On a class of Finsler metrics with special curvature properties

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Abstract. The χ-curvature is an important non-Riemannian quantity. It interacts with the flag curvature in a mysterious way. In this paper, we study Finsler metrics with vanishing χ-curvature.

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1 Introduction

Finsler geometry is just Riemannian geometry without the quadratic restriction on its metrics[3]. There are several important non-Riemannian quantities in Finsler geometry, such as the cartan torsion, the S-curvature, the **H**-curvature and the χ-curvature. The χ-curvature is determined by the S-curvature in the following way[10]

$$\chi_i := S_{\cdot i|j} y^j - S_{|i}, \tag{1.1}$$

where $\chi := \chi_i dx^i$ and S denotes the χ-curvature and the S-curvature of F, " $\cdot$ " and "$|$" denote the vertical and horizontal covariant derivatives, respectively, with respect to the Chern connection. These quantities vanish for Riemannian metrics, hence they are said to be *non-Riemannian.* The χ-curvature gives a measure of failure of a Finsler metric of scalar curvature to be of isotropic flag curvature. Thus the quantity χ deserves further investigation.

One of the fundamental problems in Finsler geometry is to understand Finsler metrics of special curvature properties. Many Finslerian geometers have studied Finsler metrics with special curvature properties [2, 6, 10, 13, 16, 18]. Furthermore, the χ-curvature is closely related to the Riemann tensor in the following way [4, 5, 10]

$$\chi_i = -\frac{1}{3}\sum_j \left(2R^j{}_{i\cdot j} + R^j{}_{j\cdot i}\right).$$

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This identity leads to a well known result which is proved by Z. Shen: if a n-dimensional Finsler metric F is of scalar curvature. Then for a 1-form θ, the χ-curvature χ almost vanishes given by

$$\chi_i = -(n+1)F^2\big(\frac{\theta}{F}\big)_{y^i} \tag{1.2}$$

if and only if the flag curvature is weakly isotropic given by

$$K = \frac{3\theta}{F} + \sigma(x). \tag{1.3}$$

In particular, $\chi = 0$ if and only if $K = \sigma$(=constant when $n \geq 3$). Moreover, Z. Shen proved that for a compact Finsler manifold with $\chi = 0$, if the flag curvature $K < 0$, then it must be Riemannian[10]. Moreover, the χ-curvature is also closely related to the other non-Riemannian quantities such as S-curvature, H-curvature and Cartan tensor, we refer to the reader to [2, 7, 10].

One of the fundamental problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature because Finsler metrics of constant flag curvature are the natural extension of Riemannian metrics of constant sectional curvature. From the above results, we can see that Finsler metrics with vanishing χ-curvature is closely related to Finsler metrics of constant flag curvature. The phenomenon inspires us to study Finsler metrics with vanishing χ-curvature.

In this paper, we mainly study general (α, β)-metrics vanishing χ-curvature, which were introduced by C. Yu and the author [14]. This class of Finsler metrics conclude (α, β)-metrics, spherically symmetric Finsler metrics [17, 16], part of Bryant's metrics[1] and part of fourth root metrics. That is to say, general (α, β)-metrics make up of a much large class of Finsler metrics, which makes it possible to find out more Finsler metrics to be of great properties. Firstly, We shall make the following assumption:

A: α is a Riemannian metric with constant sectional curvature μ and β is a 1-form satisfying

$$^{\alpha}R^i{}_j = \mu(\alpha^2\delta^i{}_j - y^i y_j), \quad b_{i|j} = \lambda a_{ij}, \tag{1.4}$$

where $\lambda = \lambda(x)$ is a scalar function with $\lambda^2 = \kappa - \mu b^2 > 0$ for some constant κ.

