Some results for slant submanifolds in generalized Sasakian space forms

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Abstract. In this paper we obtain relationships between the Ricci curvature, the scalar curvature, the squared mean curvature and the Riemannian invariant of constant slant submanifolds in generalized Sasakian space forms. We give an example of a constant slant submanifold in a generalized Sasakian space form.

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1 Introduction

For a submanifold of a Riemannian manifold, there exist several extrinsic associate invariants besides its intrinsic ones. The squared mean curvature is the most important, among the extrinsic invariants of a submanifold, and the Ricci curvature, the sectional curvature, δ_k-invariant and the scalar curvature are well-known among its intrinsic invariants.

One of the most fundamental challenges in the submanifold theory is the following:

Problem. *Establish a simple relationship between the main extrinsic invariants and intrinsic invariants of a submanifold.*

B. Y. Chen and I. Mihai gave some solutions to the above problem. They established sharp relationships between the Ricci curvature and the squared mean curvature of submanifolds in Riemannian space forms and in Sasakian space forms, such that the obtained inequalities provide upper bounds for the Ricci curvature (see [4, 14]).

In [3], B. Y. Chen showed that the Chen's invariant $\delta_M(=\delta_2)$ of a Riemannian submanifold in a real space form $\overline{M}(c)$ and satisfies the inequality

$$\delta_M \leq \frac{n-2}{2}\{\frac{n^2}{n-1}\|H\|^2+(n+1)c\}.$$

In [16], T. Oprea showed that δ_k-invariant $(k \geq 3)$ satisfies the same inequality.

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In this way, D. Cioroboiu and C. Udrişte obtained sharp relationships between some extrinsic invariants and intrinsic invariants (see[6, 7, 8, 9, 10, 11, 18]).

In[1], P.Alegre, D. E. Blair and A. Carriazo introduced the notion of generalized Sasakian space form such that this kind of manifold appears as a natural generalization of the well-known Sasakian space form $\overline{M}(c)$.

In [13], F. Malek and V. Nejadakbary gave other solutions to the above problem. For instance, in the following theorem, they established a sharp relationship between the Ricci curvature and the squared mean curvature of submanifolds in generalized Sasakian space forms.

Theorem 1.1. *Let $M^n(n \geq 3)$ be a submanifold tangent to the structure vector field in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$.*
(i) If L is a k-plane section $(2 \leq k \leq n-1)$ in T_pM normal to the structure vector field at p, then for all unit vectors $U \in L$, we have

$$\begin{aligned}\frac{2}{k-1}Ric_L(U) \quad \geq \quad & 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 + 2(n-1)f_3 \\ & +3(\frac{2}{k-1}\|P_{L,k}U\|^2 - \|P\|^2)f_2.\end{aligned}$$

where H and τ are the mean curvature vector and the scalar curvature of M, respectively.
(ii) The equality case holds identically if and only if with respect to a suitable orthonormal basis $\{e_1, e_2, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\}$ is a basis of T_pM, the coefficients of the fundamental form h at p take the following form

$$\left(\begin{array}{c|c} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \gamma & 0 & \dots & 0 \\ 0 & 0 & \gamma & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma \end{pmatrix}_{k\times k} & O \\ \hline O & O \end{array}\right)_{n\times n} \qquad r = n+1,$$

and

$$h^r_{ij} = 0, \quad r \geq n+2, \quad 1 \leq i, j \leq n.$$

In [12], A. Lotta introduced the notion of slant submanifolds in almost contact metric manifolds.

In this way, M. M. Tripathi, J. S. Kim and S. B. Kim established the following relationship between the Ricci curvature and the squared mean curvature of slant submanifolds in Sasakian space forms(see [17]).

Theorem 1.2. *Let M be a $(n+1)$-dimensional θ-slant submanifold isometrically immersed in a $(2m+1)$-dimensional Sasakian space form $\overline{M}^{2m+1}(c)$ such that $\xi \in TM$. Then*
(i) For each unit vector $U \in T_pM$, we have

$$\begin{aligned}4Ric(U) \quad \leq \quad & (n+1)^2\|H\|^2 + n(c+3) \\ & + \left\{3\cos^2\theta - (n-1+3\cos^2\theta)\Big(\eta(U)\Big)^2 - 1\right\}(c-1).\end{aligned}$$

(ii) If $H(p) = 0$, an unit vector $U \in T_pM$ satisfies the equality case if and only if U belongs to the relative null space N_p.
(iii) The equality case holds for all unit vectors $U \in T_pM$ if and only if M is a totally geodesic submanifold.

In this paper,

a) We obtain other inequalities between the Ricci curvature, the scalar curvature, and the squared mean curvature of constant slant submanifolds in generalized Sasakian space forms such that each inequality defines a lower bound for the Ricci curvature. Also, we obtain a sharp relationship between the scalar curvature, the Riemannian invariant Θ_k, δ_k-invariant and the squared mean curvature of constant slant submanifolds in generalized Sasakian space forms.

b) We obtain an equivalent condition for part (iii) of theorem 1.2

c) We give an example of a constant slant submanifold in a generalized Sasakian space form.

2 Preliminaries

In this section, we recall some definitions and basic formulas which we will use later.

