is an isometry. Thus, we notice that in an almost para-complex manifold, the associated $(0,2)$-tensor of a para-Hermitian metric is a 2-form, while the associated $(0,2)$-tensor of every para-Norden metric is again a para-Norden metric, called the associated *twin* para-Norden metric. Hence, in the second case we have a pair of mutually associated para-Norden metrics.

In the recent papers [14, 17, 18], some topics on (almost) Norden manifolds concerning to anti-Hermitian (or Norden) metric connections similar to Bismut connection from Hermitian geometry, as well as to an isotropy property of anti-Kählerian-Codazzi (or Kähler-Norden-Codazzi) manifolds are studied. Hence, similar problems can be taken into account for the case of almost para-Norden manifolds, however we will see that the para-Kähler-Norden-Codazzi manifolds are actually para-Kähler-Norden.

The aim of this paper is the study of some para-Norden metric connections on almost para-Norden manifolds and its content is as follows. In Section 2, we brief recall some basic facts about almost para-complex manifolds with para-Norden metric. A first result shows that the para-Kähler-Norden-Codazzi manifolds are para-Kähler-Norden (Proposition 2.1). In Section 3, starting from the Levi-Civita connection on an almost para-Norden manifold (M,I,g), which is the unique torsion free and metrical connection with respect to g, following an argument similar to [18], we look for other metric connections with torsion on (M,I,g). These connections will be called *para-Norden metric connections of the first type* and *of the second type*, respectively, and in some aspects they are similar with the para-Bismut connection from para-Hermitian geometry. Also, we will see that for the particular case of foliated Riemannian manifolds endowed with foliations for which the codimension is equal to the dimension of leaves (for instance vertical foliation on tangent or cotangent bundle), which inherits a natural para-Norden structure, the para-Norden metric connection of the first type coincides with the Schouten-Van Kampen connection, on the foliated manifold.

2 Generalities on almost para-complex manifolds with para-Norden metrics

A *para-complex* structure on a real finite dimensional vector space V is defined as an endomorphism $I \in End(V)$ which satisfy $I^2 = Id$, $I \neq \pm Id$ and the following two eigenspaces $V^{\pm} := \ker(Id \pm I)$ corresponding to the eigenvalues ± 1 of I have the same dimension. Such a pair (V,I) is called a *para-complex vector space*. Consequently, an *almost para-complex structure* on a $2n$-dimensional smooth manifold M is defined as an endomorphism $I \in End(TM)$ with the property that (T_xM, I_x) is a para-complex vector space, for every $x \in M$. Moreover, an almost para-complex structure I on M is said to be *integrable* if the distributions $T^{\pm}M = \ker(Id \mp I)$ are both integrable, and in this case I is called a *para-complex structure* on M. As usual, the Nijenhuis tensor N_I associated to an almost para-complex structure I is defined by

$$N_I(X,Y) := [IX,IY] - I[IX,Y] - I[X,IY] + [X,Y], \tag{2.1}$$

for every $X,Y \in \Gamma(TM)$. The para-complex structure I is integrable if and only if $N_I = 0$ (see [4]).

Definition 2.1. ([15, 16]) A *para-Norden metric* on an almost para-complex manifold (M, I) is a Riemannian metric g on M which satisfies

$$g(IX, Y) = g(X, IY) \;\Leftrightarrow\; g(IX, IY) = g(X, Y)\,,\; \forall\, X, Y \in \Gamma(TM). \tag{2.2}$$

Such a metric is also known as an almost product Riemannain metric, see for instance [7, 9, 10, 12, 13, 19], or para-B-metric. In this paper the triple (M, I, g) is called an *almost para-Norden manifold*.