The condition **A** is natural[12]. Note that if α and β satisfy (1.4) with $\lambda = 0$, then β is parallel with respect to α. We shall only consider the case when $\lambda^2 > 0$ and show the following:

Theorem 1.1. *Let $F = \alpha\phi(b^2, s)$, $s = \frac{\beta}{\alpha}$, be a general (α, β)-metric on an n-dimensional manifold M. Suppose that α and β satisfy (1.4) with $\lambda^2 > 0$. Then F has vanishing χ-curvature if and only if*

$$(n+1)\Phi + (b^2 - s^2)C_2 = 0, \tag{1.5}$$

where $C_2 = \frac{\partial C}{\partial s}$, C is given by (2.11) and

$$\Phi := \lambda^2\big\{2(E_1 - sE_{12}) - E_{22} + 2H\big[E - sE_2 + (b^2 - s^2)E_{22}\big]\big\} - \mu(E - sE_2), \tag{1.6}$$

where E and H are given by (2.6) and (2.7), respectively. Here $E_1 := \frac{\partial E}{\partial b^2}$.

Especially, take $\alpha = |y|$ and $\beta = \langle x, y\rangle$, we have the following [9]:

Corollary 1.2. *Let $F = |y|\phi(|y|^2, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finsler metric on an n-dimensional manifold M. Then F has vanishing χ-curvature if and only if*

$$(n+1)R_1 + (b^2 - s^2)[R_2]_s = 0,$$

where

$$\begin{aligned} R_1 : &= 2(E_1 - sE_{12}) - E_{22} + 2H\big[E - sE_2 + (b^2 - s^2)E_{22}\big], \\ R_2 : &= 2(2H_1 - sH_{12}) - H_{22} + 2H(2H - sH_2) + (2HH_{22} - H_2^2)(b^2 - s^2). \end{aligned}$$

The paper is organized as follows. In Section 3, we show the equivalence property between $\chi = 0$ and $\mathbf{H} = 0$ for general (α, β)-metrics under the condition (1.4). In Section 4, we discuss the relationship between $\chi = 0$ and almost constant S-curvature. In Section 5, we give some special solutions to (1.5).

2 Preliminaries

In local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by

$$\frac{d^2x^i}{dt^2} + 2G^i\big(x, \frac{dx}{dt}\big) = 0,$$

where $G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\right\}$ are the geodesic coefficients of F. The *Riemann curvature* of F is a family of endomorphism $R_y = R^i{}_j dx^j \otimes \frac{\partial}{\partial x^i} : T_xM \to T_xM$, defined by

$$R^i{}_j := 2\frac{\partial G^i}{\partial x^j} - y^k\frac{\partial^2 G^i}{\partial x^k \partial y^j} + 2G^k\frac{\partial^2 G^i}{\partial y^k \partial y^j} - \frac{\partial G^i}{\partial y^k}\frac{\partial G^k}{\partial y^j}. \tag{2.1}$$

The *Ricci curvature* is the trace of Riemann curvature, which is defined by

$$Ric = R^i{}_i.$$

A Finsler metric on a manifold M in the following form

$$F = \alpha\phi(b^2, s), \quad s = \frac{\beta}{\alpha}, \quad b = \|\beta\|_\alpha$$

is said to be of *general* (α, β)*-type* where α is a Riemannian metric, β is a 1-form and $\phi(b^2, s)$ is a positive smooth function satisfying

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \quad |s| \le b < b_o, \tag{2.2}$$

when $n \ge 3$ or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \quad |s| \le b < b_o, \tag{2.3}$$

when $n = 2$ [14].

Let $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and $\nabla\beta=b_{i|j}dx^i\otimes dx^j$. Set

$$r_{ij}=\tfrac{1}{2}(b_{i|j}+b_{j|i}),\ s_{ij}=\tfrac{1}{2}(b_{i|j}-b_{j|i}),\ r_{00}=r_{ij}y^iy^j,\ s^i{}_0=a^{ij}s_{jk}y^k,$$
$$r_i=b^jr_{ji},\ s_i=b^js_{ji},\ r_0=r_iy^i,\ s_0=s_iy^i,\ r^i=a^{ij}r_j,\ s^i=a^{ij}s_j,\ r=b^ir_i,$$

where $(a^{ij}):=(a_{ij})^{-1}$ and $b^i:=a^{ij}b_j$. It is easy to see that β is closed if and only if $s_{ij}=0$.