Let (M^n, g) be a Riemannian manifold and $L \subseteq T_pM$ be a k-plane section $(2 \leq k \leq n)$ and $U \in L$ be an unit vector. If we choose local orthonormal basis $\{e_1, \ldots, e_k\}$ for L such that $e_1 = U$, then the *Ricci curvature* of L at U is defined by

$$Ric_L(U) := \sum_{i=1}^{k} K(U, e_i),$$

in which $K(U, e_i)$ is the sectional curvature of the 2-plane section spanned by $\{U, e_i\}$. If $k = n$, then $Ric_L(U)$ denoted by $Ric(U)$. For each integer $2 \leq k \leq n$, *the Riemannian invariant* Θ_k on M is defined by:

$$\Theta_k := \frac{1}{k-1} \inf_{L,U} Ric_L(U),$$

where L runs over all k-plane section fields in TM and U runs over all unit vector fields in L. *The scalar curvature* τ at $p \in M$ is given by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i, e_j),$$

where $K(e_i, e_j)$ is the sectional curvature of the 2-plane section is spanned by e_i and e_j. Since $K(e_i, e_i) = 0$ and $K(e_i, e_j) = K(e_j, e_i)$, therefore

$$2\tau(p) = \sum_{1 \leq i \neq j \leq n} K(e_i, e_j) = \sum_{i,j=1}^{n} K(e_i, e_j).$$

For an integer $k \geq 0$, let $S(n,k)$ denote the set consisting of unordered k-tuples $(n_1, n_2, \cdots, n_k)$ of integers ≥ 2 such that $\sum_{i=1}^{k} n_i \leq n$. Denote by $S(n)$ the set of all $S(n,k)$ with $k \geq 0$ for a fixed n. In [3], Chen defined the following invariant

$$\delta(n_1, n_2, \cdots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \tau(L_2) + \cdots + \tau(L_k)\}, \qquad p \in M,$$

where $(n_1, n_2, \cdots, n_k) \in S(n)$ and $L_1, L_2, \cdots, L_k$ run over all k mutually orthogonal subspace of T_pM such that $\dim L_i = n_i$. In [16], T. Oprea extended $\delta_M = \delta(2) = \tau - \inf\Big(\tau(L)\Big)$ to

$$\delta_k = \tau - \Theta_k,$$

where Θ_k is the Riemannian invariant of M.

A $(2n+1)$-dimensional Riemannian manifold $(\overline{M}, g)$ is said to be an *almost contact metric manifold* if there exist on $\overline{M}$ a (1,1)-tensor field ϕ, a vector field ξ(is called the *structure vector field*) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for any vector fields X,Y on $\overline{M}$. Also in an almost contact metric manifold we have $\phi\xi = 0$ and $\eta \circ \phi = 0$ and for any $X \in \tau(\overline{M})$, $\eta(X) = g(X, \xi)$(see for instance [2]).We denote an almost contact metric manifold by $(\overline{M}, \phi, \xi, \eta, g)$. An almost contact metric manifold is called a *contact metric* manifold if

$$g(X, \phi Y) = d\eta(X,Y) \quad X, Y \in T\overline{M}.$$

A $(2n)$-dimensional smooth manifold M is said to be an *almost complex manifold* if there exist on M a (1,1)-tensor field J such that for any vector field $X \in TM$,

$$J^2X = -X.$$

(1,1)-tensor field J is called *almost complex structure*.

Let M be a submanifold of an almost contact metric manifold $(\overline{M}, \phi, \xi, \eta, g)$. For any vector field X tangent to M, we put

$$\phi X = PX + FX,$$

in which PX and FX are tangent and normal components of ϕX, respectively. A submanifold M of an almost contact metric manifold is called an *anti-invariant* submanifold if

$$\phi_p(T_pM) \subset T_p^{\perp}M \quad p \in M.$$

In other words, for all $X \in T_pM$, $PX = 0$. If a submanifold M in a contact metric manifold is normal to the structure vector field ξ, then it is anti-invariant. Also, submanifold M is called an *invariant* submanifold if

$$\phi_p(T_pM) \subset T_pM \quad p \in M.$$

In other words, for all $X \in T_pM$, $FX = 0$.

M is called a *constant slant submanifold (or θ-slant submanifold)* in an almost contact metric manifold if for any $0 \neq X \in T_pM$, linearly independent of ξ_p, the angle between ϕX and T_pM is a constant $\theta \in [0, \frac{\pi}{2}]$. The angle θ is called the *slant angle* of M. It is obvious that invariant and anti-invariant submanifolds are θ-slant submanifold with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. In a θ-slant submanifold M tangent to ξ, for any vector field X and Y tangent to M, we have

$$g(PX, PY) = \cos^2\theta g(\phi X, \phi Y), \qquad g(FX, FY) = \sin^2\theta g(\phi X, \phi Y),$$

and therefore for unit vector field U tangent to M, we have

$$\|PU\|^2 = g(PU, PU) = \cos^2\theta\Big(1 - \eta(U)^2\Big).$$

An almost contact metric manifold is called a *Sasakian manifold* if

$$\Big(\overline{\nabla}_X\phi\Big)(Y) = \eta(Y)X - g(X, Y)\xi,$$

where $\overline{\nabla}$ is the *Riemannian connection* of $\overline{M}$. It is easy to see that a Sasakian manifold is a contact metric manifold (see [2]).