A classification of the Riemannian almost product manifolds is introduced in [13], where six classes of these manifolds are characterized according to the properties of the tensor field F of type $(0, 3)$, defined by

$$F(X, Y, Z) = g((\nabla_X I)Y, Z),\; \forall\, X, Y, Z \in \Gamma(TM), \tag{2.3}$$

where ∇ is the Levi-Civita connection of g. The tensor F has the following important properties (see [19]):

$$F(X, Y, Z) = F(X, Z, Y)\,,\; F(X, IY, IZ) = -F(X, Y, Z). \tag{2.4}$$

Remark 2.2. The tensor F can be expressed in terms of the Tachibana operator which is an useful tool in the study of almost analyticity of tensor fields, as is pointed in [15, 16]. More exactly, we can consider the operator $\Phi_I : \mathcal{T}^0_r(M) \to \mathcal{T}^0_{r+1}(M)$, (see [21]):

$$(\Phi_I\omega)(X, Y_1, \ldots, Y_r) = IX(\omega(Y_1, \ldots, Y_r)) - X(\omega(IY_1, Y_2, \ldots, Y_r))$$

$$+\omega((L_{Y_1}I)X, Y_2, \ldots, Y_r) + \ldots + \omega(Y_1, Y_2, \ldots, (L_{Y_r}I)X), \tag{2.5}$$

for every vector fields $X, Y_1, \ldots, Y_r$, where L_X denotes the Lie derivative with respect to X. Then, according to [15], we have

$$F(X, Y, Z) = \frac{1}{2}\left[(\Phi_I g)(Z, X, Y) + (\Phi_I g)(Y, X, Z)\right]. \tag{2.6}$$

Also, we notice that in [19] a classification of the Riemannian almost product manifolds (M, I, g) with $\mathrm{Tr}\, I = 0$ is given with respect to the tensor field F, and the intersection of the basic classes in this classification is the class $\mathcal{W}_0$ determined by the condition $F(X, Y, Z) = 0$ or equivalently $\nabla I = 0$.

In the present paper, if the almost para-complex structure I is parallel with respect to the Levi-Civita connection of g, then (M, I, g) is called a *para-Kähler-Norden manifold* (according to the terminology used in [15]). Thus the para-Kähler-Norden manifolds are in the class $\mathcal{W}_0$.

Concerning the non-integrability of almost para-complex structures on almost para-Norden manifolds there is the class of quasi-para-Kähler-Norden manifolds. According to [15], an almost para-Norden manifold (M, I, g) is called a *quasi-para-Kähler-Norden manifold* if

$$F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0, \tag{2.7}$$

and this type of manifolds are just in the class $\mathcal{W}_3$ in the classification given in [19], which are also intensively studied in [7, 9, 10].

Moreover, given an almost para-Norden manifold (M, I, g) we can associate another para-Norden metric $\widetilde{g}$ on (M, I) given by

$$\widetilde{g}(X,Y) = (g \circ I)(X,Y) := g(IX,Y)\,,\ \forall\, X,Y \in \Gamma(TM), \tag{2.8}$$

which is called the *twin para-Norden metric*, (see [15]), and it plays a role similar to the para-Kähler form in para-Hermitian geometry. It is easy to see that the twin para-Norden metric of $\widetilde{g}$ is just g.

Now, for a given almost para-Norden manifold (M, I, g) we can consider the Codazzi type equation with respect to $\widetilde{g}$, that is

$$(\nabla_X \widetilde{g})(Y,Z) = (\nabla_Y \widetilde{g})(X,Z),\ \forall\, X,Y,Z \in \Gamma(TM), \tag{2.9}$$

where ∇ is the Levi-Civita connection of g, and this equation is equivalent with

$$(\nabla_X I)Y = (\nabla_Y I)X\,,\ \forall\, X,Y \in \Gamma(TM). \tag{2.10}$$

Using a terminology from [17, 18] for the almost anti-Hermitian manifolds, if the almost para-complex structure of an almost para-Norden manifold satisfies (2.10), then the triple (M, I, g) is called a *para-Kähler-Norden-Codazzi manifold.*

In what follows we will prove that the para-Kähler-Norden-Codazzi manifolds are actually para-Kähler-Norden. In fact, we have

Proposition 2.1. *In a para-Kähler-Norden-Codazzi manifold (M, I, g) the fundamental tensor F vanishes.*

Proof. Using (2.10), it is easy to see that

$$F(X,Y,Z) = F(Y,X,Z). \tag{2.11}$$

Then, according to (2.4) we obtain

$$\begin{aligned} F(X,Y,Z) &= -F(X,IY,IZ) = -F(IY,X,IZ) = F(IY,IX,Z) = F(IY,Z,IX) \\ &= F(Z,IY,IX) = -F(Z,Y,X) = -F(Z,X,Y) = -F(X,Z,Y) \\ &= -F(X,Y,Z) \end{aligned}$$

which end the proof. □

Let us denote by $\widetilde{\nabla}$ the Levi-Civita connection of $\widetilde{g}$. Then, the following important tensor field on an almost para-Norden manifold (M, I, g) can be defined by

$$\Phi(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y,\ \forall\, X,Y \in \Gamma(TM), \tag{2.12}$$

and it is called the *potential* of $\widetilde{\nabla}$ with respect to ∇ (see [19]).

