Gauge theory on contact metric manifolds

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Abstract. In this paper we develop the gauge theory on a contact manifold. We consider a Lagrangian which is supposed to be invariant under a global action of a Lie group and we obtain the equation of motion and the conservation laws. In order to get a local gauge invariant Lagrangian, we introduced some gauge fields and determine what form have to take such an invariant Lagrangian.

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1 Introduction

This paper is about Lagrangians depending by r scalar fields on a contact metric manifold. It answers at the question: "What have to be the form of a Lagrangian to be invariant at the local action of a Lie group, also called infinitesimal transformation?".

The concept of a non-abelian gauge theory as a generalization of Maxwell's theory of electromagnetism, was introduced by Yang and Mills. A brief survey of interaction between the work of physics community and the mathematicians about gauge theory and differential manifolds could be found in [9]. In monographies [7], [8] were given the basics facts and tehnics of gauge field theories. Some topological aspects of gauge theory on contact 3-manifolds were studied in [10], [14], [15]. The topic of gauge-invariant Lagrangians in complex geometry was discussed in [11], [12], [13]. The gauge-invariance of Lagrangians and the gauge fields for tangent bundle and for foliated manifolds are the subjects of [1], [5], respectively.

Following the general case of foliated manifolds from [5], for a Lagrangian invariant at coordinates transformation, in this paper we express the equation of motions and the conservation laws for the scalar fields using some adapted connections on a contact manifold. So, the first section of the paper is devoted to determine the adapted connections. In the second section we consider a Lagrangian invariant at the coordinate transformation and we study what form it has to take for being invariant at global action of a Lie group, in subsection 2.1, then to be invariant at local action, in subsection 2.2. Here we need to introduce some new fields, called gauge fields, to ensure the local invariance. The last subsection is devoted to study the behaviour of gauge fields at local action of the Lie group.

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2 Contact metric manifolds

2.1 An adapted frame field on a contact metric manifold

Let M be a $(2n+1)$-dimensional manifold and (φ, ξ, η) an *almost contact structure* on M. That is, φ is a tensor field of type $(1,1)$, ξ a vector field, called the *Reeb vector field* on M, and η a 1-form on M, such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

Moreover, if the $(2n+1)$-form $\eta \wedge (d\eta)^n$ doesn't vanishes everywhere on M then (M, η) is a *contact manifold*.

A Riemannian metric compatible with the almost contact structure (φ, ξ, η) is a Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.2}$$

A manifold M endowed with an almost contact structure and a Riemannian metric compatible with it is called an *almost contact metric manifold*.

There are well-known the following properties which derive from the conditions (2.1) and (2.2):

(2.3)

$$(a) \quad \varphi\xi = 0\,,\ (b) \quad \varphi^3 = -\varphi\,,\ (c) \quad \eta\circ\varphi = 0\,,\ (d) \quad \eta(X) = g(X, \xi)\,,\ (e) \quad d\eta(\xi, X) = 0,$$

for every $X \in \Gamma(TM)$. Also, if the almost contact metric manifold is contact, then we have

$$d\eta(X, Y) = \phi(X, Y)\,,\ \forall X, Y \in \Gamma(TM), \tag{2.4}$$

where ϕ is the *fundamental* (or *Sasaki*) 2-form on M given by

$$\phi(X, Y) = g(X, \varphi Y)\,,\ \forall X, Y \in \Gamma(TM). \tag{2.5}$$

Moreover, the almost contact metric manifold is said to be: *K–contact* if it is contact and ξ is Killing; *normal* if $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$; *Sasakian* if it is contact and normal. If M is Sasakian manifold then it is K–contact [6].

Also, we consider the *contact distribution* $\mathcal{D}$ defined by the subspaces

$$\mathcal{D}_x = \{X_x \in T_xM \,|\, \eta_x(X_x) = 0\},$$

which is the transversal distribution to the *characteristic foliation* $\mathcal{F}_\xi$ (1-dimensional foliation determined by the Reeb vector field ξ). Then, the structural distribution of characteristic foliation $\mathcal{F}_\xi$ is $T\mathcal{F}_\xi := \langle\xi\rangle = \{f\xi \,|\, f \in C^\infty(M)\}$.

According to the general theory of foliations, [17, 18, 19], we can choose a local coordinate system $(U, x = (x^0, x^i))$, $i \in \{1, \dots, 2n\}$, adapted to foliation $\mathcal{F}_\xi$, that is $\xi = \partial/\partial x^0$ on U.

Then, by $\eta(\xi) = 1$, we deduce that

$$\eta = dx^0 + \eta_i dx^i\,,\ \eta_i = \eta\left(\frac{\partial}{\partial x^i}\right)\,,\ \forall\, i \in \{1, \dots, 2n\}. \tag{2.6}$$

This allows us to consider on U the local basis $\{\partial/\partial x^0, \delta/\delta x^i\}$, $i = 1, \ldots, 2n$, called adapted to $\mathcal{F}_\xi$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \eta_i \frac{\partial}{\partial x^0}\,, \forall\, i \in \{1, \ldots, 2n\}. \tag{2.7}$$

Obviously, the set $\{\delta/\delta x^i\}$, $i \in \{1, \ldots, 2n\}$ is a local basis in $\Gamma(\mathcal{D}|_U)$, and the dual basis of $\{\partial/\partial x^0, \delta/\delta x^i\}$ is

$$\{\eta, dx^1, dx^2, \ldots, dx^{2n}\}\,. \tag{2.8}$$

In the following we shall evaluate the Lie brackets for the vector fields from the adapted basis $\{\partial/\partial x^0, \delta/\delta x^i\}$. By relation (2.3)(e) and

$$2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta[X, Y], \forall X, Y \in \Gamma(TM),$$

it follows $\eta[\xi, X] = -\xi(\eta(X))$. So, for any $X \in \Gamma(\mathcal{D})$, we have $[\xi, X] \in \Gamma(\mathcal{D})$. That means

$$\left[\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\right] \in \Gamma(\mathcal{D}).$$

On the other hand, a direct computation give us

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^0}\right] = \frac{\partial \eta_i}{\partial x^0}\frac{\partial}{\partial x^0}.$$

We obtain that

$$\left[\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\right] = 0, \quad \eta_i = \eta_i(x^1, x^2, ..., x^{2n}). \tag{2.9}$$

Then, for $\delta/\delta x^i$ from (2.7), we can compute

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = \left(\frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i}\right)\frac{\partial}{\partial x^0}.$$

Also, we have the relations (2.4), (2.5) and (2.6), which express locally that

$$d\eta\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = \frac{1}{2}\left(\frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i}\right) = \phi_{ji} = g_{jk}\varphi_i^k, \tag{2.10}$$

where we put

$$\phi_{ij} = \phi\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right), \quad \varphi\left(\frac{\delta}{\delta x^i}\right) = \varphi_i^j \frac{\delta}{\delta x^i}, \quad g_{ij} = g\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right). \tag{2.11}$$

Obviously, $\phi_{ij} = -\phi_{ji}$ and $g_{ij} = g_{ji}$, for all $i, j \in \{1, \ldots, 2n\}$. Hence, we obtain

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = 2\phi_{ji}\frac{\partial}{\partial x^0}. \tag{2.12}$$

By second relation (2.9) and (2.10) we remark that function ϕ_{ij} doesn't depends by x^0, for every $i, j \in \{1, \ldots, 2n\}$. In the end of this subsection, we notice that the metric g can be expressed with respect to adapted cobasis $\{dx^i, \eta\}$, $i \in \{1, \ldots, 2n\}$ in the form

$$g = g_{ij}dx^i \otimes dx^j + \eta \otimes \eta. \tag{2.13}$$

2.2 Adapted connections on a contact metric manifold

Let us consider a contact metric manifold $(M\varphi, \xi, \eta, g)$ as in the previous subsection and the Reeb foliation $\mathcal{F}_\xi$ on it, generated by ξ. According to the orthogonal decomposition $TM = \mathcal{D} \oplus \langle\xi\rangle$, we consider the projection morphisms v and h of $\Gamma(TM)$ on $\Gamma(\langle\xi\rangle)$ and $\Gamma(\mathcal{D})$, respectively.