Lemma 2.1. *([14]) The spray coefficients G^i of a general (α,β)-metric $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ are related to the spray coefficients ${}^\alpha G^i$ of α and given by*

$$\begin{aligned}G^i &= {}^\alpha G^i+\alpha Qs^i{}_0+\big\{\Theta(-2\alpha Qs_0+r_{00}+2\alpha^2Rr)+\alpha\Omega(r_0+s_0)\big\}\frac{y^i}{\alpha}\\ &\quad+\big\{\Psi(-2\alpha Qs_0+r_{00}+2\alpha^2Rr)+\alpha\Pi(r_0+s_0)\big\}b^i-\alpha^2R(r^i+s^i),\end{aligned}\tag{2.4}$$

where

$$Q=\frac{\phi_2}{\phi-s\phi_2},\quad R=\frac{\phi_1}{\phi-s\phi_2},$$
$$\Theta=\frac{(\phi-s\phi_2)\phi_2-s\phi\phi_{22}}{2\phi\big(\phi-s\phi_2+(b^2-s^2)\phi_{22}\big)},\quad \Psi=\frac{\phi_{22}}{2\big(\phi-s\phi_2+(b^2-s^2)\phi_{22}\big)},$$
$$\Pi=\frac{(\phi-s\phi_2)\phi_{12}-s\phi_1\phi_{22}}{(\phi-s\phi_2)\big(\phi-s\phi_2+(b^2-s^2)\phi_{22}\big)},\quad \Omega=\frac{2\phi_1}{\phi}-\frac{s\phi+(b^2-s^2)\phi_2}{\phi}\Pi.$$

Suppose that β is conformal with respect to α and $d\beta=0$, then by (2.4), the spray coefficients G^i of F is given by

$$G^i={}^\alpha G^i+\lambda\alpha Ey^i+\lambda\alpha^2Hb^i,\tag{2.5}$$

where

$$E:=\frac{\phi_2+2s\phi_1}{2\phi}-H\frac{s\phi+(b^2-s^2)\phi_2}{\phi},\tag{2.6}$$

$$H:=\frac{\phi_{22}-2(\phi_1-s\phi_{12})}{2\big[\phi-s\phi_2+(b^2-s^2)\phi_{22}\big]}.\tag{2.7}$$

By a direct computation, we obtain the formula of the Riemannian curvature of F as follows [8]:

Lemma 2.2. *Let $F=\alpha\phi(b^2,\frac{\beta}{\alpha})$ be a general (α,β)-metric on a manifold M, where α and β satisfy (1.4) with $\lambda^2>0$. Then the Riemannian curvature of F is given by*

$$R^i{}_j=\alpha^2A\delta^i{}_j+\alpha Bb_jy^i-\alpha sCy_jb^i+\alpha^2Cb_jb^i-(A+sB)y_jy^i,\tag{2.8}$$

where

$$A=\mu(1+sE)+\lambda^2\big\{E^2-2sE_1-E_2+2H\big[1+sE+(b^2-s^2)E_2\big]\big\},\tag{2.9}$$

$$\begin{aligned}B=\lambda^2\big\{&2(2E_1-sE_{12})-EE_2-E_{22}-H_2\big[1+sE+E_2(b^2-s^2)\big]\\ &+2H\big[E-sE_2+E_{22}(b^2-s^2)\big]\big\}-\mu(2E-sE_2),\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}C=\lambda^2\big[&2(2H_1-sH_{12})-H_{22}+2H(2H-sH_2)\\ &+(2HH_{22}-H_2^2)(b^2-s^2)\big]-\mu(2H-sH_2).\end{aligned}\tag{2.11}$$

3 χ-curvature

In this section, we will give the χ-curvature of general (α, β)-metrics. Subsequently we are going to show the equivalence property between $\chi = 0$ and $\mathbf{H} = 0$ for general (α, β)-metrics under the condition (1.4).