Let $(\overline{M}, \phi, \xi, \eta, g)$ be an almost contact metric manifold. The plane $\pi_p \subset T_p\overline{M}$ spanned by $\{X, \phi X\}$, where $0 \neq X \in T_p\overline{M}$ is normal to ξ_p, is called a *ϕ-section* of $\overline{M}$ at p and the sectional curvature $K(\pi_p)$ is called the *ϕ-sectional curvature* of π_p. The Sasakian manifold $\overline{M}$ is called the *Sasakian space form*, if there exists constant c such that for any $p \in \overline{M}$ and for any ϕ-section π_p of $\overline{M}$, $K(\pi_p) = c$, and denote it by $\overline{M}(c)$. The submanifold M of Sasakian space form $\overline{M}(c)$ is called *C-totally real*, if the structure vector field of $\overline{M}(c)$ be normal to M. It is proved that in a Sasakian space form $\overline{M}(c)$, the curvature tensor satisfies the following equality([15])

$$\begin{aligned}\overline{R}(X, Y,)Z &= \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + \frac{c-1}{4}\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &\quad + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &\quad - g(Y, Z)\eta(X)\xi\}.\end{aligned}$$

An almost contact manifold is called *generalized Sasakian space form* if

$$\begin{aligned}\overline{R}(X, Y,)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &\quad - g(Y, Z)\eta(X)\xi\},\end{aligned} \tag{2.1}$$

where f_1, f_2, f_3 are differentiable functions on $\overline{M}$. We denote this kind of manifolds by $\overline{M}(f_1, f_2, f_3)$(see [1]). It is clear that every Sasakian space form is a generalized Sasakian space form, but the converse is not necessarily true.

Let M^n be a submanifold of $\overline{M}^{2m+1}$ and h is the second fundamental form of M, $\overline{R}$ and R are the curvature tensors of $\overline{M}$ and M, respectively. The Gauss equation is given by

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) - g\Big(h(X, W), h(Y, Z)\Big) + g\Big(h(X, Z), h(Y, W)\Big), \tag{2.2}$$

for any vector fields X, Y, Z, W on M. Let $\{e_1, \cdots, e_n, \cdots, e_{2m+1}\}$ be a local orthonormal basis of $T_p\overline{M}$ such that $\{e_1, \cdots, e_n\}$ is a local orthonormal basis of T_pM. The *mean curvature vector* $H(p)$ is

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

and thus

$$n^2\|H\|^2 = \sum_{i,j=1}^{n} g\Big(h(e_i, e_i), h(e_j, e_j)\Big). \tag{2.3}$$

The submanifold M is called *totally geodesic* if $h = 0$, and is called *minimal* if H vanishes identically. We set

$$h_{ij}^r = g\Big(h(e_i, e_j), e_r\Big),\ i, j \in \{1, \cdots, n\},\ r \in \{n+1, \cdots, 2m+1\},$$

the coefficients of the second fundamental form h with respect to $\{e_1, \cdots, e_n, \cdots, e_{2m+1}\}$, and

$$\|h\|^2 = \sum_{i,j=1}^{n} g\Big(h(e_i, e_j), h(e_i, e_j)\Big) = \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2. \tag{2.4}$$

Now by (2.3), (2.4) and the Gauss equation (2.2), we have

$$\sum_{1\leq i,j\leq n} \overline{R}(e_j, e_i, e_i, e_j) = 2\tau - n^2\|H\|^2 + \|h\|^2. \tag{2.5}$$

Let M^n be a submanifold of an almost contact metric manifold $(\overline{M}^{2m+1}, \phi, \xi, \eta, g)$. For any local orthonormal frame $\{e_1, \ldots, e_{2m+1}\}$ such that $e_1, \ldots, e_n$ are tangent to M, we have $g(e_i, \phi e_j) = g(e_i, Pe_j)$ for any $i, j \in \{1, \ldots, n\}$. Therefore the squared norm of P is given by

$$\|P\|^2 = \sum_{i,j=1}^{n}\Big(g(e_i, Pe_j)\Big)^2 = \sum_{i,j=1}^{n}\Big(g(e_i, \phi e_j)\Big)^2.$$

Let $L \subseteq T_pM$ be a k-plane section. For any unit vector $U \in L$, we choose a local orthonormal basis $\{e_1, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $e_1, \ldots, e_k$ are tangent to L and $e_1 = U$. We define

$$\|P_{k,L}U\|^2 := \sum_{j=1}^{k}\Big(g(U, Pe_j)\Big)^2.$$

If $L = T_pM$, we denote $\|P_{k,L}U\|$ by $\|P_nU\|$.

We recall the following result of B.Y.Chen for later use.

Lemma 2.1. *([5]). Let $n \geq 2$ and $a_1, \cdots, a_n$ and b are real numbers such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + b\Big).$$

Then $2a_1a_2 \geq b$,with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

3 The Ricci curvature of constant slant submanifolds tangent to the structure vector field in generalized Sasakian space forms

In this section, we prove sharp relationships between the Ricci curvature, the squared mean curvature, the scalar curvature, the Riemannian invariant Θ_k and δ_k-invariant of $(n \geq 3)$-dimensional constant slant submanifolds M in generalized Sasakian space forms $\overline{M}(f_1, f_2, f_3)$.

Theorem 3.1. *Let $M^n (n \geq 3)$ be a θ-slant submanifold tangent to the structure vector field in generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$.*

a) *If $L \subseteq T_pM$ be a k-plane section $(k \geq 2)$ tangent to ξ_p and unit vector $U \in L$ is linearly independent of ξ_p, then*

$$\begin{aligned} 2Ric_L(U) \;\; \geq \;\; & (k-1)\mathcal{A} + \Bigg(2(n-2)(k-1)\Big(1 - \sec^2\theta \|PU\|^2\Big) \\ & + 2\lambda^2\Big((k-1)(n-1) - 1\Big)\sec^4\theta\|PU\|^4 \Bigg) f_3 + 6\|P_{k,L}U\|^2 f_2. \end{aligned} \tag{3.1}$$

b) *If $L \subset T_pM$ be a k-plane section $(k \geq 2)$ such that $\xi_p \in T_pM \setminus L$ and $U \in L$ be an unit vector, then*

$$\begin{aligned} 2Ric_L(U) \;\; & \geq (k-1)\mathcal{A} + (k-1)\Bigg(2(n-2)\Big(1 - \sec^2\theta\|PU\|^2\Big) \\ & + 2\lambda^2(n-1)\sec^4\theta\|PU\|^4 \Bigg) f_3 + 6\|P_{k,L}U\|^2 f_2, \end{aligned} \tag{3.2}$$

in which

$$\mathcal{A} := 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 - 3\|P\|^2 f_2,$$

$$\lambda := \frac{1}{\|\xi - \eta(U)U\|},$$

H and τ are the mean curvature vector and the scalar curvature of M at p, respectively.