Since both connections $\widetilde{\nabla}$ and ∇, respectively, are torsion-free, we get that Φ is symmetric, that is

$$\Phi(X,Y) = \Phi(Y,X). \tag{2.13}$$

Also, the tensors F and Φ can be related by the following identity:

$$F(X,Y,Z) = \widetilde{g}(\Phi(X,Y),Z) + \widetilde{g}(\Phi(X,Z),Y). \tag{2.14}$$

Indeed, we have
(2.15)
$$F(X,Y,Z) = g((\nabla_X I)Y, Z) = g(\nabla_X IY, Z) - g(I\nabla_X Y, Z) = g(\nabla_X IY, Z) - \widetilde{g}(\nabla_X Y, Z),$$

and taking into account $\nabla g = 0$ it follows $Xg(IY,Z) - g(\nabla_X IY, Z) - g(IY, \nabla_X Z) = 0$. Thus, we have

$$g(\nabla_X IY, Z) = X\widetilde{g}(Y,Z) - \widetilde{g}(Y, \nabla_X Z) = \widetilde{g}(\widetilde{\nabla}_X Y, Z) + \widetilde{g}(Y, \widetilde{\nabla}_X Z) - \widetilde{g}(Y, \nabla_X Z),$$

where the last equality is from $(\widetilde{\nabla}_X \widetilde{g})(Y,Z) = 0$. Hence, the relation (2.14) follows by the above relation, (2.12) and (2.15).

Also, we have

$$(\nabla_X \widetilde{g})(Y,Z) = X\left(\widetilde{g}\,(Y,Z)\right) - \widetilde{g}\,(\nabla_X Y, Z) - \widetilde{g}\,(Y, \nabla_X Z)\,,$$

and, from $\widetilde{g}\,(Y,Z) = g(Y, IZ)$, $\widetilde{g}\,(\nabla_X Y, Z) = g\,(\nabla_X Y, IZ)$ and $(\nabla_X g)\,(Y, IZ) = 0$, we obtain

$$(\nabla_X \widetilde{g})(Y,Z) = g\,(Y, \nabla_X (IZ)) - g\,(Y, I\,(\nabla_X Z)) = g\,((\nabla_X I)\,Z, Y)\,. \tag{2.16}$$

Remark 2.3. The relations (2.16) and (2.3) show that ∇ is a metric connection with respect to the twin metric $\widetilde{g}$ if and only if the fundamental tensor vanishes. In this case we also have $\widetilde{\nabla} = \nabla$. Thus, it results that in a para-Kähler-Norden (or para-Kähler-Norden-Codazzi) manifold the potential Φ vanishes.

3 Para-Norden metric connections

The Levi-Civita connection ∇ on an almost para-Norden manifold (M, I, g) is the unique connection which is torsion free and metrical with respect to g. In this section, following an argument similar from [18] for anti-Hermitian manifolds, we look for other metric connections with torsion on (M, I, g) which will be called *para-Norden metric connections.*

Let ∇^* be a linear connection on an almost para-Norden manifold (M, I, g) with torsion T^* and a potential Φ^* with respect to the Levi-Civita connection ∇, that is

$$T^*(X,Y) = \nabla^*_X Y - \nabla^*_Y X - [X,Y]\,,\ \Phi^*(X,Y) = \nabla^*_X Y - \nabla_X Y.$$

