Pointwise hemi-slant submanifolds of almost contact metric 3-structures

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Abstract. In this paper we introduce pointwise hemi 3-slant submanifolds of almost contact metric 3-structures. We characterize these submanifolds and give non-trivial examples of such submanifolds. In addition, we prove that the distribution spanned by the structure vector fields is totally geodesic and integrable. Moreover, we investigate the integrability conditions for other distributions.

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1 Introduction

As a generalization of slant submanifolds, F. Etayo [5] and B.-Y. Chen and O.J. Garay [4] introduced and characterized pointwise (quasi) slant submanifolds of Hermitian manifolds. After that, many authors have studied pointwise slant submanifolds in various structures [6, 11, 13].

On the other hand, hemi-slant submanifolds are the special case of bi-slant submanifolds [1, 3, 8, 9]. In this paper, we have extended that concept to the pointwise hemi-slant submanifolds of almost metric contact 3-structures. In fact, these submanifolds are the generalizations of invariant, anti-invariant, semi-invariant, slant and pointwise slant submanifolds of almost contact and almost contact 3-structure manifolds.

Let (N, g) be a submanifold of a Riemannian manifold (M, g). We denote the Levi-Civita connection of M and N by $\tilde{\nabla}$ and ∇, respectively. If $T^{\perp}N$ is the normal bundle of N, then for all $X, Y \in TN$ and $U \in T^{\perp}N$, the Gauss and Weingarten formulas imply

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \tilde{\nabla}_Y U = D_X U - A_U X, \tag{1.1}$$

where σ is the second fundamental form of the submanifold, A_U is the shape operator in the direction of U and D denotes covariant differentiation with respect to the

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normal connection. Moreover, it is well-known that the second fundamental form σ and shape operator A_U satisfy in the following relation

$$g(\sigma(X,Y),U) = g(A_U X, Y), \tag{1.2}$$

and if they be equal to zero on N, then N is said to be totally geodesic.

Let (M, g) be an odd dimensional Riemannian manifold with a vector field ξ, a 1-form η and a (1,1)-tensor field φ such that for all $X, Y \in TM$

$$\eta(\xi) = 1, \quad \varphi^2(Y) = -Y + \eta(Y)\xi, \tag{1.3}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{1.4}$$

Then $(M, g, \xi, \eta, \varphi)$ is an almost contact metric manifold [2].

Let M admits three almost contact metric structures $(\xi_r, \eta_r, \varphi_r)$, $r = 1,2,3$, satisfying in the following equations:

$$\eta_r(\xi_s) = 0, \; \varphi_r(\xi_s) = -\varphi_s(\xi_r) = \xi_t, \; \eta_r(\varphi_s) = -\eta_s(\varphi_r) = \eta_t, \tag{1.5}$$

$$\varphi_r o \varphi_s - \eta_s \otimes \xi_r = -\varphi_s o \varphi_r + \eta_r \otimes \xi_s = \varphi_t, \tag{1.6}$$

$$g(\varphi_r X, \varphi_r Y) = g(X,Y) - \eta_r(X)\eta_r(Y), \tag{1.7}$$

in which (r, s, t) is a cyclic permutation of $(1,2,3)$ and $X, Y \in TM$. Then M has an almost contact metric 3-structure $(\xi_r, \eta_r, \varphi_r)_{r\in\{1,2,3\}}$ [14]. By using Equation (1.7), one can easily prove that φ_r is skew symmetric with respect to the metric g, i.e.

$$g(\varphi_r X, Y) = -g(X, \varphi_r Y). \tag{1.8}$$

In the present paper, we first define a pointwise hemi 3-slant submanifold of an almost contact manifold with 3-structures and give some examples to show the existence of such submanifolds in Section 2. In addition, we characterize these submanifolds. In Section 3, we study pointwise hemi 3-slant submanifold of 3-cosymplectic manifolds and show that the distribution which is spanned by $\{\xi_1, \xi_2, \xi_3\}$ is totally geodesic and integrable. Finally, the integrability of other distributions of a pointwise hemi 3-slant submanifold of 3-cosymplectic manifolds is investigated.