c) *The equality case of (3.1) and (3.2) holds identically if and only if respect to a suitable orthonormal basis $\{e_1, e_2, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\}$ is a basis of T_pM, the coefficients of the fundamental form h at p take the following form*

$$\left(\begin{array}{c|c} \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & \gamma & 0 & \ldots & 0 \\ 0 & 0 & \gamma & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma \end{pmatrix}_{k\times k} & O \\ \hline O & O \end{array} \right)_{n\times n} \qquad r = n+1,$$

and $h^r_{ij} = 0$, $r \geq n+2$, $1 \leq i, j \leq n$.

Proof. **a)** We choose a local orthonormal basis $\{e_1, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\} \subset T_pM$, L spanned by $\{e_1, \ldots, e_k\}$, $e_1 = U$, $e_2 = \lambda\Big(\xi - \eta(U)U\Big)$ and e_{n+1} parallel to H at p. For $k \geq 3$, from (2.3), with respect to this basis we have

$$n^2\|H\|^2 = \sum_{i,j=1}^{n} g\Big(h(e_i, e_i), h(e_j, e_j)\Big) = \Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2. \tag{3.3}$$

Also, from (2.1) and (2.5), we have

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - n(n-1)f_1 - 3\|P\|^2 f_2 + 2(n-1)\Big(\eta(e_1)^2 + \eta(e_2)^2\Big)f_3. \tag{3.4}$$

Set

$$\begin{aligned}\delta := 2\tau - \tfrac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 - 3\|P\|^2 f_2 \\ +2(n-2)\Big(\eta(e_1)^2 + \eta(e_2)^2\Big)f_3.\end{aligned} \tag{3.5}$$

Therefore from (3.4) and (3.5), we have

$$n^2\|H\|^2 = (n-1)\Bigg(\|h\|^2 + \delta - 2f_1 + 2\Big(\eta(e_1)^2 + \eta(e_2)^2\Big)f_3\Bigg).$$

From (2.4), (3.4) and the above equality, we have

$$\begin{aligned}\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 &= (n-1)\Bigg(\textstyle\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 \\ &\quad +\delta - 2f_1 + 2\Big(\eta(e_1)^2 + \eta(e_2)^2\Big)f_3\Bigg).\end{aligned}$$

We set

$$b := \delta - 2f_1 + 2\Big(\eta(e_1)^2 + \eta(e_2)^2\Big)f_3 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2,$$

and $a_1 = h_{11}^{n+1}$ and $a_2 = h_{\alpha\alpha}^{n+1}$, for $\alpha \in \{2, \ldots, n\}$, then from lemma 2.1, we have $a_1a_2 \geq \frac{b}{2}$. Therefore

$$\begin{aligned}h_{11}^{n+1}h_{\alpha\alpha}^{n+1} &\geq \tfrac{\delta}{2} - \Bigg(f_1 - \Big(\eta(e_1)^2 + \eta(e_2)^2\Big)f_3\Bigg) + \textstyle\sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 \\ &\quad +\tfrac{1}{2}\textstyle\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^{r})^2.\end{aligned} \tag{3.6}$$

On the other hand by setting $X = W = e_1$ and $Y = Z = e_2$ in (2.2) and using (2.1), we have

$$\begin{aligned}f_1 + 3\Big(g(e_2, \phi e_1)\Big)^2 f_2 - \Big(\eta(e_1)^2 + \eta(e_2)^2\Big)f_3 &= K(e_1, e_2) - \textstyle\sum_{r=n+1}^{2m+1} h_{11}^{r}h_{22}^{r} \\ &\quad + \textstyle\sum_{r=n+1}^{2m+1}(h_{12}^{r})^2,\end{aligned}$$

therefore

$$f_1-\Big(\eta(e_1)^2+\eta(e_2)^2\Big)f_3+h_{11}^{n+1}h_{22}^{n+1} = K(e_1,e_2)-3\Big(g(e_2,\phi e_1)\Big)^2 f_2 - \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r + \sum_{r=n+1}^{2m+1}(h_{12}^r)^2.$$

From (3.6) and the above equality, we have

$$\begin{aligned} K(e_1,e_2) &- 3\Big(g(e_2,\phi e_1)\Big)^2 f_2 - \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r + \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 \\ &\geq \frac{\delta}{2} + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^r)^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2. \end{aligned} \tag{3.7}$$

On the other hand

$$\sum_{r=n+2}^{2m+1} h_{11}^r h_{\alpha\alpha}^r = \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r+h_{\alpha\alpha}^r)^2 - \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r)^2 - \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{\alpha\alpha}^r)^2. \tag{3.8}$$

Now by substituting (3.8) (for $\alpha=2$) in (3.7) after simplification we get

$$\begin{aligned} K(e_1,e_2) &\geq \frac{\delta}{2} + 3\Big(g(e_2,\phi e_1)\Big)^2 f_2 + \sum_{\substack{1\leq i<j\leq n\\ i\neq 1\vee j\neq 2}}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=3}^{n}(h_{ii}^r)^2 \\ &\quad + \sum_{r=n+2}^{2m+1}\sum_{\substack{1\leq i<j\leq n\\ i\neq 1\vee j\neq 2}}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r+h_{22}^r)^2 \\ &\geq \frac{\delta}{2} + 3\Big(g(e_2,\phi e_1)\Big)^2 f_2. \end{aligned} \tag{3.9}$$