Also, we consider the associated tensors of type $(0,3)$ with respect to g which are defined by

$$\mathcal{T}^*(X,Y,Z) = g(T^*(X,Y),Z)\,,\ \Psi^*(X,Y,Z) = g(\Phi^*(X,Y),Z).$$

It is easy to see that

$$T^*(X,Y) = \Phi^*(X,Y) - \Phi^*(Y,X),\ \forall\, X,Y \in \Gamma(TM), \tag{3.1}$$

and the covariant derivative of g with respect to ∇^* can be written in the following form (see [18] for the general Riemannian manifolds)

$$(\nabla^*_X g)(Y,Z) = -\Psi^*(X,Y,Z) - \Psi^*(X,Z,Y)\,,\ \forall\, X,Y,Z \in \Gamma(TM).$$

Consequently, $\nabla^* g = 0$ if and only if

$$\Psi^*(X,Y,Z) + \Psi^*(X,Z,Y) = 0\,,\ \forall\, X,Y,Z \in \Gamma(TM). \tag{3.2}$$

Now, from (3.1), we have

$$\mathcal{T}^*(X,Y,Z) = \Psi^*(X,Y,Z) - \Psi^*(Y,X,Z),$$

and similarly,

$$\mathcal{T}^*(Z,X,Y) = \Psi^*(Z,X,Y) - \Psi^*(X,Z,Y),$$
$$\mathcal{T}^*(Z,Y,X) = \Psi^*(Z,Y,X) - \Psi^*(Y,Z,X).$$

Using (3.2), the last three relations yields

$$\Psi^*(X,Y,Z) = \frac{1}{2}\left(\mathcal{T}^*(X,Y,Z) + \mathcal{T}^*(Z,X,Y) + \mathcal{T}^*(Z,Y,X)\right). \tag{3.3}$$

Proposition 3.1. *If $\widetilde{g}$ is the twin para-Norden metric associated with g, then $\nabla^*\widetilde{g} = 0$ if and only if*

$$(\nabla_X \widetilde{g})(Y,Z) = \Psi^*(X,Y,IZ) + \Psi^*(X,Z,IY)\,,\ \forall\, X,Y,Z \in \Gamma(TM). \tag{3.4}$$

Proof. By direct calculus, we have

$$\begin{aligned}(\nabla^*_X\widetilde{g})(Y,Z) &= X\widetilde{g}(Y,Z) - \widetilde{g}(\nabla^*_X Y,Z) - \widetilde{g}(Y,\nabla^*_X Z)\\ &= (\nabla_X\widetilde{g})(Y,Z) - \widetilde{g}(\Phi^*(X,Y),Z) - \widetilde{g}(Y,\Phi^*(X,Z))\\ &= (\nabla_X\widetilde{g})(Y,Z) - g(\Phi^*(X,Y),IZ) - g(\Phi^*(X,Z),IY)\\ &= (\nabla_X\widetilde{g})(Y,Z) - \Psi^*(X,Y,IZ) - \Psi^*(X,Z,IY),\end{aligned}$$

which end the proof. □

We notice that the connection ∇^* is not completely determined by (3.2) and (3.4). So we will introduce some other conditions concerning the potential Φ^* and we will try to solve the equation with respect to Φ^*.

In what follows, we suppose only that $\nabla^*\widetilde{g} = 0$ and make no use of $\nabla^* g = 0$, that is (3.2) holds. The equation (3.2) will be satisfied in some special cases, which will be discussed in the sequel.

Using the terminology from [18] for the case of the almost complex manifolds with Norden (anti-Hermitian) metrics, we give the following definition.

Definition 3.1. A metric connection $\nabla^* = \nabla + \Phi^*$ with respect to a para-Norden metric g on an almost para-complex manifold (M,I), i.e. $\nabla^* g = 0$, is called a *para-Norden metric connection of the first type* if it is also metric with respect to the associated twin para-Norden metric, i.e $\nabla^*\widetilde{g} = 0$, and

$$\Psi^*(X,Y,IZ) = \Psi^*(X,Z,IY)\,,\ \forall\, X,Y,Z \in \Gamma(TM). \tag{3.5}$$

Now, using (3.4) and (3.5), we obtain

$$\Psi^*(X,Y,IZ) = \frac{1}{2}(\nabla_X\widetilde{g})(Y,Z)\,,\ \forall\, X,Y,Z \in \Gamma(TM).$$