2 Pointwise hemi-slant submanifolds

Let N be a submanifold of an almost contact metric 3-structure manifold $(M, g, \xi_r, \eta_r, \varphi_r)_{r\in\{1,2,3\}}$. Then for any $Y \in TN$, we put $\varphi_r Y = F_r Y + P_r Y$ where F_r is tangential projection of φ_r on TN and P_r is normal projection of φ_r on $T^\perp N$.

Moreover, if $W \in T^\perp N$, we write $\varphi_r W = f_r W + p_r W$ in which f_r and p_r are tangential and normal projection of φ_r, respectively.

By using Equation (1.8), for any $X, Y \in TN$ and $V, W \in T^{\perp}N$, we can describe the behavior of maps F_r, P_r, f_r and p_r with respect to the metric g as following:

$$g(F_rX, Y) = -g(X, F_rY), \quad g(p_rV, W) = -g(V, p_rW), \tag{2.1}$$

$$g(P_rX, V) = -g(X, f_rV), \quad g(f_rV, Y) = -g(V, p_rY). \tag{2.2}$$

Definition 2.1. [7] A submanifold N of an almost contact 3-structure $(M, g, \xi_r, \eta_r, \varphi_r)$ is said to be a pointwise 3-slant submanifold if at any point $x \in N$ the angle between $\varphi_r(Y_x)$ and T_xN is independent of choice of Y_x for any non-zero $Y_x \in T_xN \backslash \{\xi_{r_x}\}$ and $r = 1, 2, 3$.

Definition 2.2. Let N be a submanifold of an almost contact 3-manifold $(M, g, \xi_r, \eta_r, \varphi_r)$. Then, N is called a pointwise hemi 3-slant submanifold if there exist three orthogonal distributions $\mathcal{D}_\theta, \mathcal{D}^{\perp}$ and Ξ on N such that

(a) $TN = \mathcal{D}_\theta \oplus \mathcal{D}^{\perp} \oplus \Xi$, where $\Xi = Span\{\xi_1, \xi_2, \xi_3\}$;

(b) $\mathcal{D}^{\perp}$ is anti-invariant with respect to $\varphi_r, \forall r = 1, 2, 3$, i.e. $\varphi_r(\mathcal{D}^{\perp}) \subset T^{\perp}N$;

(c) $\mathcal{D}_\theta$ is a pointwise 3-slant distribution. That means for any $Y \in \mathcal{D}_\theta$ the angle between $\varphi_r(Y)$ and $\mathcal{D}_\theta$ is independent of the choice of Y.

It should be noted that if $dim(\mathcal{D}^{\perp}) = 0$, then N is a pointwise 3-slant submanifold and if $dim(\mathcal{D}_\theta \oplus \Xi) = 0$, then N is an anti-invariant submanifold. In this paper the dimensions of all the three distributions are non-zero and in this case the submanifold is called a *proper pointwise hemi 3-slant* submanifold.

Now, we give two examples of proper pointwise hemi 3-slant submanifolds of almost contact metric 3-manifolds.

Example 2.3. Let the (1,1)-tensor fields $\varphi_1, \varphi_2, \varphi_3$ are defined on $M = \mathbb{R}^{15}$ and $g = \Sigma_{i=1}^{15} dx^i \otimes dx^i$ as following

$$\varphi_1(\partial_{4k+1}) = \partial_{4k+2}, \varphi_1(\partial_{4k+2}) = -\partial_{4k+1}, \varphi_1(\partial_{4k+3}) = \partial_{4k+4}, \varphi_1(\partial_{4k+4}) = -\partial_{4k+3},$$

$$\varphi_1(\partial_{13}) = \partial_{14}, \varphi_1(\partial_{14}) = -\partial_{13}, \varphi_1(\partial_{15}) = 0,$$

$$\varphi_2(\partial_{4k+1}) = \partial_{4k+3}, \varphi_2(\partial_{4k+2}) = -\partial_{4k+4}, \varphi_2(\partial_{4k+3}) = -\partial_{4k+1}, \varphi_2(\partial_{4k+4}) = \partial_{4k+2},$$

$$\varphi_2(\partial_{13}) = \partial_{15}, \varphi_2(\partial_{15}) = -\partial_{13}, \varphi_2(\partial_{14}) = 0,$$

for $k = 0, 1, 2$. Moreover, $\xi_1 = \partial_{15}, \xi_2 = \partial_{14}, \xi_3 = \partial_{13}$ and η_r's be the dual of ξ_r's for $r = 1, 2, 3$ and $\varphi_3 = \varphi_1 o \varphi_2 - \eta_2 \otimes \xi_1$. $(M, g, \xi_r, \eta_r, \varphi_r)_{r \in \{1,2,3\}}$ is an almost contact metric 3-structure manifold.