For $\alpha\geq 3$ from Gauss equation (2.2) and (2.1), we have

$$f_1-\eta(e_1)^2 f_3 + h_{11}^{n+1}h_{\alpha\alpha}^{n+1} = K(e_1,e_\alpha) - 3\Big(g(e_\alpha,\phi e_1)\Big)^2 f_2 - \sum_{r=n+2}^{2m+1} h_{11}^r h_{\alpha\alpha}^r + \sum_{r=n+1}^{2m+1}(h_{1\alpha}^r)^2.$$

From (3.6), (3.8) and the above equality with similar computation as above, we get

$$\begin{aligned} K(e_1,e_\alpha) &\geq \frac{\delta}{2} + \eta(e_2)^2 f_3 + 3\Big(g(e_\alpha,\phi e_1)\Big)^2 f_2 + \sum_{\substack{1\leq i<j\leq n\\ i\neq 1\vee j\neq \alpha}}(h_{ij}^{n+1})^2 \\ &\quad + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\substack{i=2\\ i\neq\alpha}}^{n}(h_{ii}^r)^2 + \sum_{r=n+2}^{2m+1}\sum_{\substack{1\leq i<j\leq n\\ i\neq 1\vee j\neq \alpha}}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r+h_{\alpha\alpha}^r)^2 \\ &\geq \frac{\delta}{2} + 3\Big(g(e_\alpha,\phi e_1)\Big)^2 f_2 + \eta(e_2)^2 f_3. \end{aligned} \tag{3.10}$$

Therefore from (3.7) and the above inequality and by substituting δ from (3.5), we have

$$\begin{aligned} 2Ric_L(U) &= 2\sum_{\alpha=2}^{k} K(e_1, e_\alpha) = 2\left\{K(e_1, e_2) + \sum_{\alpha=3}^{k} K(e_1, e_\alpha)\right\}. \\ &\geq (k-1)\mathcal{A} + 6\|P_{k,L}U\|^2 f_2 \\ &\quad + \left\{2(n-2)(k-1)\eta(e_1)^2 + 2\Big((k-1)(n-1)-1\Big)\eta(e_2)^2\right\} f_3. \end{aligned} \tag{3.11}$$

Since

$$\eta(e_1)^2 = 1 - (1-\eta(e_1)^2), \quad \eta(e_2) = \lambda(1-\eta(e_1)^2), \quad \sec^2\theta\|PU\|^2 = 1-\eta(e_1)^2, \tag{3.12}$$

therefore from (3.10), (3.11) and (3.12), we get (3.1).

For $k = 2$, with similar computation, we get (3.9). Since

$$Ric_L(U) = K(e_1, e_2),$$

from (3.9) and by substituting δ from (3.5), we get (3.1).

b) Let $L' \subseteq T_pM$ be a $(k+1)$-plane section such that $L \subset L'$ and ξ_p be tangent to L'. We choose local orthogonal basis $\{e_1, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\} \subset T_pM$, L' spanned by $\{e_1, \ldots, e_{k+1}\}$, L spanned by $\{e_1, \ldots, e_k\}$, $e_1 = U$, $e_{k+1} = \lambda\Big(\xi - \eta(U)U\Big)$ and e_{n+1} is parallel to H at p. For $\alpha \in \{2, \ldots, k\}$, with similar computation, we have

$$\begin{aligned} K(e_1, e_\alpha) &\geq \frac{\delta}{2} + \eta(e_{k+1})^2 f_3 + 3\Big(g(e_\alpha, \phi e_1)\Big)^2 f_2 + \sum_{\substack{1\leq i<j\leq n \\ i\neq 1 \vee j\neq\alpha}} (h_{ij}^{n+1})^2 \\ &\quad + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\substack{i=2 \\ i\neq\alpha}}^{n} (h_{ii}^r)^2 + \sum_{r=n+2}^{2m+1}\sum_{\substack{1\leq i<j\leq n \\ i\neq 1 \vee j\neq\alpha}} (h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1} (h_{11}^r + h_{\alpha\alpha}^r)^2 \\ &\geq \frac{\delta}{2} + 3\Big(g(e_\alpha, \phi e_1)\Big)^2 f_2 + \eta(e_{k+1})^2 f_3. \end{aligned}$$

Therefore from (3.5) and the above inequality, we have

$$\begin{aligned} 2Ric_L(U) = 2\sum_{\alpha=2}^{k} K(e_1, e_\alpha) &\geq (k-1)\mathcal{A} + 6\|P_{k,L}U\|^2 f_2 \\ &\quad + (k-1)\left\{2(n-2)\eta(e_1)^2 + 2(n-1)\eta(e_{k+1})^2\right\} f_3. \end{aligned}$$

From (3.12) and the above inequality, we get (3.2).

c) Assume that the equality case of (3.1) for all unit vectors $U \in L$ is true. From

$$\begin{aligned} 2Ric_L(U) &= 2\sum_{\alpha=2}^{k} K(e_1, e_\alpha) = 2\left\{K(e_1, e_2) + \sum_{\alpha=3}^{k} K(e_1, e_\alpha)\right\}. \\ &= (k-1)\mathcal{A} + 6\|P_{k,L}U\|^2 f_2 \\ &\quad + \left\{2(n-2)(k-1)\eta(e_1)^2 + 2\Big((k-1)(n-1)-1\Big)\eta(e_2)^2\right\} f_3, \end{aligned}$$