According to the general theory of adapted connections on semi-Riemannian foliations, see [5], an adapted connection for the foliation $\mathcal{F}_\xi$ (that means a linear connection on M which induces linear connections on both distributions $\mathcal{D}$, $\langle\xi\rangle$), is given by

$$\nabla_X Y = h\widetilde{\nabla}_X hY + v\widetilde{\nabla}_X vY + h(Q(X, hY)) + v(Q(X, vY), \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$, where $\widetilde{\nabla}$ is an arbitrary linear connection on M and Q is an arbitrary tensor field of type $(1,2)$ on M.

In order to find some adapted connection on the contact metric manifold M, we shall use relation (2.14) for $\widetilde{\nabla}$ the Levi-Civita connection of the metric g. Firstly, we compute the local coefficients of $\widetilde{\nabla}$ with respect to adapted local frame $\left\{\partial/\partial x^0, \delta/\delta x^i\right\}$, using the well-known Koszul formula

$$2g(\widetilde{\nabla}_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(Y,X)) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y),$$

and we obtain the following local expression of $\widetilde{\nabla}$:

$$\begin{aligned}
&\widetilde{\nabla}_{\frac{\delta}{\delta x^j}} \tfrac{\delta}{\delta x^i} = F^k_{ij} \tfrac{\delta}{\delta x^k} + \left(\phi_{ij} - \tfrac{1}{2}\tfrac{\partial g_{ij}}{\partial x^0}\right)\tfrac{\partial}{\partial x^0},\\
&\widetilde{\nabla}_{\frac{\delta}{\delta x^j}} \tfrac{\partial}{\partial x^0} = \widetilde{\nabla}_{\frac{\partial}{\partial x^0}} \tfrac{\delta}{\delta x^j} = \left(\tfrac{1}{2} g^{kl}\tfrac{\partial g_{lj}}{\partial x^0} - \varphi^k_j\right)\tfrac{\delta}{\delta x^k},\\
&\widetilde{\nabla}_{\frac{\partial}{\partial x^0}} \tfrac{\partial}{\partial x^0} = 0,
\end{aligned} \tag{2.15}$$

where

$$F^k_{ij} = \frac{1}{2} g^{kh}\left(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}\right), \tag{2.16}$$

and $\left(g^{ij}\right)_{2n\times 2n}$ is the inverse matrix of $(g_{ij})_{2n\times 2n}$ given in (2.11).

For an adapted connection $\overset{\alpha}{\nabla}$, we denote its local coefficients by

$$\begin{aligned}
&\overset{\alpha}{\nabla}_{\frac{\delta}{\delta x^j}} \tfrac{\delta}{\delta x^i} = \overset{\alpha}{F}{}^k_{ij} \tfrac{\delta}{\delta x^k}, \quad \overset{\alpha}{\nabla}_{\frac{\partial}{\partial x^0}} \tfrac{\delta}{\delta x^i} = \overset{\alpha}{D}{}^k_i \tfrac{\delta}{\delta x^k},\\
&\overset{\alpha}{\nabla}_{\frac{\delta}{\delta x^i}} \tfrac{\partial}{\partial x^0} = \overset{\alpha}{L}_i \tfrac{\partial}{\partial x^0}, \quad \overset{\alpha}{\nabla}_{\frac{\partial}{\partial x^0}} \tfrac{\partial}{\partial x^0} = \overset{\alpha}{C} \tfrac{\partial}{\partial x^0}.
\end{aligned} \tag{2.17}$$

Following some idea from [5], we consider four adapted connections on the contact metric manifold as follows.

The *first adapted connection* on the contact metric manifold M is defined by

$$\overset{1}{\nabla}_X Y = h\widetilde{\nabla}_X hY + v\widetilde{\nabla}_X vY, \quad \forall X, Y \in \Gamma(TM). \tag{2.18}$$

We notice that every vector field $Y \in \Gamma(TM)$ admits a decomposition with respect to the adapted basis $\left\{\partial/\partial x^0, \delta/\delta x^i\right\}$ in the form

$$Y = Y^i \frac{\delta}{\delta x^i} + Y^0 \frac{\partial}{\partial x^0}, \tag{2.19}$$

where, from $\delta/\delta x^i \in \Gamma(\mathcal{D}) = \operatorname{Ker} \eta$, we have $Y^0 = \eta(Y)$, so the projections of Y on $\Gamma(\langle\xi\rangle)$ and $\Gamma(\mathcal{D})$ respectively, are given by

$$vY = \eta(Y) \frac{\partial}{\partial x^0}, \quad hY = Y^i \frac{\delta}{\delta x^i}. \tag{2.20}$$

It results the following local form for (2.18):

$$\overset{1}{\nabla}_X Y = X(\eta(Y)) \frac{\partial}{\partial x^0} + \eta(Y) v\widetilde{\nabla}_X \frac{\partial}{\partial x^0} + X(Y^i) \frac{\delta}{\delta x^i} + Y^i h\widetilde{\nabla}_X \frac{\delta}{\delta x^i}. \tag{2.21}$$

Using (2.21), by direct computation, we obtain the local coefficients of the first adapted connection $\overset{1}{\nabla}$ as

$$\overset{1}{F}{}^k_{ij} = F^k_{ij}, \quad \overset{1}{D}{}^k_i = \frac{1}{2} g^{kl} \frac{\partial g_{li}}{\partial x^0} - \varphi^k_i, \quad \overset{1}{L}_i = \overset{1}{C} = 0. \tag{2.22}$$

The *second adapted connection* is defined by

$$\overset{2}{\nabla}_X Y = h\widetilde{\nabla}_X hY - h\widetilde{\nabla}_{hY} vX + v\widetilde{\nabla}_X vY - v\widetilde{\nabla}_{vY} hX, \, \forall\, X, Y \in \Gamma(TM), \tag{2.23}$$

and, by direct computation, its local coefficients are

$$\overset{2}{F}{}^k_{ij} = F^k_{ij}, \quad \overset{2}{D}{}^k_i = 0, \quad \overset{2}{L}_i = \overset{2}{C} = 0. \tag{2.24}$$

The *third adapted connection* is defined by

$$\overset{3}{\nabla}_X Y = \overset{1}{\nabla}_X Y + hQ(vX, hY), \quad \forall X, Y \in \Gamma(TM), \tag{2.25}$$

where the tensor field Q is defined by

$$g\left(hQ(vX, hY), hZ\right) = g\left(v[hY, hZ], vX\right).$$

Denoting by a^k_i the local components of the projection on $\Gamma(\mathcal{D})$ of the vector field $Q\left(\partial/\partial x^0, \delta/\delta x^i\right)$, the above condition give us $a^k_i = \varphi^k_i$, so we obtain

$$hQ\left(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\right) = \varphi^k_i \frac{\delta}{\delta x^k}.$$