Now, let $f, h \in C^{\infty}(\mathbb{R}^{15})$. Then we define a 6-dimensional submanifold N given by the immersion

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = (t_1 f, t_2 h, t_2 h, t_2 h, t_3, 0, 0, 0, t_1 h, 0, 0, 0, t_4, t_5, t_6).$$

By taking $\mathcal{D}_\theta = \mathrm{Span}\{\mathrm{X}_1 = \mathrm{f}\partial_1 + \mathrm{h}\partial_9, \mathrm{X}_2 = \mathrm{h}(\partial_2 + \partial_3 + \partial_4)\}$, $\mathcal{D}^{\perp} = \mathrm{Span}\{\mathrm{X}_3 = \partial_5\}$ and $\Xi = \mathrm{Span}\{\mathrm{X}_4 = \partial_{13}, \mathrm{X}_5 = \partial_{14}, \mathrm{X}_6 = \partial_{15}\}$. Then it is clear that $\mathcal{D}_\theta$ is a pointwise 3-slant distribution by slant function $\Theta = cos^{-1}(\frac{h}{\sqrt{3}\sqrt{h^2+f^2}})$ and $\mathcal{D}^{\perp}$ is an anti-invariant distribution. Therefore, N is a pointwise hemi 3-slant submanifold of $\mathbb{R}^{15}$.

Example 2.4. Let N be a pointwise 3-slant submanifold of a 3-structure manifold $(M, g, \ \xi_r, \eta_r, \varphi_r)_{r\in\{1,2,3\}}$. For instance, it can be the structure in the Example 2 of [7], i.e. $(M, g) = (\mathbb{R}^{11}, \Sigma_{i=1}^{11} dx^i \otimes dx^i)$ and $N = (v \sin f, 0, 0, 0, ku \sin f, ku \sin f, ku \sin f, v \cos f, 0, 0, 0)$.

Now, let $N' = (y, 0, 0, 0,)$ be a submanifold of a 4m-dimensional hyper-Kaehler manifold $(M' = \mathbb{R}^{4m}, g', I, J, K)$ which is introduced in a example of [10]. It is obvious that $I(TN') \subset T^{\perp}N'$, $J(TN') \subset T^{\perp}N'$, $K(TN') \subset T^{\perp}N'$.

By using the above notations, we suppose that $(\tilde{M}, \tilde{g}) = (M \times M', g \otimes g')$ and $\tilde{N} = N \times N'$. Therefore, $\tilde{M}$ is a $(4m + 11)$-dimensional almost contact 3-structure manifold. We take $\mathcal{D}_\theta \otimes \Xi = TN$ and $\mathcal{D}^{\perp} = TN'$. Thus $\tilde{N}$ is a pointwise hemi 3-slant submanifold of $\tilde{M}$. The slant function of slant distribution $\mathcal{D}_\theta$ is $\Theta = \cos^{-1}(\frac{\cos \tilde{f}}{k\sqrt{3}})$, where $k \in \mathbb{R}^+$ and $\tilde{f} \in C^\infty(\tilde{M})$ is the smooth extension of the function f.

Theorem 2.1. *[7] Let N be a submanifold of an almost contact metric 3-structure manifold $(M, g, \xi_r, \eta_r, \varphi_r)$ such that ξ_r's are normal to N for $r = 1, 2, 3$. Then, N is a pointwise 3-slant submanifold if and only if there exists a real function Θ on N such that*

$$F_r F_s Y = -\cos^2 \Theta Y, \ \forall Y \in TN, \forall\, r, s \in \{1, 2, 3\}. \tag{2.3}$$

By using Theorem 2.1, it is easy to prove the following corollary.