(3.9) and (3.10), we have

$$\begin{array}{lll} h_{ii}^r = 0 & r \geq n+2, & 1 \leq i \leq n, \\ h_{ij}^r = 0 & r \geq n+1, & 1 \leq i < j \leq n, \end{array}$$

therefore

$$\sum_{\alpha=2}^{k} h_{11}^{n+1} h_{\alpha\alpha}^{n+1} = \sum_{\beta=2}^{k} \frac{b}{2}.$$

Since $h_{11}^{n+1} h_{\alpha\alpha}^{n+1} \geq \frac{b}{2}$, from the above equality, we have

$$h_{11}^{n+1} h_{\alpha\alpha}^{n+1} = \frac{b}{2},$$

in which $\alpha \in \{2, \cdots, k\}$. Therefore by lemma 2.1, we can complete the proof. The converse statement is straightforward. The proof of equality case of (3.2) is similar. □

Theorem 3.2. *Let $M^n (n \geq 3)$ be a θ-slant submanifold tangent to ξ in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$.*

a) *If $L \subseteq T_pM$ be a k-plane section($k \geq 2$) such that $\xi_p \in L$, then*

$$\begin{aligned} 2Ric_L(\xi) \;\geq\; & (k-1)\Big(2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 \\ & -3\|P\|^2 f_2 + 2(n-2)f_3\Big), \end{aligned} \tag{3.13}$$

in which H and τ are the mean curvature vector and the scalar curvature of M at p, respectively.

b) *The equality case holds identically if and only if respect to a suitable orthonormal basis $\{e_1, e_2, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\}$ is a basis of T_pM, the coefficients of the fundamental form h at p take the following form*

$$\left(\begin{array}{c|c} \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & \gamma & 0 & \ldots & 0 \\ 0 & 0 & \gamma & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma \end{pmatrix}_{k\times k} & O \\ \hline O & O \end{array}\right)_{n\times n} \qquad r = n+1,$$

and

$$h_{ij}^r = 0, \quad r \geq n+2, \quad 1 \leq i, j \leq n.$$

Proof. We choose local orthonormal basis $\{e_1, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\} \subset T_pM$, L spanned by $\{e_1, \ldots, e_k\}$, $e_1 = \xi$ and e_{n+1} is parallel to H at p. From (2.1), (2.2) and (2.5), we have

$$n^2\|H\|^2 = 2\tau - n(n-1)f_1 - 3\|P\|^2 f_2 + 2(n-1)f_3 + \|h\|^2, \tag{3.14}$$

Set

$$\delta := 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 - 3\|P\|^2 f_2 + 2(n-2)f_3. \tag{3.15}$$

Then from (3.14) we have

$$n^2\|H\|^2 = (n-1)(\|h\|^2 + \delta - 2f_1 + 2f_3), \tag{3.16}$$

and substituting (2.3) and (2.4) in the above equality, we get

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big(\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{1\le i\ne j\le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 \quad +\delta - 2f_1 + 2f_3\Big).$$

Now set

$$b := \delta - 2f_1 + 2f_3 + \sum_{1\le i\ne j\le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2,$$

$a_1 = h_{\alpha\alpha}^{n+1}$ and $a_2 = h_{11}^{n+1}$ for $\alpha \in \{2,\dots,n\}$, then from lemma 2.1, we have $a_1a_2 \ge \frac{b}{2}$. Therefore

$$\begin{aligned} h_{\alpha\alpha}^{n+1}h_{11}^{n+1} + f_1 - f_3 \quad \ge \quad & \frac{\delta}{2} + \sum_{1\le i<j\le n}(h_{ij}^{n+1})^2 \\ & +\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\le i<j\le n}(h_{ij}^{r})^2. \end{aligned} \tag{3.17}$$

On the other hand from (2.1) and the Gauss equation, for $\alpha \in \{2,\dots,n\}$, we have

$$f_1 - f_3 = K(e_1, e_\alpha) - \sum_{r=n+1}^{2m+1} h_{11}^{r}h_{\alpha\alpha}^{r} + \sum_{r=n+1}^{2m+1}(h_{1\alpha}^{r})^2.$$

Therefore

$$f_1 - f_3 + h_{11}^{n+1}h_{\alpha\alpha}^{n+1} = K(e_1, e_\alpha) - \sum_{r=n+2}^{2m+1} h_{11}^{r}h_{\alpha\alpha}^{r} + \sum_{r=n+1}^{2m+1}(h_{1\alpha}^{r})^2.$$

By comparing the above equality and (3.17), we obtain

$$\begin{aligned} K(e_1, e_\alpha) \quad - \quad & \sum_{r=n+2}^{2m+1} h_{11}^{r}h_{\alpha\alpha}^{r} + \sum_{r=n+1}^{2m+1}(h_{1\alpha}^{r})^2 \\ \ge \quad & \frac{\delta}{2} + \sum_{1\le i<j\le n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\le i<j\le n}(h_{ij}^{r})^2. \end{aligned}$$

By using (3.8), we have

$$\begin{aligned} K(e_1, e_\alpha) \quad \ge \quad & \frac{\delta}{2} + \sum_{\substack{1\le i<j\le n \\ i\ne 1 \vee j\ne\alpha}}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\substack{i=2 \\ i\ne\alpha}}^{n}(h_{ii}^{r})^2 \\ & + \sum_{r=n+2}^{2m+1}\sum_{\substack{1\le i<j\le n \\ i\ne 1 \vee j\ne\alpha}}(h_{ij}^{r})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^{r} + h_{\alpha\alpha}^{r})^2 \ge \frac{\delta}{2}. \end{aligned}$$

Therefore

$$Ric_L(\xi) = \sum_{\alpha=2}^{k} K(e_1, e_\alpha) \geq (k-1)\frac{\delta}{2}.$$