By direct computation, the local coefficients of the third adapted connections are

$$\overset{3}{F}{}^k_{ij} = F^k_{ij}, \quad \overset{3}{D}{}^k_i = \frac{1}{2} g^{kl} \frac{\partial g_{li}}{\partial x^0} + \varphi^k_i, \quad \overset{3}{L}_i = \overset{3}{C} = 0. \tag{2.26}$$

Finally, the *fourth adapted connection* on the contact metric manifold M, is defined as the average connection between $\overset{1}{\nabla}$ and $\overset{3}{\nabla}$, that is

$$\overset{4}{\nabla}_X Y = \frac{1}{2}\left(\overset{1}{\nabla}_X Y + \overset{3}{\nabla}_X Y\right), \quad \forall X, Y \in \Gamma(TM), \tag{2.27}$$

and it has the same local coefficients with $\overset{1}{\nabla}$, excepting $\overset{4}{D_i^k} = g^{kl}\frac{\partial g_{li}}{\partial x^0}$.

Remark 2.1. (i) The horizontal coefficients of all four adapted connections coincides, that is $\overset{\alpha}{F_{ij}^k} = F_{ij}^k$, $\alpha \in \{1, 2, 3, 4\}$.

(ii) The first adapted connection $\overset{1}{\nabla}$ is just the Schouten-Van Kampen connection associated to the Reeb foliation $\mathcal{F}_\xi$, see for instance [3], p.107.

(iii) The second adapted connection $\overset{2}{\nabla}$ is just the $\mathcal{D}$-connection on a contact metric manifold (introduced in [2]). Also, it can be viewed as the Vrănceanu connection or Vaisman connection associated to the Reeb foliation $\mathcal{F}_\xi$, see [3, 18].

(iv) If M is K-contact, then $\partial g_{ij}/\partial x^0 = 0$, and then $\overset{2}{\nabla} = \overset{4}{\nabla}$.

In the next section the adapted connections $\overset{\alpha}{\nabla}$ will be used to express the Euler-Lagrange equation for a Lagrangian on a contact manifold.

We finish this subsection with some considerations about basic connections (with respect to Reeb foliation) on a contact manifold. Generally speaking, on the foliated manifold $(M, \mathcal{F})$ there is an adapted atlas whose coordinate system on the open set $U \subset M$ is $(x^i) = (x^a, x^u)$, where $a \in \{1, \ldots, q\}$, $u \in \{q+1, \ldots, m\}$, such that the points in the same leaf $\mathcal{L} \cap U$ have their first q coordinates equal, and are distinguished by their last $(m-q)$ coordinates. Locally, the structural bundle F is spanned by $\{\partial/\partial x^u\}$, $u \in \{q+1, \ldots, m\}$.

Also, if we consider the canonical exact sequence associated to the foliation given by the integrable subbundle F, namely

$$0 \longrightarrow F \xrightarrow{i_F} TM \xrightarrow{\pi_{QF}} QF \longrightarrow 0,$$

then we recall that a connection ∇ on the normal bundle QF is said to be *basic* if

$$\nabla_X Y = \pi_{QF}[X, \widetilde{Y}] \tag{2.28}$$

for any $X \in \Gamma(F)$, $\widetilde{Y} \in \Gamma(TM)$ such that $\pi_{QF}(\widetilde{Y}) = Y$. Obviously, the right-hand side of (2.28) does not depend by choice of vector field $\tilde{Y}$, because the integrability of F.

Now, returning to the contact metric manifold M endoweed with the characteristic foliation $\mathcal{F}_\xi$, and taking into account that $Q\mathcal{F}_\xi \cong \mathcal{D}$, a linear connection ∇ on M is basic if and only if

$$\nabla_{\frac{\partial}{\partial x^0}} Y = h\left[\frac{\partial}{\partial x^0}, \widetilde{Y}\right], \tag{2.29}$$

where $\widetilde{Y} \in \Gamma(TM)$ such that $h(\widetilde{Y}) = Y$. Locally, let $\widetilde{Y} = Y^0\partial/\partial x^0 + Y^i\delta/\delta x^i$. Then $h(\widetilde{Y}) = Y^i\delta/\delta x^i$ and relation (2.29) locally becomes

$$\frac{\partial Y^i}{\partial x^0} + Y^k D^i_k = \frac{\partial Y^i}{\partial x^0},$$

where D^i_k are transversal (horizontal) components of $\nabla_{\frac{\partial}{\partial x^0}}\frac{\delta}{\delta x^i}$. But the above equality means

Proposition 2.1. *The connection ∇ is basic if and only if all horizontal components of $\nabla_{\frac{\partial}{\partial x^0}}\frac{\delta}{\delta x^i}$ vanish.*

Concerning now to adapted connections $\overset{1}{\nabla}$, $\overset{2}{\nabla}$, $\overset{3}{\nabla}$ and $\overset{4}{\nabla}$, respectively, their locally coefficients given in (2.22), (2.24) and (2.26) show that

Proposition 2.2. *From the four defined above connections, only the second connection, $\overset{2}{\nabla}$, is basic with respect to the characteristic foliation $\mathcal{F}_\xi$ determined by the Reeb vector field ξ.*

3 Invariance of Lagrangians on a contact metric manifold

In [5], the equation of motion for r scalar fields Q^A, $A \in \{1,\ldots,r\}$, on a semi-Riemannian foliated manifold $(M,\mathcal{F},g)$, are expressed using covariant derivative with respect to the Vrănceanu connection on that manifold. In this section we apply that idea for the case of the contact metric manifold $(M\varphi,\xi,\eta,g)$, endowed with the characteristic foliation $\mathcal{F}_\xi$.

We start with a Lagrangian depending by r scalar fields $Q^A = Q^A(x)$, $A \in \{1,\ldots,r\}$, on the contact metric manifold $(M\varphi,\xi,\eta,g)$, that is

$$\mathcal{L}(x) = \mathcal{L}\left(Q^A(x), \frac{\delta Q^A}{\delta x^i}(x), \frac{\partial Q^A}{\partial x^0}(x)\right), \tag{3.1}$$

which is invariant under the coordinate transformations on M.

Let us consider the function H, locally defined by $H(x) = \sqrt{|\det(g_{ij}(x))|}$, $i,j \in \{1,\ldots,2n\}$. From direct computation, we have the following transformation law in the intersection $U \cap \widetilde{U} \neq \varnothing$ of two domains of local chart of M

$$\widetilde{H} = \left|\det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}\right)\right| \cdot H\,,\; i,j \in \{1,\ldots,2n\}.$$

Then

$$\mathcal{L}_0(x) = H(x)\cdot\mathcal{L}(x), \tag{3.2}$$

is a Lagrangian density on M. Thus, the functional

$$I(\Omega) = \int_\Omega \mathcal{L}_0(x)dx^1 \wedge \ldots \wedge dx^{2n} \wedge \eta, \tag{3.3}$$

where Ω is a compact domain of M, does not depend of the coordinates on M.