Corollary 2.2. *Let $\mathcal{D}$ be a distribution on a submanifold of an almost contact metric 3-structure manifold $(M, g, \xi_r, \eta_r, \varphi_r)$ such that $\mathcal{D}$ is orthogonal to the distribution $< \xi_1, \xi_2, \xi_3 >$. Then $\mathcal{D}$ is a pointwise 3-slant distribution if and only if there exists a function $\rho \in [-1, 0)$ such that for all $Y \in \mathcal{D}$, $F_r F_s Y = \rho Y, \forall\, r, s \in \{1, 2, 3\}$. Furthermore, if Θ is the slant function, then $\rho = -\cos^2 \Theta$.*

Theorem 2.3. *Let N be a submanifold of an almost contact metric 3-structure manifold $(M, g, \xi_r, \eta_r, \varphi_r)$ which ξ_r's are tangent to N for $r = 1, 2, 3$. Then N is a pointwise hemi 3-slant submanifold if and only if there exists a real-valued function $\rho \in [-1, 0)$ such that for all $r, s \in \{1, 2, 3\}$, the following conditions hold:*

(a) $\mathcal{D} = \{Y \in TN \backslash < \xi_1, \xi_2, \xi_3 > | F_r F_s Y = \rho Y\}$ is a distribution on N;

(b) $\forall Y \in TN$ orthogonal to distribution $\mathcal{D} \oplus < \xi_1, \xi_2, \xi_3 >$, $F_r Y = 0$. Moreover, in this case if Θ is the slant function, then $\rho = -\cos^2 \Theta$.

Proof. Let N be a pointwise hemi 3-slant submanifold and $TN = \mathcal{D}_\theta \oplus \mathcal{D}^\perp \oplus \Xi$. From Corollary 2.2, for all $Y \in \mathcal{D}_\theta$, we have $F_r F_s Y = \rho Y$. By taking $\mathcal{D} = \mathcal{D}_\theta$, since $\mathcal{D}^\perp$ is anti-invariant, $\forall Z \in \mathcal{D}^\perp$, $F_r Z = 0$.

Conversely, from (a) and Corollary 2.2, we get $\mathcal{D}$ is a pointwise 3-slant distribution. On the other hand, (b) implies that there exists an anti-invariant distribution on N and since $\Xi \subset TN$ and does not satisfies both the conditions; we conclude that N is a pointwise hemi 3-slant submanifold. □

3 Pointwise hemi 3-slant submanifolds of 3-cosymplectic manifolds

An almost contact metric 3-structure manifold $(M, g, \xi_r, \eta_r, \varphi_r)$ is called a 3-cosymplectic manifold if

$$(\tilde{\nabla}_X \varphi_r)W = 0, \quad \forall\, X, W \in TM,$$

for $r = 1, 2, 3$. It is well known that [12] in 3-cosymplectic manifolds the following equation holds:

$$\tilde{\nabla}_W \xi_r = 0, \ \forall\, W \in TM. \tag{3.1}$$

For any $X, W \in TN$, the covariant derivative of the maps F_r and P_r are defined by

$$(\nabla_W F_r)X = \nabla_W F_r X - F_r \nabla_W X, \tag{3.2}$$

$$(D_W P_r)X = D_W P_r X - P_r \nabla_W X. \tag{3.3}$$

Let N be a submanifold of a 3-cosymplectic manifold $(M, g, \xi_r, \eta_r, \varphi_r)$. Then from (3.2), (3.3) and Gauss and Weingarten formulas, we get

$$(\nabla_W F_r)X = A_{P_r X} W + f_r \sigma(X, W), \tag{3.4}$$

$$(D_W P_r)X = P_r \sigma(X, W) - \sigma(W, F_r X). \tag{3.5}$$

Remark 3.1. In Example 2.3, according to the definition of M and g, the connection $\tilde{\nabla}$ is flat. So, we have $\tilde{\nabla}\varphi_r = 0$ and the 3-structure manifold $(M, g, \xi_r, \eta_r, \varphi_r)$ is 3-cosymplectic.

Theorem 3.1. *Let N be a pointwise hemi 3-slant submanifold of a 3-cosymplectic manifold $(M, g, \xi_r, \eta_r, \varphi_r)$. Then the distribution Ξ is integrable and totally geodesic.*

Proof. Since M is a 3-cosymplectic manifold and the connection $\tilde{\nabla}$ is symmetric, thus we have

$$[\xi_r, \xi_s] = \tilde{\nabla}_{\xi_s} \xi_r - \tilde{\nabla}_{\xi_r} \xi_s,$$

and (3.1) implies that $[\xi_r, \xi_s] = 0 \in \Xi$. Hence, Ξ is an integrable distribution.