By Lemma 2.2, we can easily get a formula for the Ricci curvature

$$Ric = \sum_{i=1}^{n} R^i{}_i = \big[(n-1)A + (b^2 - s^2)C\big]\alpha^2, \tag{3.1}$$

where A and C are given by (2.9) and (2.11), respectively. Note that

$$\alpha_{y^i} = \frac{y_i}{\alpha}, \quad s_{y^i} = \frac{\alpha b_i - s y_i}{\alpha^2}. \tag{3.2}$$

By (3.2), we have

$$\sum_j R^j{}_{j\cdot i} = \frac{\partial Ric}{\partial y^i} = 2\big[(n-1)A+(b^2-s^2)C\big]y_i+\big[(n-1)A_2-2sC+(b^2-s^2)C_2\big](\alpha b_i-sy_i). \tag{3.3}$$

By simple calculations, we have

$$s_{y^i} b^i = \frac{b^2 - s^2}{\alpha}, \quad s_{y^i} y^i = 0, \quad \frac{\partial y_j}{\partial y^i} = a_{ij}. \tag{3.4}$$

By Lemma 2.2 and (3.4), we obtain

$$\sum_i R^j{}_{i\cdot j} = -\big[(n-1)A+(b^2-s^2)C\big]y_i + \big[(n+1)B+A_2+sC+(b^2-s^2)C_2\big](\alpha b_i - sy_i). \tag{3.5}$$

Below is a delicate relationship between χ-curvature and Riemann curvature:

Lemma 3.1. *[4, 5, 10]*

$$\chi_i = -\frac{1}{3}\sum_j \big(2R^j{}_{i\cdot j} + R^j{}_{j\cdot i}\big). \tag{3.6}$$

Plugging (3.3) and (3.5) into (3.6), we obtain the following formula for χ-curvature:

$$\chi_i = -\frac{1}{3}\big[(n+1)(A_2+2B) + 3(b^2-s^2)C_2\big](\alpha b_i - sy_i). \tag{3.7}$$

Note that

$$\Phi := \frac{1}{3}(A_2+2B) = \lambda^2\big\{2(E_1-sE_{12})-E_{22}+2H\big[E-sE_2+(b^2-s^2)E_{22}\big]\big\}-\mu(E-sE_2). \tag{3.8}$$

Proof of Theorem 1.1. From (3.7) and (3.8), the proof of Theorem 1.1 immediately follows. □

The **H**-curvature $\mathbf{H} = H_{ij} dx^i \otimes dx^j$ is defined by $H_{ij} := E_{ij|m} y^m$, where E_{ij} is the mean Berwald curvature of F. $\mathbf{H}$ can be expressed in terms of χ_i by

$$H_{ij} = \frac{1}{4}\{\chi_{i\cdot j} + \chi_{j\cdot i}\}. \tag{3.9}$$

Let

$$M := -\frac{1}{3}\big[(n+1)(A_2 + 2B) + 3(b^2 - s^2)C_2\big]. \tag{3.10}$$

Together with (3.10), differentiating (3.7) with respect to y^j yields

$$\begin{aligned} \chi_{i\cdot j} &= M_2 s_{y^j}(\alpha b_i - s y_i) + M(\alpha_{y^j} b_i - s_{y^j} y_i - a_{ij} s) \\ &= M_2 \alpha^{-2}(\alpha b_i - s y_i)(\alpha b_j - s y_j) + M\big[\alpha^{-1}(y_j b_i - y_i b_j) + \alpha^{-2} s y_i y_j - s a_{ij}\big], \end{aligned} \tag{3.11}$$

where we have used (3.2) and the third equality of (3.4). It follows from (3.9) and (3.11) that

$$\begin{aligned} H_{ij} &= M_2 \alpha^{-2}(\alpha b_i - s y_i)(\alpha b_j - s y_j) + sM\big[\alpha^{-2} y_i y_j - a_{ij}\big] \\ &= \alpha^{-2}\big[M_2(\alpha b_i - s y_i)(\alpha b_j - s y_j) - sM(a_{ij}\alpha^2 - y_i y_j)\big]. \end{aligned} \tag{3.12}$$

Lemma 3.2. *Let $F = \alpha\phi(b^2, s)$, $s = \frac{\beta}{\alpha}$, be a general (α, β)-metric on an n-dimensional manifold M. Suppose that α and β satisfy (1.4). Then $\chi = 0$ if and only if $\mathbf{H} = 0$.*

Proof. By (3.9), we can see that the necessity is obvious. In the following, it suffices to show that χ-curvature vanishes if H-curvature vanishes.