Taking into account that $\nabla\widetilde{g} = \nabla g \circ I + g \circ \nabla I = g \circ \nabla I$, the above relation reads as

$$g(\Phi^*(X,Y), IZ) = \frac{1}{2} g((\nabla_X I)Y, Z)\,,\ \forall\, X, Y, Z \in \Gamma(TM), \tag{3.6}$$

or, equivalently

$$g(I\Phi^*(X,Y), Z) = \frac{1}{2} g((\nabla_X I)Y, Z)\,,\ \forall\, X, Y, Z \in \Gamma(TM). \tag{3.7}$$

The last relation implies

$$I\Phi^*(X,Y) = \frac{1}{2}(\nabla_X I)Y\,,\ \forall\, X, Y \in \Gamma(TM), \tag{3.8}$$

or, equivalently

$$\Phi^*(X,Y) = \frac{1}{2} I(\nabla_X I)Y\,,\ \forall\, X, Y \in \Gamma(TM). \tag{3.9}$$

Moreover, if we put $Z \mapsto IZ$ in (3.6), we have

$$\Psi^*(X,Y,Z) = \frac{1}{2} g((\nabla_X I)Y, IZ) = \frac{1}{2} F(X,Y,IZ),\ \forall\, X, Y, Z \in \Gamma(TM), \tag{3.10}$$

and, using $F(X,Z,IY) = -F(X,Y,IZ)$, we have

$$\Psi^*(X,Y,Z) + \Psi^*(X,Z,Y) = \frac{1}{2}\left[F(X,Y,IZ) + F(X,Z,IY)\right] = 0.$$

Thus, the tensor Ψ^* in the form (3.10) satisfies (3.2), that is $\nabla^* g = 0$, which yields

Theorem 3.2. *The connection*

$$\nabla^* = \nabla + \frac{1}{2} I(\nabla I) \tag{3.11}$$

is a para-Norden metric connection of the first type.

Remark 3.2. The para-Norden metric connection of the first type from (3.11) is also defined and studied in [11] under the name of *P-connection on Riemannian almost product manifolds*. Also, taking into account that in a para-Kähler-Norden (or para-Kähler-Norden-Codazzi) manifold, we have $\nabla I = 0$, then the para-Norden metric connection of the first type coincides with the Levi-Civita connection of g.

Example 3.3. (*Foliated manifolds with almost para-Norden structure.*)
Let us consider a particular case of almost para-Norden manifold. Let (M, g) be a Riemannian $2n$-dimensional manifold endowed with a n-codimensional foliation $\mathcal{F}$ (for instance the vertical foliation on tangent or cotangent bundle). Obviously, there is the decomposition $TM = T\mathcal{F} \oplus T^{\perp}\mathcal{F}$, where $T\mathcal{F}$ is the structural bundle (the tangent bundle along the leaves of $\mathcal{F}$) and $T^{\perp}\mathcal{F}$ is the transversal bundle of $\mathcal{F}$.

If v, h are projections of TM on $T\mathcal{F}$ and $T^{\perp}\mathcal{F}$, respectively, then

$$I = v - h,$$

satisfies $I^2 = Id$, so it is an almost para-complex structure on M. By direct calculus, we have

$$g(IX, IY) = g(vX - hX, vY - hY) = g(vX, vY) + g(hX, hY) = g(X, Y),$$

for every $X, Y \in \Gamma(TM)$, and thus, (M, g, I) is an almost para-Norden manifold.

The para-Norden metric connection $\nabla^* = \nabla + \frac{1}{2}I(\nabla I)$ given in (3.11) is in case of foliated manifold $(M, \mathcal{F})$,

$$\begin{aligned}\nabla^*_X Y &= \nabla_X Y + \frac{1}{2}I(\nabla_X I)Y\\ &= \frac{1}{2}\left[\nabla_X Y + I\left(\nabla_X(IY)\right)\right]\\ &= \frac{1}{2}\left[v\nabla_X Y + h\nabla_X Y + (v-h)\left(\nabla_X vY - \nabla_X hY\right)\right],\end{aligned}$$

hence

$$\nabla^*_X Y = v\left(\nabla_X vY\right) + h\left(\nabla_X hY\right). \tag{3.12}$$

The above relation shows that this para-Norden metric connection is exactly the Schouten-Van Kampen connection, on the foliated manifold (see [2]).