Moreover, from Gauss and Weingarten formulas, we get $0 = \tilde{\nabla}_{\xi_s} \xi_r = \nabla_{\xi_s} \xi_r + \sigma(\xi_s, \xi_r)$. This means that $\sigma(\xi_s, \xi_r) = 0$ and therefore Ξ is totally geodesic. □

Theorem 3.2. *The distribution $\mathcal{D}_\theta \oplus \mathcal{D}^\perp$ of a pointwise hemi 3-slant submanifold of a 3-cosymplectic manifold $(M, g, \xi_r, \eta_r, \varphi_r)$ is integrable.*

Proof. Since $\tilde{\nabla}$ is symmetric and compatible with respect to g, $\forall\, Y, Z \in \mathcal{D}_\theta \oplus \mathcal{D}^\perp$ and $r = 1, 2, 3$, we have

$$g(\xi_r, [Y, Z]) = g(\xi_r, \tilde{\nabla}_Z Y - \tilde{\nabla}_Y Z) = -g(Y, \tilde{\nabla}_Z \xi_r) + g(Z, \tilde{\nabla}_Y \xi_r). \tag{3.6}$$

Using (3.1) and (3.6), we find that $g(\xi_r, [Y, Z]) = 0$, thus $[Y, Z] \in \mathcal{D}_\theta \oplus \mathcal{D}^\perp$, which means that the distribution $\mathcal{D}_\theta \oplus \mathcal{D}^\perp$ is integrable. □

Lemma 3.3. *Let N be a pointwise hemi 3-slant submanifold of 3-cosymplectic manifold $(M, g, \xi_r, \eta_r, \varphi_r)$. Then for any $Y, W \in \mathcal{D}^\perp$ the shape operator satisfies*

$$A_{P_rY}W = A_{P_rW}Y, \tag{3.7}$$

where P_r is the normal projection of φ_r.

Proof. Let $(M, g, \xi_r, \eta_r, \varphi_r)$ be a 3-cosymplectic manifold and $Y, W \in \mathcal{D}^\perp$. Then, for any $X \in TN$, we have

$$g(A_{\varphi_rY}W, X) = g(\sigma(X, W), \varphi_rY). \tag{3.8}$$

Using Gauss formula, we derive

$$g(A_{\varphi_rY}W, X) = -g(\varphi_r\tilde{\nabla}_XW, Y) + g(\nabla_XW, \varphi_rY).$$

Since $\mathcal{D}^\perp$ is anti-invariant, then $\varphi_rY \in T^\perp M$ and hence the last term in the right and side of above relation is identically zero. Then, by using cosymplectic characteristic equation we find

$$g(A_{\varphi_rY}W, X) = -g(\tilde{\nabla}_X\varphi_rW, Y) = g(A_{\varphi_rW}X, Y).$$

Since A is symmetric, so we get

$$g(A_{\varphi_rY}W, X) = g(A_{P_rW}Y, X),$$

and the proof is complete. □

Theorem 3.4. *The anti-invariant distribution $\mathcal{D}^\perp$ of a pointwise hemi 3-slant submanifold of 3-cosymplectic manifold $(M, g, \xi_r, \eta_r, \varphi_r)$ is always integrable.*

Proof. Let $Y, Z \in \mathcal{D}^\perp$. Then for $r = 1, 2, 3$, we have

$$\varphi_r[Y, Z] = F_r[Y, Z] + P_r[Y, Z] = F_r\nabla_ZY - F_r\nabla_YZ + P_r[Y, Z].$$

On the other hand, since $(M, g, \xi_r, \eta_r, \varphi_r)$ is a 3-cosymplectic manifold and $F_rZ = 0$, then we have

$$(\tilde{\nabla}_YF_r)Z = F_r\nabla_YZ - A_{P_rZ}Y = 0.$$

So, $\varphi_r[Y, Z] = -A_{P_rZ}Y + A_{P_rY}Z + P_r[Y, Z]$. From Lemma 3.3, we obtain

$$\varphi_r[Y, Z] = P_r[Y, Z],$$

thus $[Y, Z] \in \mathcal{D}^\perp$. □

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