Suppose that $\mathbf{H} = 0$, then $H_{ij} = 0$. Contracting (3.12) with $b^i b^j$ yields

$$H_{ij} b^i b^j = \big[(b^2 - s^2)M_2 - sM\big](b^2 - s^2) = 0. \tag{3.13}$$

Hence,

$$(b^2 - s^2)M_2 - sM = 0. \tag{3.14}$$

By (3.12), (3.14) and $H_{ij} = 0$, we have

$$H_{ij} = M_2 \alpha^{-2}\big[(\alpha b_i - s y_i)(\alpha b_j - s y_j) - (b^2 - s^2)(a_{ij}\alpha^2 - y_i y_j)\big]. \tag{3.15}$$

Obviously, we have $M_2 = 0$. Hence, by (3.14), we obtain $M = 0$. It follows from (3.7) and (3.10) that $\chi_i = 0$. Furthermore, $\chi = 0$. □

4 The relationship of $\chi = 0$ and almost isotropic constant S-curvature

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. In a local coordinate system (x^i, y^i), the Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}(x)dx$ is given by

$$\sigma_{BH}(x) = \frac{\mathrm{Vol}(\mathbf{B}^n(1))}{\mathrm{Vol}\{(y^i) \in R^n | F(x, y^i \frac{\partial}{\partial x^i}) < 1\}},$$

where Vol denotes the Euclidean volume and $\mathbf{B}^n(1)$ is a unit ball in R^n. The Holmes-Thompson volume form $dV_{HT} = \sigma_{HT}(x)dx$ is defined as

$$\sigma_{HT}(x) = \frac{1}{\mathrm{Vol}(\mathbf{B}^n(1))} \int_{F(x,y^i\frac{\partial}{\partial x^i})<1} \det(g_{ij}(x,y))dy.$$

When F is a Riemannian metric, both volume forms are reduced to the same Riemannian volume form

$$dV_{BH} = dV_{HT} = \sqrt{\det(g_{ij}(x))}dx.$$

In [19], we have obtained the following lemma:

Lemma 4.1. *Let $F = \alpha\phi(b^2, s)$, $s = \frac{\beta}{\alpha}$, be a general (α, β)-metric on an n-dimensional manifold M. Let $dV = dV_{BH}$ or dV_{HT}. Let*

$$f(b^2) := \begin{cases} \dfrac{\int_0^\pi \sin^{n-2} t dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi(b^2, b\cos t)^n} dt}, & if\ dV = dV_{BH}, \\ \dfrac{\int_0^\pi (\sin^{n-2} t)\mu(b^2, b\cos t)dt}{\int_0^\pi \sin^{n-2} t dt}, & if\ dV = dV_{HT}, \end{cases} \tag{4.1}$$

where $\mu(b^2, s) := \phi(\phi - s\phi_2)^{n-2}\big[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big]$. Then the volume form dV is given by

$$dV = f(b^2)dV_\alpha,$$

where $dV_\alpha = \sqrt{det(a_{ij})}dx$ denotes the Riemannian volume form of α.

The distortion $\tau = \tau(x, y)$ on TM with respect to a given volume form $dV = \sigma(x)dx$ is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

The S-curvature is the rate of change of the distortion along geodesics. More precisely, it is defined by

$$S(x, y) = \frac{d}{dt}\big[\tau(c(t), \dot{c}(t))\big]|_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$.

Definition 4.1. Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. Let S denote the S-curvature of F with respect to the volume form $dV = \sigma(x)dx$.

(a) F is of *almost isotropic S-curvature* if

$$S = (n + 1)cF + \eta, \tag{4.2}$$

where $c = c(x)$ is a scalar function and η is a 1-form on M with $d\eta = 0$;

(b)F is of *isotropic S-curvature* if $c = c(x)$ is a scalar function and $\eta = 0$;

(c) F is of *constant S-curvature* if $c = c(x)$ is a constant and $\eta = 0$.

If we denote the spray coefficients G^i of F, then the S-curvature with respect to the volume form $dV = \sigma(x)dx$ is given by

$$S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\big(\ln\sigma\big). \tag{4.3}$$

Lemma 4.2. *Let $F = \alpha\phi(b^2, s)$, $s = \frac{\beta}{\alpha}$, be a general (α, β)-metric on an n-dimensional manifold M and $dV = \sigma(x)dx$ be a given volume form. Suppose that α and β satisfy (1.4). Then the S-curvature of the given volume form is given by*

$$S = \lambda\alpha\big[(n+1)E + 2sH + H_2(b^2 - s^2) - 2sg\big], \tag{4.4}$$

where $g = \frac{f'(b^2)}{f(b^2)}$, E and H are given by (2.6) and (2.7), respectively.