By substituting δ from (3.15) in the above equality, we get (3.13). □

Corollary 3.3. *Let $M^n (n \geq 3)$ be a θ-slant submanifold tangent to ξ in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$.*

a) *For any integer $2 \leq k \leq n$, we have*

$$2\Theta_k \geq 2\tau - \mathcal{B} + \inf_{L,U} \{B_1,\ B_2,\ B_3\},$$

in which L runs over all k-plane section in T_pM and U runs over all unit vectors in L and

$$\begin{aligned}
\mathcal{B} &:= \tfrac{n^2(n-2)}{n-1}\|H\|^2 + (n+1)(n-2)f_1 + 3\|P\|^2 f_2,\\
B_1 &:= \tfrac{1}{k-1}\Big\{\Big(2(n-2)(k-1)\Big(1-\sec^2\theta\|PU\|^2\Big)\\
&\quad +2\lambda^2\Big((k-1)(n-1)-1\Big)\sec^4\theta\|PU\|^4\Big)f_3 + 6\|P_{k,L}U\|^2 f_2\Big\},\\
B_2 &:= \Big(2(n-2)\Big(1-\sec^2\theta\|PU\|^2\Big) + 2\lambda^2(n-1)\sec^4\theta\|PU\|^4\Big)f_3 + \tfrac{6}{k-1}\|P_{k,L}U\|^2 f_2,\\
B_3 &:= 2(n-2)f_3, \quad \lambda := \tfrac{1}{\|\xi-\eta(U)U\|},
\end{aligned}$$

where Θ_k, H and τ are the Riemannian invariant, the mean curvature vector and the scalar curvature of M at p, respectively.

b) *For any integer $2 \leq k \leq n$, we have*

$$2\delta_k \leq \mathcal{B} - \inf_{L,U} \{B_1,\ B_2,\ B_3\},$$

where $\delta_k = \tau - \Theta_k$.

c) *The equality case of (a) and (b) holds identically if and only if respect to a suitable orthonormal basis $\{e_1, e_2, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\}$ is a basis of T_pM, the coefficients of the fundamental form h at p take the following form*

$$\left(\begin{array}{c|c}
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0\\
0 & \gamma & 0 & \ldots & 0\\
0 & 0 & \gamma & \ldots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \ldots & \gamma
\end{pmatrix}_{k\times k} & O\\
\hline
O & O
\end{array}\right)_{n\times n} \qquad r = n+1,$$

and $h^r_{ij} = 0$, $r \geq n+2$, $1 \leq i, j \leq n$.

Corollary 3.4. *Let $M^n (n \geq 3)$ be a θ-slant submanifold tangent to ξ in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$.*

a) *If $U \in T_pM$ be an unit vector, linearly independent of ξ_p, then*

$$2Ric(U) \geq (n-1)\mathcal{A} + 2(n-2)\Big((n-1)\Big(1 - \sec^2\theta\|PU\|^2\Big) + \lambda^2 n \sec^4\theta\|PU\|^4\Big) f_3 + 6\|P_nU\|^2 f_2. \tag{3.18}$$

in which

$$\mathcal{A} := 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 - 3\|P\|^2 f_2,$$
$$\lambda := \frac{1}{\|\xi - \eta(U)U\|},$$

H and τ are the Riemannian invariant, the mean curvature vector and the scalar curvature of M at p, respectively.

b) *The equality case holds identically if and only if respect to a suitable orthonormal basis $\{e_1, e_2, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\}$ is a basis of T_pM, the coefficients of the fundamental form h at p take the following form*

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & \gamma & 0 & \ldots & 0 \\ 0 & 0 & \gamma & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma \end{pmatrix}_{n\times n} \qquad r = n+1,$$

and $h^r_{ij} = 0$, $r \geq n+2$, $1 \leq i, j \leq n$.

Proof. In theorem 3.1, let $L = T_pM$. We get (3.18) from (3.1) with $k = n$. □

Corollary 3.5. *Let $M^n (n \geq 3)$ be a θ-slant submanifold tangent to ξ in a Sasakian space form $\overline{M}^{2m+1}(c)$.*

a) *If $U \in T_pM$ be an unit vector, linearly independent of ξ_p, then*

$$G \geq Ric(U) \geq G - \Big\{\frac{n^2(2n-3)}{4}\|H\|^2 + \frac{n(n-1)^2}{2}(\frac{c+3}{4}) - (n-1)\tau - 3\Big\{\|P_nU\|^2 - \frac{1}{2}\|P\|^2 - \|PU\|^2 + \frac{(n-1)^2}{3} - \frac{n(n-2)}{3}\sec^2\theta\|PU\|^2 + \frac{\lambda^2}{3}n(n-2)\sec^4\theta\|PU\|^4\Big\}(\frac{c-1}{4})\Big\}$$

in which

$$G = \frac{n^2}{4}\|H\|^2 + (n-1)(\frac{c+3}{4}) + \big\{(1-n) + (n-2)\sec^2\theta\|PU\|^2 + 3\|PU\|^2\big\}(\frac{c-1}{4}),$$

$$\lambda := \frac{1}{\|\xi - \eta(U)U\|},$$

H and τ are the mean curvature vector and the scalar curvature of M at p, respectively.

b) *The following assertions are equivalent:*
(i) For each unit vector $U \in T_pM$, we have $Ric(U) = G$ at p.

(ii) On M, we have

$$\tau = \frac{1}{n-1}\Big\{\frac{n^2(2n-3)}{4}\|H\|^2 + \frac{n(n-1)^2}{2}(\frac{c+3}{4}) - 3\Big\{\|P_nU\|^2 - \frac{1}{2}\|P\|^2 - \|PU\|^2 + \frac{(n-1)^2}{3} - \frac{n(n-2)}{3}\sec^2\theta\|PU\|^2 + \frac{\lambda^2}{3}n(n-2)\sec^4\theta\|PU\|^4\Big\}(\frac{c-1}{4})\Big\}$$

(iii) p is a totally geodesic point.