As usual, we assume that the equations of motion for the fields $Q^A(x)$ follow from the variational principle $\delta(I(\Omega)) = 0$. Hence, the Euler-Lagrange equations for fields Q^A are

$$\frac{\partial \mathcal{L}_0}{\partial Q^A} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial x^i}\right)}\right) - \frac{\partial}{\partial x^0}\left(\frac{\partial \mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}\right) = 0. \tag{3.4}$$

Taking into account the relation (2.7), we obtain (from (3.1))

$$\frac{\partial \mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)} = \frac{\partial \mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct.} - \eta_i \frac{\partial \mathcal{L}_0}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}. \tag{3.5}$$

Then, the equations (3.4) become

$$\frac{\partial \mathcal{L}_0}{\partial Q^A} - \frac{\delta}{\delta x^i}\left(\frac{\partial \mathcal{L}_0}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}\right) - \frac{\partial}{\partial x^0}\left(\frac{\partial \mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct}\right) = 0, \tag{3.6}$$

and, from (3.2), we get

$$\begin{aligned} &H\cdot\left\{\frac{\partial \mathcal{L}}{\partial Q^A} - \frac{\delta}{\delta x^i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}\right) - \frac{\partial}{\partial x^0}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct}\right)\right\} \\ &= \frac{\delta H}{\delta x^i}\cdot\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)} + \frac{\partial H}{\partial x^0}\cdot\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct}. \end{aligned} \tag{3.7}$$

Now we denote

$$Q^i_A := \frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}, \quad Q^0_A := \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct}, \; i \in \{1,\ldots,2n\}. \tag{3.8}$$

From the changing rules for horizontal vector fields, that is

$$\frac{\delta}{\delta x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\delta}{\delta \widetilde{x}^j},$$

it follows that Q^i_A are components of r horizontal vector fields

$$hQ_A = Q^i_A \frac{\delta}{\delta x^i}.$$

Taking into account relations (2.17), the covariant derivatives of Q^i_A, Q^0_A with respect to an adapted connection $\overset{\alpha}{\nabla}$, $\alpha = 1,2,3,4$, given locally in subsection 2.2 are

$$Q^i_{A|_j} = \frac{\delta Q^i_A}{\delta x^j} + Q^k_A F^i_{kj}, \quad Q^i_{A|_0} = \frac{\partial Q^i_A}{\partial x^0} + Q^k_A \overset{\alpha}{D}{}^i_k, \quad Q^0_{A|_0} = \frac{\partial Q^0_A}{\partial x^0}, \quad Q^0_{A|_i} = \frac{\delta Q^0_A}{\delta x^i}.$$

Then, the equation (3.7) could be rewritten in the form

$$\frac{\partial \mathcal{L}}{\partial Q^A} - Q^i_{A|_i} - Q^0_{A|_0} = \left(\frac{1}{H}\frac{\delta H}{\delta x^i} - F^j_{ij}\right)Q^i_A + \frac{1}{H}\frac{\partial H}{\partial x^0}\cdot Q^0_A. \tag{3.9}$$

But, by direct calculus, we have

$$\frac{1}{H}\frac{\delta H}{\delta x^i}=\frac{1}{2}g^{js}\frac{\delta g_{js}}{\delta x^i},\quad \frac{1}{H}\frac{\partial H}{\partial x^0}=\frac{1}{2}g^{js}\frac{\partial g_{js}}{\partial x^0}.$$

On the other hand, taking into account the relations (2.16) it follows

$$F^j_{ij}=\frac{1}{H}\frac{\delta H}{\delta x^i}.$$

Hence, we obtain that the equation of motion for the scalar fields Q^A have the form

$$\frac{\partial \mathcal{L}}{\partial Q^A}-Q^i_{A|i}-Q^0_{A|0}=\frac{1}{2}g^{js}\frac{\partial g_{js}}{\partial x^0}Q^0_A, \tag{3.10}$$

where the covariant derivatives of Q^i_A, Q^0_A are taken with respect to one of the adapted connections introduced in subsection 2.2.

Remark 3.1. If M is K-contact then the equation of motion for the scalar fields Q^A simplify in the nice form

$$\frac{\partial \mathcal{L}}{\partial Q^A}-Q^i_{A|i}-Q^0_{A|0}=0. \tag{3.11}$$

3.1 Globally gauge invariance

In this section we study the invariance of the Lagrangian (3.1) under the action of an arbitrary m-dimensional Lie group G on the physical fields $Q^A(x)$. We also consider that G admits a r-dimensional representation ρ.

A Lie group G is essentially uniquely determnined by its Lie algebra, defined by the basis $\{X_a\}$, $a\in\{1,\ldots,m\}$. The representation ρ assigns to every vector field X_a a $r\times r$-matrix $([X_a]^A_B)_{r\times r}$, $A,B\in\{1,\ldots,r\}$. There are well known relations

$$[X_a,X_b]=C^c_{ab}X_c,\quad C^c_{ab}=-C^c_{ba},$$

where the structure constants C^c_{ab} obey the Jacobi identity

$$C^d_{ab}C^e_{dc}+C^d_{bc}C^e_{da}+C^d_{ca}C^e_{db}=0.$$

Moreover, the matrices generators are satisfying

$$[X_a]^A_B[X_b]^B_C-[X_b]^A_B[X_a]^B_C=C^c_{ab}[X_c]^A_C. \tag{3.12}$$

Now, according to [3, 5], the group G being given, for any vector field $X=\varepsilon^aX_a$ on G, a *global gauge action* of G on the scalar physical fields $Q^A(x)$, $A\in\{1,\ldots,r\}$, is given by the infinitesimal transformations

$$Q'^A(x)=Q^A(x)+\delta(Q^A(x)),\quad \delta(Q^A(x))=\varepsilon^a[X_a]^A_BQ^B(x). \tag{3.13}$$

Applying the operators $\delta/\delta x^i$ and $\partial/\partial x^0$ to (3.13), we obtain

$$\begin{aligned}&\tfrac{\delta Q'^A}{\delta x^i}=\tfrac{\delta Q^A}{\delta x^i}+\delta\left(\tfrac{\delta Q^A}{\delta x^i}\right),\quad \delta\left(\tfrac{\delta Q^A}{\delta x^i}\right)=\varepsilon^a[X_a]^A_B\tfrac{\delta Q^B}{\delta x^i}\\ &\tfrac{\partial Q'^A}{\partial x^0}=\tfrac{\partial Q^A}{\partial x^0}+\delta\left(\tfrac{\partial Q^A}{\partial x^0}\right),\quad \delta\left(\tfrac{\partial Q^A}{\partial x^0}\right)=\varepsilon^a[X_a]^A_B\tfrac{\partial Q^B}{\partial x^0}.\end{aligned} \tag{3.14}$$

Now, we suppose that the Lagrangian (3.1) is *globally gauge G-invariant*, that is, $\mathcal{L}$ is invariant under the infinitesimal transformations (3.13) and (3.14) This means that $\delta\mathcal{L} = 0$, or equivalently, $\mathcal{L}$ does not depend by ε^a. It follows that

$$\left[\frac{\partial\mathcal{L}}{\partial Q^A}Q^B + \frac{\partial\mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}\frac{\delta Q^B}{\delta x^i} + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}\Big|_{\frac{\delta Q^A}{\delta x^i}=ct}\frac{\partial Q^B}{\partial x^0}\right][X_a]^A_B = 0, \tag{3.15}$$

or, equivalently

$$\left[\frac{\partial\mathcal{L}}{\partial Q^A}Q^B + Q^i_A\frac{\delta Q^B}{\delta x^i} + Q^0_A\frac{\partial Q^B}{\partial x^0}\right][X_a]^A_B = 0, \tag{3.16}$$

with the notations (3.8).