Proof. Suppose that α and β satisfy (1.4), then the spray coefficients G^i of F is given by (2.5). Differentiating (2.5) with respect to y^i yields

$$\begin{aligned}\big[G^m\big]_{y^m} &= \big[{}^{\alpha}G^m\big]_{y^m} + \lambda E\alpha_{y^m}y^m + \lambda\alpha E_2 s_{y^m}y^m + n\lambda\alpha E + \lambda H[\alpha^2]_{y^m}b^m + \lambda\alpha^2 H_2 s_{y^m}b^m \\ &= \big[{}^{\alpha}G^m\big]_{y^m} + \lambda\alpha\big[(n+1)E + 2sH + H_2(b^2 - s^2)\big].\end{aligned} \tag{4.5}$$

From Lemma 4.1, we obtain $dV = \sigma dx = f(b^2)\sigma_\alpha dx$. Hence,

$$y^m\frac{\partial}{\partial x^m}(\ln\sigma) = \frac{f'(b^2)}{f(b^2)}y^m\frac{\partial b^2}{\partial x^m} + y^m\frac{\partial}{\partial x^m}(\ln\sigma_\alpha). \tag{4.6}$$

By the second equality of (1.4), we have $r_o = \lambda\alpha s$ and $s_o = 0$. Hence,

$$y^m\frac{\partial b^2}{\partial x^m} = 2(r_o + s_o) = 2r_o = 2\lambda\alpha s. \tag{4.7}$$

Plugging (4.7) into (4.6) yields

$$y^m\frac{\partial}{\partial x^m}(\ln\sigma) = 2\lambda\alpha s\frac{f'(b^2)}{f(b^2)} + y^m\frac{\partial}{\partial x^m}(\ln\sigma_\alpha). \tag{4.8}$$

Note that $\big[{}^{\alpha}G^m\big]_{y^m} = y^m\frac{\partial}{\partial x^m}(\ln\sigma_\alpha)$. Inserting (4.5) and (4.8) into (4.2), we have (4.4). □

By Lemma 4.2, we obtain the following

Lemma 4.3. *Let $F = \alpha\phi(b^2, s)$, $s = \frac{\beta}{\alpha}$, be a general (α, β)-metric on an n-dimensional manifold M. Suppose that α and β satisfy (1.4). Then F has almost isotropic S-curvature if and only if*

$$\lambda\alpha\big[(n+1)E + 2sH + H_2(b^2 - s^2) - 2sg\big] = (n+1)c\alpha\phi + \eta, \tag{4.9}$$

where η is a closed form, $g = \frac{f'(b^2)}{f(b^2)}$, E and H are given by (2.6) and (2.7), respectively.

The χ-curvature is determined by S-curvature, precisely

$$\chi_i := S_{\cdot i|j}y^j - S_{|i}. \tag{4.10}$$

According to (4.10), we obtain the following

Lemma 4.4. *Let F be a Finsler metric on an n-dimensional manifold M. If F is of almost isotropic S-curvature, then F is of almost vanishing χ-curvature.*

Proof. If F is of almost isotropic S-curvature, then by (4.2), a direct computation yields

$$S_{\cdot i} = (n+1)cF_{\cdot i} + \eta_{\cdot i}, \quad S_{\cdot i|j} = (n+1)c_{|j}F_{\cdot i} + \eta_{\cdot i|j}, \quad S_{|i} = (n+1)c_{|i}F + \eta_{|i}, \tag{4.11}$$

where we have used $F_{|i} = 0$. Plugging (4.11) into (4.10) yields

$$\chi_i = (n+1)\big[c_{x^j}y^jF_{\cdot i} - c_{x^i}F\big] + \eta_{\cdot i|j}y^j - \eta_{|i}, \tag{4.12}$$

where $c_{|i} = c_{x^i}$. Because $\eta = \eta_i y^i$ is a 1-form and $d\eta = 0$,

$$\eta_{\cdot i|j}y^j - \eta_{|i} = \eta_{i|j}y^j - \eta_{j|i}y^j = \big(\eta_{i|j} - \eta_{j|i}\big)y^j = 0. \tag{4.13}$$