Proof. It is obvious from Theorem 1.2 and Corollary 3.4 by taking $f_1 = \frac{(c+3)}{4}$ and $f_2 = f_3 = \frac{(c-1)}{4}$. □

4 The Ricci curvature of invariant and anti-invariant submanifolds tangent to structure vector field in generalized Sasakian space forms

F. Malek and V. Nejadakbary obtained some results for anti-invariant submanifolds in generalized Sasakian space forms (see [13]). If M be an invariant submanifold tangent to ξ in a generalized Sasakian space form and $L \subset T_pM$ be a k-plane section normal to ξ_p, we obtain same result as theorem 1.1. In this section we are going to prove other results for invariant submanifolds in generalized Sasakian space forms.

Corollary 4.1. *Let $M^n (n \geq 3)$ be an invariant submanifold tangent to ξ in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$.*

a) *If $L \subseteq T_pM$ be a k-plane section $(k \geq 2)$ tangent to ξ_p and unit vector $U \in L$ is normal to ξ_p, then*

$$2Ric_L(U) \geq (k-1)\mathcal{A} + +2\Big((k-1)(n-1)-1\Big)f_3 + 6\|P_{k,L}U\|^2 f_2.$$

in which

$$\mathcal{A} := 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 - 3\|P\|^2 f_2,$$

H and τ are the mean curvature vector and the scalar curvature of M at p, respectively.

b) *The equality case holds identically if and only if respect to a suitable orthonormal basis $\{e_1, e_2, \ldots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \ldots, e_n\}$ is a basis of T_pM, the* *coefficients of the fundamental form h at p take the following form*

$$\left(\begin{array}{c|c} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \gamma & 0 & \dots & 0 \\ 0 & 0 & \gamma & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma \end{pmatrix}_{k\times k} & O \\ \hline O & O \end{array}\right)_{n\times n} \qquad r = n+1,$$

and $h^r_{ij} = 0$, $r \geq n+2$, $1 \leq i,j \leq n$.

Proof. It is obvious from theorem 3.1 because $\eta(U) = 0$, $\eta\Big(\xi_p - \eta(U)U\Big) = 1$, $\lambda = 1$, and from (3.12), $\sec^2\theta \|PU\|^2 = 1$. □

Corollary 4.2. *Let* $M^n (n \geq 3)$ *be an invariant submanifold tangent to* ξ *in a generalized Sasakian space form* $\overline{M}^{2m+1}(f_1, f_2, f_3)$.

a) *If* $L \subset T_pM$ *be a* $(n-1)$*-plane section normal to* ξ_p*, then for all unit vectors* $U \in L$*, we have*

$$\frac{2}{n-2} Ric_L(U) \geq 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 + 2(n-1)f_3 - 3\|P\|^2 f_2,$$

in which H *and* τ *are the mean curvature vector and the scalar curvature of* M *at* p*, respectively.*

b) *The equality case holds identically if and only if with respect to a suitable orthonormal basis* $\{e_1, e_2, \dots, e_{2m+1}\}$ *of* $T_p\overline{M}$ *such that* $\{e_1, \dots, e_n\}$ *is a basis of* T_pM*, the coefficients of the fundamental form* h *at* p *take the following form*

$$\left(\begin{array}{c|c} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \gamma & 0 & \dots & 0 \\ 0 & 0 & \gamma & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma \end{pmatrix}_{(n-1)\times (n-1)} & O \\ \hline O & O \end{array}\right)_{n\times n} \qquad r = n+1,$$

and $h^r_{ij} = 0$, $r \geq n+2$, $1 \leq i,j \leq n$.

Proof. Let $U \in L$ be an unit vector. Since ξ is normal to L, therefore $\eta(U) = 0$, $\eta\Big(\xi_p - \eta(U)U\Big) = 1$, $\lambda = 1$, and from (3.12), $\sec^2\theta\|PU\|^2 = 1$. Also $\phi(U)$ is normal to U. We choose local orthonormal basis $\{e_1, \dots, e_{2m+1}\}$ of $T_p\overline{M}$ such that $\{e_1, \dots, e_n\} \subset T_pM$, L spanned by $\{e_1, \dots, e_{n-1}\}$, $e_1 = U$, $e_2 = \phi(U)$, $e_n = \xi_p$ and e_{n+1} is parallel to H at p. Therefore $\|P_{(n-1),L}U\|^2 = 0$. The proof is completed from part (b) of theorem 3.1. □

Example 4.1. The standard complex Euclidean space $\mathbb{C}^n$ with coordinates $(z_1, z_2, \dots, z_n)$ such that for any $1 \leq i \leq n, z_i \in \mathbb{C}$ is an almost complex manifold with almost structure J induced by multiplication by $\sqrt{-1}$. In [1], it is shown that $\overline{M} = \mathbb{R} \times_f \mathbb{C}^n$ is a generalized Sasakian space form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t) > 0$. We define submanifold $M = \mathbb{R} \times_f I^n$ of $\overline{M}$ such that $I = (-1,1) \times (-1,1)$. It is easy to see that M is an invariant submanifold in $\overline{M}(f_1, f_2, f_3)$. A lower bound for the Ricci curvature of this submanifold and any k-plane section $L \subseteq T_pM$ can be obtained by Theorems 3.1, 3.2 and Corollary 3.4. Also The lower bound for the Riemannian invariant Θ_k on M can be obtained by Corollary 3.3.

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