From relation (3.10) it follows

$$\frac{\partial\mathcal{L}}{\partial Q^A}[X_a]^A_B Q^B = Q^i_{A|_i}[X_a]^A_B Q^B + Q^0_{A|_0}[X_a]^A_B Q^B + \frac{1}{2}g^{js}\frac{\partial g_{js}}{\partial x^0}Q^0_A[X_a]^A_B Q^B. \tag{3.17}$$

Then it is natural to consider the scalar fields

$$J^i_a = -Q^i_A[X_a]^A_B Q^B, \quad J^0_a = -Q^0_A[X_a]^A_B Q^B,$$

which are components of m horizontal vector fields $hJ_a = J^i_a\delta/\delta x^i$, and m colinear vertical vector fields $vJ_a = J^0_a\xi$, called *horizontal currents* and *vertical currents*, respectively.

Taking into account relations (2.17), the covariant derivatives of J^i_a, J^0_a with respect to an adapted connection $\overset{\alpha}{\nabla}$, $\alpha = 1,2,3,4$, given locally in subsection 2.2, are given by

$$J^i_{a|_j} = \frac{\delta J^i_a}{\delta x^j} + J^k_a F^i_{kj}\,,\ J^0_{a|_0} = \frac{\partial J^0_a}{\partial x^0}.$$

But, we have

$$\begin{aligned}\frac{\delta J^i_a}{\delta x^j} &= -\frac{\delta Q^i_A}{\delta x^j}[X_a]^A_B Q^B - Q^i_A[X_a]^A_B\frac{\delta Q^B}{\delta x^j},\\ \frac{\partial J^0_a}{\partial x^0} &= -\frac{\partial Q^0_A}{\partial x^0}[X_a]^A_B Q^B - Q^0_A[X_a]^A_B\frac{\partial Q^B}{\partial x^0},\end{aligned}$$

and replacing (3.17) in (3.16) and, using also the expressions of covariant derivatives of fields Q^i_A, Q^0_A with respect to the same connection $\overset{\alpha}{\nabla}$, we obtain the conservation laws

$$J^i_{a|_i} + J^0_{a|_0} = \frac{1}{2}g^{js}\frac{\partial g_{js}}{\partial x^0}Q^0_A[X_a]^A_B Q^B. \tag{3.18}$$

According to the terminology from [3, 5], the vector fields hJ_a and vJ_a will be called called the *horizontal* and the *vertical currents* on the contact metric manifold M, respectively.

Remark 3.2. If M is K-contact then the above conservation laws reduce in the simple form

$$J^i_{a|_i} + J^0_{a|_0} = 0. \tag{3.19}$$

3.2 Locally gauge invariance

A group of global transformations is characterized by the parameters ϵ^a being independent by the coordinates (x^0, x^i). In this subsection we suppose now that the parameters of the group are coordinates dependent, that means that the action of G on fields $Q^A(x)$ is local. In this situation, the scalar fields $Q^A(x)$ transform according to

$$Q'^A(x) = Q^A(x) + \overset{*}{\delta}(Q^A(x)),\ \overset{*}{\delta}(Q^A(x)) = \varepsilon^a(x)[X_a]^A_B Q^B(x). \tag{3.20}$$

Then, from above relations, we have

$$\frac{\delta Q'^A}{\delta x^i} = \frac{\delta Q^A}{\delta x^i} + \varepsilon^a(x)[X_a]^A_B \frac{\delta Q^B}{\delta x^i} + \frac{\delta \varepsilon^a}{\delta x^i}[X_a]^A_B Q^B, \tag{3.21}$$

$$\frac{\partial Q'^A}{\partial x^0} = \frac{\partial Q^A}{\partial x^0} + \varepsilon^a(x)[X_a]^A_B \frac{\partial Q^B}{\partial x^0} + \frac{\partial \varepsilon^a}{\partial x^0}[X_a]^A_B Q^B. \tag{3.22}$$

Now, we have to remark that a globally invariant Lagrangian may be not invariant under the local transformations (3.20). The variation of the Lagrangian is

$$\begin{aligned}
\overset{*}{\delta}(\mathcal{L}) &= \frac{\partial \mathcal{L}}{\partial Q^A}\overset{*}{\delta}(Q^A) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}\overset{*}{\delta}\left(\frac{\delta Q^A}{\delta x^i}\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct}\overset{*}{\delta}\left(\frac{\partial Q^B}{\partial x^0}\right) \\
&= \varepsilon^a\left[\frac{\partial \mathcal{L}}{\partial Q^A}Q^B + \frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}\frac{\delta Q^B}{\delta x^i} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct}\frac{\partial Q^B}{\partial x^0}\right][X_a]^A_B \\
&\quad + \left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}\frac{\delta \varepsilon^a}{\delta x^i} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct}\frac{\partial \varepsilon^a}{\partial x^0}\right]Q^B[X_a]^A_B.
\end{aligned}$$

Taking into account that the Lagrangian satisfy relation (3.15) (the global invariance), we obtain the variation of $\mathcal{L}$ by the form

$$\overset{*}{\delta}(\mathcal{L}) = \left[Q^i_A\frac{\delta \varepsilon^a}{\delta x^i} + Q^0_A\frac{\partial \varepsilon^a}{\partial x^0}\right]Q^B[X_a]^A_B.$$

Hence, we need to add some new fields, called *gauge fields*, see [7, 8], to obtain a locally invariant Lagrangian.

More exactly, we consider the horizontal and vertical 1-forms

$$H^a = H^a_i(x)dx^i,\ i \in \{1, \ldots, 2n\},\ a \in \{1, \ldots, r\} \tag{3.23}$$

and

$$\zeta^a = \sigma^a(x)\eta,\ a \in \{1, \ldots, r\}, \tag{3.24}$$

respectively, where $H^a_i, \sigma^a \in C^\infty(M)$.

Since η is a global 1-form on M, the functions σ^a are globally defined on M, while H_i^a are locally defined functions on M, and they have to transform as it follows (at the local coordinate changing on M)

$$H_i^a = \frac{\partial \widetilde{x}^j}{\partial x^i} \widetilde{H}_j^a. \tag{3.25}$$