It follows from (4.13) that

$$\chi_i = (n+1)\big[c_{x^j}y^jF_{\cdot i} - c_{x^i}F\big] = -(n+1)F^2\big(\frac{\theta}{F}\big)_{\cdot i}, \tag{4.14}$$

where $\theta = c_{x^i}y^i$. Hence, F is of almost vanishing χ-curvature. □

By Lemma 4.4, we have the following

Corollary 4.5. *Let F be a Finsler metric on an n-dimensional manifold M. If F is of almost constant S-curvature, then F is of vanishing χ-curvature.*

Note that for Lemma 4.4 and Corollary 4.5, the converse is not true. Let us see an example:

Example 4.2. Let

$$\phi(b^2, s) = \frac{\big(\sqrt{1-b^2+s^2}+s\big)^2}{(1-b^2)^2\sqrt{1-b^2+s^2}}$$

and take $\alpha = |y|$ and $\beta = \langle x, y\rangle$. Then the Finsler metric $F = \alpha\phi(b^2, s)$ has vanishing χ-curvature. However, F isn't of almost constant S-curvature.

From Theorem 1.1 and Lemma 4.3, by a direct computation, we can verify Example 4.2.

5 Special solutions

We now look at the special case when $H = 0$. In this case, according to (2.5), we have the following

Proposition 5.1. *Let $F = \alpha\phi(b^2, s)$ be a Finsler metric. Suppose that α and β satisfy (1.4), then F is projectively flat if and only if $H = 0$, namely,*

$$\phi_{22} - 2(\phi_1 - s\phi_{12}) = 0.$$

In [8], we obtain the following result

Lemma 5.2. *Let $F = \alpha\phi(b^2, s)$ be a general (α, β)-metric on a manifold where α and β satisfy (1.4). Then F is of scalar flag curvature if and only if $C = 0$.*

The χ-curvature is an important non-Riemannian quantity. Its importance lies in the following[10]

Lemma 5.3. *Let (M, F) be a Finsler manifold of scalar flag curvature. Then F has isotropic flag curvature if and only if the χ-curvature vanishes.*

By Theorem 1.1, Lemma 5.2 and Lemma 5.3, we obtain the following

Theorem 5.4. *Let $F = \alpha\phi(b^2, s)$, $s = \frac{\beta}{\alpha}$, be a general (α, β)-metric on an n-dimensional manifold $M(n \geq 3)$. Suppose that α and β satisfy (1.4) with $\lambda^2 > 0$. Then F is of constant flag curvature if and only if*

$$\Phi = 0, \quad C = 0.$$

According to Lemma 2.2, we obtain the following [15]

Corollary 5.5. *Let $F = \alpha\phi(b^2, s)$, $s = \frac{\beta}{\alpha}$, be a general (α, β)-metric on an n-dimensional manifold $M(n \geq 3)$ with $H = 0$, namely,*

$$\phi_{22} - 2(\phi_1 - s\phi_{12}) = 0. \tag{5.1}$$

Suppose that α and β satisfy (1.4) with $\lambda^2 > 0$. Then F is of constant flag curvature K if and only if

$$\mu(1 + s\psi) + (\kappa - \mu b^2)\big\{\psi^2 - 2s\psi_1 - \psi_2\big\} = K\phi^2, \tag{5.2}$$

where $\psi := \frac{\phi_2 + 2s\phi_1}{2\phi}$.