Definition 3.4. A metric connection $\nabla^* = \nabla + \Phi^*$ with respect to a para-Norden metric g on an almost para-complex manifold (M, I), i.e. $\nabla^* g = 0$, is called a *para-Norden metric connection of the second type* if it is also metric with respect to the associated twin para-Norden metric, i.e $\nabla^*\widetilde{g} = 0$, and

$$\Psi^*(X, Y, IZ) = \Psi^*(Z, Y, IX)\,,\ \forall\, X, Y, Z \in \Gamma(TM). \tag{3.13}$$

Using (3.4) and taking into account (3.13), we have

$$(\nabla_X\widetilde{g})(Y, Z) - (\nabla_Y\widetilde{g})(Z, X) + (\nabla_Z\widetilde{g})(X, Y) = 2\Psi^*(X, Y, IZ) = 2g(I\Phi^*(X, Y), Z), \tag{3.14}$$

for every $X, Y, Z \in \Gamma(TM)$.

On the other hand, the Tachibana operator Φ_I applied to g reduces to

$$(\Phi_I g)(Y, Z, X) = (\nabla_X\widetilde{g})(Y, Z) - (\nabla_Y\widetilde{g})(Z, X) + (\nabla_Z\widetilde{g})(X, Y)\,,\ \forall\, X, Y, Z \in \Gamma(TM). \tag{3.15}$$

Indeed, according to formula (2.8) from [16], we have

$$(\Phi_I g)(Y, Z, X) = -g((\nabla_Y I)Z, X) + g((\nabla_Z I)Y, X) + g((\nabla_X I)Y, Z),$$

for every $\forall\, X, Y, Z \in \Gamma(TM)$. Also, we have

$$\begin{aligned}(\nabla_X\widetilde{g})(Y, Z) &= Xg(IY, Z) - g(I\nabla_X Y, Z) - g(IY\nabla_X Z)\\ &= g(\nabla_X IY, Z) + g(IY, \nabla_X Z) - g(I\nabla_X Y, Z) - g(IY\nabla_X Z)\\ &= g((\nabla_X I)Y, Z),\end{aligned}$$

and similarly, $(\nabla_Y\widetilde{g})(Z, X) = g((\nabla_Y I)Z, X)$ and $(\nabla_Z\widetilde{g})(X, Y) = g((\nabla_Z I)X, Y)$.

But, $g((\nabla_Z I)X, Y) = g((\nabla_Z I)Y, X)$ (or equivalently $F(Z, X, Y) = F(Z, Y, X)$), and thus the relation (3.15) holds.

Now, it is well known that the para-Kähler-Norden condition ($\nabla I = 0$) is equivalent with $(\Phi_I g)(X, Y, Z) = 0$, see Theorem 2 in [16]. Thus, from (3.14) we have $\Phi^* = 0$ and the metric connection ∇^* reduces to the Levi-Civita connection ∇ and it is clear that the tensor $\Psi^* = 0$ satisfies the relation (3.2). Consequently, the connection ∇^* is a para-Norden metric connection. Thus, we have

Theorem 3.3. *If (M, I, g) is a para-Kähler-Norden manifold, then the para-Norden metric connection ∇^* of the second type coincides with the Levi-Civita connection ∇.*

Finally, let us consider the case when (M, I, g) is a quasi-para-Kähler-Norden manifold, that is the relation (2.7) holds. Then, from (3.14), we obtain $\Phi^*(X, Y) = I(\nabla_X I)Y$ which satisfies (3.2). This implies

Theorem 3.4. *If (M, I, g) is a quasi-para-Kähler-Norden manifold, then the para-Norden metric connection of the second type ∇^* have the form $\nabla^* = \nabla + I(\nabla I)$.*

Remark 3.5. As it is noticed in [18] for anti-Hermitian case, we have the same conclussion for para-Norden manifolds case. More precisely, for a given almost para-Hermitian manifold (M, I, g), there is an unique connection ∇^* which preserves g, I and has a skew-symmetric torsion. It is known as the para-Bismut connection, see [1]. For the almost para-Norden manifolds, from $\nabla^* g = 0$ and $\nabla^* \widetilde{g} = 0$ we have $\nabla^* I = 0$, hence in some aspects the para-Norden metric connections of the first and second type studied above are similar to the para-Bismut connection.

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