Now, we ask that $\mathcal{L} = \mathcal{L}\left(Q^A, \delta Q^A/\delta x^i, \partial Q^A/\partial x^0, H_i^a, \sigma^a\right)$ is an invariant Lagrangian to the local action of G. According to [7], the gauge fields have to transform as it follows:

$$\overset{*}{\delta}\left(H_i^a\right) = \varepsilon^b C_{bc}^a H_i^c + \frac{\delta \varepsilon^a}{\delta x^i}, \quad \overset{*}{\delta}\left(\sigma^a\right) = \varepsilon^b C_{bc}^a \sigma^c + \frac{\partial \varepsilon^a}{\partial x^0}, \tag{3.26}$$

and

$$\begin{aligned} \overset{*}{\delta}(\mathcal{L}) \quad = \quad & \frac{\partial \mathcal{L}}{\partial Q^A} \overset{*}{\delta}\left(Q^A\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)} \overset{*}{\delta}\left(\frac{\delta Q^A}{\delta x^i}\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}\Big|_{\frac{\delta Q^A}{\delta x^i}=ct} \overset{*}{\delta}\left(\frac{\partial Q^A}{\partial x^0}\right) \\ & + \frac{\partial \mathcal{L}}{\partial H_i^a} \overset{*}{\delta}\left(H_i^a\right) + \frac{\partial \mathcal{L}}{\partial \sigma^a} \overset{*}{\delta}\left(\sigma^a\right), \end{aligned}$$

must vanishes. That is

$$\begin{aligned} & \left[\frac{\partial \mathcal{L}}{\partial Q^A} Q^B + \frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)} \frac{\delta Q^B}{\delta x^i} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}\Big|_{\frac{\delta Q^A}{\delta x^i}=ct} \frac{\partial Q^B}{\partial x^0}\right] [X_a]_B^A \varepsilon^a \\ & + \left[\frac{\partial \mathcal{L}}{\partial H_i^a} C_{bc}^a H_i^c + \frac{\partial \mathcal{L}}{\partial \sigma^a} C_{bc}^a \sigma^c\right] \varepsilon^b + \left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)} [X_a]_B^A Q^B + \frac{\partial \mathcal{L}}{\partial H_i^a}\right] \frac{\delta \varepsilon^a}{\delta x^i} \\ & + \left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}\Big|_{\frac{\delta Q^A}{\delta x^i}=ct} [X_a]_B^A Q^B + \frac{\partial \mathcal{L}}{\partial \sigma^a}\right] \frac{\partial \varepsilon^a}{\partial x^0} = 0. \end{aligned}$$

Taking into account that parameters functions $\varepsilon^a(x)$ are arbitrary, we obtain the following equivalent conditions for the vanishing of $\overset{*}{\delta}(\mathcal{L})$:

$$\frac{\partial \mathcal{L}}{\partial Q^A} [X_a]_B^A Q^B + Q_A^i [X_a]_B^A \frac{\delta Q^B}{\delta x^i} + Q_A^0 [X_a]_B^A \frac{\partial Q^B}{\partial x^0} + \frac{\partial \mathcal{L}}{\partial H_i^b} C_{ac}^b H_i^c + \frac{\partial \mathcal{L}}{\partial \sigma^b} C_{ac}^b \sigma^c = 0, \tag{3.27}$$

$$Q_A^i Q^B [X_a]_B^A + \frac{\partial \mathcal{L}}{\partial H_i^a} = 0, \quad Q_A^0 Q^B [X_a]_B^A + \frac{\partial \mathcal{L}}{\partial \sigma^a} = 0. \tag{3.28}$$

In order to obtain identities (3.27) and (3.28), it is enough to add some additional fields enter into Lagrangian from some expressions like covariant derivatives, called the *horizontal and vertical gauge-covariant derivatives* of physical fields:

$$D_i Q^A = \frac{\delta Q^A}{\delta x^i} - H_i^a [X_a]_B^A Q^B, \; D_0 Q^A = \frac{\partial Q^A}{\partial x^0} - \sigma^a [X_a]_B^A Q^B. \tag{3.29}$$

Indeed, if we take the Lagrangian in the form

$$\mathcal{L}=\mathcal{L}\left(Q^A,D_iQ^A,D_0Q^A\right), \tag{3.30}$$

then we have

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}=\frac{\partial\mathcal{L}}{\partial D_iQ^A},\quad \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}|_{\frac{\delta Q^A}{\delta x^i}=ct}=\frac{\partial\mathcal{L}}{\partial D_0Q^A},$$

$$\frac{\partial\mathcal{L}}{\partial H_i^a}=-\frac{\partial\mathcal{L}}{\partial D_iQ^A}[X_a]_A^BQ^B,\quad \frac{\partial\mathcal{L}}{\partial\sigma^a}=-\frac{\partial\mathcal{L}}{\partial D_0Q^A}[X_a]_A^BQ^B,$$

$$\frac{\partial\mathcal{L}}{\partial Q^A}=\frac{\partial\mathcal{L}}{\partial Q^A}-\frac{\partial\mathcal{L}}{\partial D_iQ^B}H_i^a[X_a]_A^B-\frac{\partial\mathcal{L}}{\partial D_0Q^B}\sigma^a[X_a]_A^B.$$

By a direct computation it follows that conditions (3.28) are satisfied and condition (3.27) is equivalent to

$$\left[\frac{\partial\mathcal{L}}{\partial Q^A}Q^B+\frac{\partial\mathcal{L}}{\partial D_iQ^A}D_iQ^B+\frac{\partial\mathcal{L}}{\partial D_0Q^A}D_0Q^B\right][X_a]_A^B=0,$$

which is true from the global invariance of the Lagrangian (3.30).

Remark 3.3. The local gauge invariance of a Lagrangian on a contact metric manifold M endowed with the characteristic foliation $\mathcal{F}_\xi$ is obtained from a global gauge invariant Lagrangian just by replacing the usual derivatives $\delta Q^A/\delta x^i$ and $\partial Q^A/\partial x^0$ by the horizontal and vertical gauge-covariant derivatives D_iQ^A and D_0Q^A, respectively. In this way we obtain the *minimal replacement principle* for contact metric manifolds.

3.3 Lagrangians for gauge fields

The Lagrangian (3.30) is made up of the free Lagrangian for scalar fields Q^A and the interaction of the scalar fields with the gauge fields H_i^a, σ^a. Now we shall find the expression for the Lagrangian of the gauge fields which is invariant under the group action.

This Lagrangian depends on the gauge fields as well as on their derivatives, so it is given by

$$\mathcal{L}_1=\mathcal{L}_1\left(H_i^a,\sigma^a,\frac{\delta H_j^a}{\delta x^i},\frac{\partial H_j^a}{\partial x^0},\frac{\delta\sigma^a}{\delta x^i},\frac{\partial\sigma^a}{\partial x^0}\right). \tag{3.31}$$

The condition for invariance of $\mathcal{L}_1$ is

$$\begin{aligned}\overset{*}{\delta}(\mathcal{L}_1)=\frac{\partial\mathcal{L}_1}{\partial H_i^a}\overset{*}{\delta}(H_i^a)+\frac{\partial\mathcal{L}_1}{\partial\left(\frac{\delta H_i^a}{\delta x^j}\right)}\overset{*}{\delta}\left(\frac{\delta H_i^a}{\delta x^j}\right)+\frac{\partial\mathcal{L}_1}{\partial\left(\frac{\partial H_i^a}{\partial x^0}\right)}|_{\frac{\delta H_i^a}{\delta x^j}=ct}\overset{*}{\delta}\left(\frac{\partial H_i^a}{\partial x^0}\right)+\\ +\frac{\partial\mathcal{L}_1}{\partial\sigma^a}\overset{*}{\delta}(\sigma^a)+\frac{\partial\mathcal{L}_1}{\partial\left(\frac{\delta\sigma^a}{\delta x^j}\right)}\overset{*}{\delta}\left(\frac{\delta\sigma^a}{\delta x^j}\right)+\frac{\partial\mathcal{L}_1}{\partial\left(\frac{\partial\sigma^a}{\partial x^0}\right)}|_{\frac{\delta\sigma^a}{\delta x^j}=ct}\overset{*}{\delta}\left(\frac{\partial\sigma^a}{\partial x^0}\right)=0,\end{aligned} \tag{3.32}$$