By Theorem 5.4, we obtain the following

Corollary 5.6. *Let $F = \alpha\phi(b^2, s)$, $s = \frac{\beta}{\alpha}$, be a general (α, β)-metric on an n-dimensional manifold $M(n \geq 3)$ with $H = 0$, namely,*

$$\phi_{22} - 2(\phi_1 - s\phi_{12}) = 0.$$

Suppose that α and β satisfy (1.4) with $\lambda^2 > 0$. Then F is of constant flag curvature if and only if

$$(\kappa - \mu b^2)\big\{2(\psi_1 - s\psi_{12}) - \psi_{22}\big]\big\} - \mu(\psi - s\psi_2) = 0.$$

Very luckily, (5.1) and (5.2) are solvable (see Yu-Zhu [15]), hence we obtain special solutions of (1.5). According to the four different cases: (i) $\kappa = 0$ and $\mu = 0$; (ii) $\kappa \neq 0$ and $\mu = 0$; (iii) $\kappa \neq 0$ and $\mu \neq 0$; (iv) $\kappa = 0$ and $\mu \neq 0$, we obtain all solutions to (5.1) and (5.2).

Proposition 5.7. *[15] The non-constant solutions of equations (5.1) and (5.2) are given by the following cases:*

1) When $\kappa \neq 0$ and $\mu = 0$, ϕ is given by one of the forms:

$$\phi = \frac{1}{2\sqrt{-\sigma}} \frac{1}{\sqrt{C - b^2 + s^2} \pm s}, \tag{5.3}$$

$$\phi = \frac{q(u)}{q^2(u)(Dq(u) + s)^2 + \sigma}, \tag{5.4}$$

where $\sigma := \frac{K}{\kappa}$ *and* $u := b^2 - s^2$*, the function* $q(u)$ *satisfies the following equation:*

$$D^2 q^4 + (u - C)q^2 - \sigma = 0,$$

where C *and* D *are constants.*

2) When $\kappa = 0$ *and* $\mu \neq 0$*,* ϕ *is given by*

$$\phi = \frac{2q(u)(\sqrt{u+s^2} \pm s)^2}{\left[q(u)(\sqrt{u+s^2} \pm s)^2 + p(u)\right]^2 + \tau}, \tag{5.5}$$

where $\tau := -\frac{K}{\mu}$ *and* $u := b^2 - s^2$*, the functions* $p(u)$ *and* $q(u)$ *are given by one of the forms:*

$$p(u) = \pm\sqrt{-\tau}, \quad q(u) = \pm\frac{(C \pm \sqrt{C^2 + 8pu})^2}{4u^2} \tag{5.6}$$

or

$$p(u) = \pm\sqrt{\frac{-(C^2 - D)\tau - C(C\tau - 2u) \pm \sqrt{D(C\tau - 2u)^2 - D(C^2 - D)\tau^2}}{2(C^2 - D)}}, \tag{5.7}$$

$$q(u) = \frac{p^2 + \tau - upp' \pm \sqrt{(p^2 + \tau - upp')^2 - (p^2 + \tau)u^2 p'}}{u^2 p'}, \tag{5.8}$$

where C *and* D *are constants.*

Remark: The case when $\kappa = 0$ and $\mu = 0$ is trivial. The case when $\kappa \neq 0$ and $\mu \neq 0$ can be reduced to the case $\kappa \neq 0$ and $\mu = 0$ by some special deformations.

Note that F is Riemannian if $D = 0$ in (5.4). It is excluded. Hence, the solutions in (5.4) can be re-expressed explicitly as follows:

(i) When $\sigma = 0$, ϕ is given by

$$\phi(b^2, s) = \frac{D}{\sqrt{C - b^2 + s^2}\left(\sqrt{C - b^2 + s^2} \pm s\right)^2}.$$

In this case, the corresponding general (α, β)-metrics are generalized Berwald metrics.

(ii) When $\sigma < 0$, ϕ is given by

$$\phi(b^2, s) = \frac{1}{2\sqrt{-\sigma}}\Big(\frac{1}{\pm\sqrt{C + 2\sqrt{-\sigma}D - b^2 + s^2} - s} - \frac{1}{\pm\sqrt{C - 2\sqrt{-\sigma}D - b^2 + s^2} - s}\Big).$$

In this case, the corresponding general (α, β)-metrics are first given by Z. Shen in [11].

(iii) When $\sigma > 0$ and q is real, ϕ is given by

$$\phi(b^2, s) = \frac{1}{\sqrt{\sigma}} Re \frac{1}{\sqrt{C + 2\sqrt{\sigma}Di + b^2 - s^2 + is}}.$$

In this case, the corresponding general (α, β)-metrics are Bryant's metrics.

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