where the gauge fields transform according to (3.26), while their derivatives follow the rules

$$\tag{3.33}\begin{aligned}
\overset{*}{\delta}\left(\frac{\delta H_i^a}{\delta x^j}\right) &= \varepsilon^b C_{bc}^a \frac{\delta H_i^c}{\delta x^j} + C_{bc}^a \frac{\delta \varepsilon^b}{\delta x^j} H_i^c + \frac{\delta^2 \varepsilon^a}{\delta x^j \delta x^i},\\
\overset{*}{\delta}\left(\frac{\delta \sigma^a}{\delta x^j}\right) &= \varepsilon^b C_{bc}^a \frac{\delta \sigma^c}{\delta x^j} + C_{bc}^a \frac{\delta \varepsilon^b}{\delta x^j} \sigma^c + \frac{\delta}{\delta x^j}\left(\frac{\partial \varepsilon^a}{\partial x^0}\right),\\
\overset{*}{\delta}\left(\frac{\partial H_i^a}{\partial x^0}\right) &= \varepsilon^b C_{bc}^a \frac{\partial H_i^c}{\partial x^0} + C_{bc}^a \frac{\partial \varepsilon^b}{\partial x^0} H_i^c + \frac{\partial}{\partial x^0}\left(\frac{\delta \varepsilon^a}{\delta x^i}\right),\\
\overset{*}{\delta}\left(\frac{\partial \sigma^a}{\partial x^0}\right) &= \varepsilon^b C_{bc}^a \frac{\partial \sigma^c}{\partial x^0} + C_{bc}^a \frac{\partial \varepsilon^b}{\partial x^0} \sigma^c + \frac{\partial^2 \varepsilon^a}{\partial x^0 \partial x^0}.
\end{aligned}$$

Replacing (3.33) in (3.32) and using the randomness of parameters ε^a, we obtain the equivalent conditions for the vanishing of $\overset{*}{\delta}(\mathcal{L}_1)$:

$$\tag{3.34}\frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta \sigma^a}{\delta x^i}\right)} + \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\partial H_i^a}{\partial x^0}\right)}\Big|_{\frac{\delta H_i^a}{\delta x^j}=ct} = 0, \quad \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\partial \sigma^a}{\partial x^0}\right)}\Big|_{\frac{\delta \sigma^a}{\delta x^j}=ct} = 0, \quad \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta H_i^a}{\delta x^j}\right)} + \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta H_j^a}{\delta x^i}\right)} = 0,$$

$$\tag{3.35}\frac{\partial \mathcal{L}_1}{\partial \sigma^a} + \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\partial H_i^b}{\partial x^0}\right)}\Big|_{\frac{\delta H_i^b}{\delta x^j}=ct} C_{ac}^b H_i^c + \phi_{ij}\frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta H_i^a}{\delta x^j}\right)} = 0,$$

$$\tag{3.36}\frac{\partial \mathcal{L}_1}{\partial H_i^a} + \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta H_j^b}{\delta x^i}\right)} C_{ac}^b H_j^c + \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta \sigma^b}{\delta x^i}\right)} C_{ac}^b \sigma^c = 0,$$

$$\tag{3.37}\frac{\partial \mathcal{L}_1}{\partial H_i^a} C_{bc}^a H_i^c + \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta H_i^a}{\delta x^j}\right)} C_{bc}^a \frac{\delta H_i^c}{\delta x^j} + \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\partial H_i^a}{\partial x^0}\right)}\Big|_{\frac{\delta H_i^a}{\delta x^j}=ct} C_{bc}^a \frac{\partial H_i^c}{\partial x^0} + \frac{\partial \mathcal{L}_1}{\partial \sigma^a} C_{bc}^a \sigma^c + \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta \sigma^a}{\delta x^i}\right)} C_{bc}^a \frac{\delta \sigma^c}{\delta x^i} = 0$$

where we have taken into account the relations (2.9) and (2.12).

Then, the additional gauge fields must enter into Lagrangian through some combinations such that the above conditions are ensured. Let us define the following differentiable functions:

$$\tag{3.38}\begin{aligned}
R_{ij}^a &= \frac{\delta H_i^a}{\delta x^j} - \frac{\delta H_j^a}{\delta x^i} + \frac{1}{2} C_{bc}^a \left(H_i^b H_j^c - H_i^c H_j^b\right) + \phi_{ji}\sigma^a,\\
P_i^a &= \frac{\partial H_i^a}{\partial x^0} - \frac{\delta \sigma^a}{\delta x^i} - \frac{1}{2} C_{bc}^a \left(\sigma^b H_i^c - H_i^b \sigma^c\right),
\end{aligned}$$

which are the local components of so called *strength fields.*

Taking into account (3.25), these functions transform (at the local coordinate changing on M) as it follows:

$$R_{ij}^a = \frac{\partial \widetilde{x}^l}{\partial x^j}\frac{\partial \widetilde{x}^k}{\partial x^i}\widetilde{R}_{kl}^a, \quad P_i^a = \frac{\partial \widetilde{x}^j}{\partial x^i}\widetilde{P}_j^a,$$

hence they are components of some adapted tensor fields, while S^a are globally defined functions on M. We obtain

$$\frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta H_i^a}{\delta x^j}\right)} = \frac{\partial \mathcal{L}_1}{\partial R_{ij}^a}, \quad \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta H_j^a}{\delta x^i}\right)} = -\frac{\partial \mathcal{L}_1}{\partial R_{ij}^a}, \quad \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\partial H_i^a}{\partial x^0}\right)}\Big|_{\frac{\delta H_i^a}{\delta x^j}=ct} = \frac{\partial \mathcal{L}_1}{\partial P_i^a}, \quad \frac{\partial \mathcal{L}_1}{\partial\left(\frac{\delta \sigma^a}{\delta x^i}\right)} = -\frac{\partial \mathcal{L}_1}{\partial P_i^a},$$

which make (3.34) true. Then, using also $C_{ab}^c = -C_{ba}^c$, we have

$$\frac{\partial \mathcal{L}_1}{\partial \sigma^a} = \phi_{ji}\frac{\partial \mathcal{L}_1}{\partial R_{ij}^a} - \frac{\partial \mathcal{L}_1}{\partial P_i^b}C_{ad}^b H_i^d,$$

$$\frac{\partial \mathcal{L}_1}{\partial H_i^a} = \frac{\partial \mathcal{L}_1}{\partial R_{ij}^b}C_{ad}^b H_j^d + \frac{\partial \mathcal{L}_1}{\partial P_i^b}C_{ad}^b \sigma^d.$$

The above relations make (3.35) and (3.36) also true. Hence, an invariant Lagrangian $\mathcal{L}_1$ must be express through fields R_{ij}^a, P_i^a such that (3.37) is satisfied.

Inspired by the Yang-Mills Lagrangian, in [7] and [5] are given such Lagrangians. According to these, we also could take the following Lagrangian, corresponding to horizontal fields R_{ij}^a and P_i^a:

$$\mathcal{L}_1 = -\frac{1}{4}C_{ac}^d C_{bd}^c\left(g^{ij}g^{kl}R_{ik}^a R_{jl}^b + 2g^{ij}P_i^a P_j^b\right), \tag{3.39}$$

which is locally gauge invariant at the local action of G.

The full Lagrangian of the system of the scalar fields Q^A and the gauge fields will be given by the sum of Lagrangian $\mathcal{L}_1$ of the gauge fields and the Lagrangian from (3.30), that is

$$\overline{\mathcal{L}}(x) = \mathcal{L}(x) + \mathcal{L}_1(x), \tag{3.40}$$

which contains the Lagrangian of scalar fields as well as the interaction between the scalar and gauge fields.

3.4 Equations of motion and conservation laws for full Lagrangian and Bianchi identities for strength fields

According to the previous discussion the Lagrangian $\overline{\mathcal{L}}$ from (3.40) is locally gauge G-invariant, hence it can be proposed as full Lagrangian for the gauge theory on the contact metric manifold M. Then, we consider the associated Lagrangian density $\overline{\mathcal{L}}_0 = H\cdot\overline{\mathcal{L}}$, and suppose the equations of motion follow from the variational principle $\delta(I(\Omega)) = 0$.

Hence, we get the following three Euler-Lagrange equations for physical scalar fields and gauge fields

$$\frac{\partial \overline{\mathcal{L}}_0}{\partial Q^A} - \frac{\partial}{\partial x^i}\left(\frac{\partial \overline{\mathcal{L}}_0}{\partial\left(\frac{\partial Q^A}{\partial x^i}\right)}\right) - \frac{\partial}{\partial x^0}\left(\frac{\partial \overline{\mathcal{L}}_0}{\partial\left(\frac{\partial Q^A}{\partial x^0}\right)}\right) = 0, \tag{3.41}$$

$$\frac{\partial \overline{\mathcal{L}}_0}{\partial H_i^a} - \frac{\partial}{\partial x^j}\left(\frac{\partial \overline{\mathcal{L}}_0}{\partial\left(\frac{\partial H_i^a}{\partial x^j}\right)}\right) - \frac{\partial}{\partial x^0}\left(\frac{\partial \overline{\mathcal{L}}_0}{\partial\left(\frac{\partial H_i^a}{\partial x^0}\right)}\right) = 0, \tag{3.42}$$

$$\frac{\partial \overline{\mathcal{L}}_0}{\partial \sigma^a} - \frac{\partial}{\partial x^i}\left(\frac{\partial \overline{\mathcal{L}}_0}{\partial\left(\frac{\partial \sigma^a}{\partial x^i}\right)}\right) - \frac{\partial}{\partial x^0}\left(\frac{\partial \overline{\mathcal{L}}_0}{\partial\left(\frac{\partial \sigma^a}{\partial x^0}\right)}\right) = 0. \tag{3.43}$$

According to the previous computations, we remark that (3.41) are equivalent with

$$\frac{\partial \overline{\mathcal{L}}}{\partial Q^A} - Q^i_A|_i - Q^0_A|_0 = \frac{1}{2}g^{js}\frac{\partial g_{js}}{\partial x^0}Q^0_A, \tag{3.44}$$

where the covariant derivatives of Q^i_A, Q^0_A are taken with respect to one of the adapted connections $\overset{\alpha}{\nabla}$, $\alpha = 1,2,3,4$, introduced in subsection 2.2.

Remark 3.4. Although, the equations (3.44) and (3.10) have the same form, we notice that they do not coincide because of the contribution of the local gauge invariant Lagrangian $\mathcal{L}(x)$ (from (3.30)) in (3.44).

On the other hand, by similar computations as in the general case of semi-Riemannian foliated manifolds [5], or of vector bundles endowed with vertical foliation [3], we get that (3.42) and (3.43) are equivalent with

$$\frac{\partial \overline{\mathcal{L}}}{\partial H^a_i} - H^{ij}_a|_j - H^i_a|_0 = \frac{1}{2}g^{js}\frac{\partial g_{js}}{\partial x^0}H^i_a, \tag{3.45}$$

and

$$\frac{\partial \overline{\mathcal{L}}}{\partial \sigma^a} - \sigma^i_a|_i = 0, \tag{3.46}$$

respectively, where

$$H^{ij}_a = \frac{\partial \overline{\mathcal{L}}}{\partial\left(\frac{\delta H^a_i}{\delta x^j}\right)}\,,\; H^{ij}_a|_j = \frac{\delta H^{ij}_a}{\delta x^j} + H^{ik}_a F^j_{kj}\,,\; H^i_a = \frac{\partial \overline{\mathcal{L}}}{\partial\left(\frac{\partial H^a_i}{\partial x^0}\right)}\Big|_{\frac{\delta H^a_i}{\delta x^j}=ct}\,,\; H^i_a|_0 = \frac{\partial H^i_a}{\partial x^0}$$

and

$$\sigma^i_a = \frac{\partial \overline{\mathcal{L}}}{\partial\left(\frac{\delta \sigma^a}{\delta x^i}\right)}\,,\; \sigma^i_a|_j = \frac{\delta \sigma^i_a}{\delta x^j} + \sigma^k_a F^i_{kj}.$$

In particular, if M is K-contact then we have the following simple form of equations of motion for full Lagrangian $\overline{\mathcal{L}}$:

$$\frac{\partial \overline{\mathcal{L}}}{\partial Q^A} - Q^i_A|_i - Q^0_A|_0 = 0\,,\; \frac{\partial \overline{\mathcal{L}}}{\partial H^a_i} - H^{ij}_a|_j - H^i_a|_0 = 0\,,\; \frac{\partial \overline{\mathcal{L}}}{\partial \sigma^a} - \sigma^i_a|_i = 0. \tag{3.47}$$

Moreover, for the horizontal and vertical currents $hJ_a = J^i_a(\delta/\delta x^i)$, $vJ_a = J^0_a\xi$ associated to the full Lagrangian $\overline{\mathcal{L}}$, we have

$$J^i_a = -Q^i_A[X_a]^A_B Q^B - H^{ji}_b C^b_{ac} H^c_j - \sigma^i_b C^b_{ac}\sigma^c\,,\; J^0_a = -Q^0_A[X_a]^A_B Q^B - H^i_b C^b_{ac} H^c_i,$$

and the conservation laws of the horizontal and vertical currents become

$$J^i_a|_i + J^0_a|_0 = \frac{1}{2}g^{js}\frac{\partial g_{js}}{\partial x^0}\left(Q^0_A[X_a]^A_B Q^B + H^i_a\right). \tag{3.48}$$

In particular, if M is K-contact, we obtain $J_a^i|_i + J_a^0|_0 = 0$.

In what follows, we are interested to obtain the Bianchi identities for the covariant derivatives of strength fields R_{ij}^a and P_i^a with respect to one of adapted connections $\overset{\alpha}{\nabla} = (\overset{\alpha}{F}{}_{ij}^k, \overset{\alpha}{D}{}_i^k, 0, 0)$, $\alpha \in \{1, 2, 3, 4\}$ introduced in subsection 2.2. Firstly, we define the following *gauge covariant derivatives* of strength fields

$$R_{ij}^a|_k = \frac{\delta R_{ij}^a}{\delta x^k} + C_{bc}^a R_{ij}^b H_k^c - R_{hj}^a \overset{\alpha}{F}{}_{ik}^h - R_{ih}^a \overset{\alpha}{F}{}_{jk}^h, \tag{3.49}$$

$$R_{ij}^a|_0 = \frac{\partial R_{ij}^a}{\partial x^0} + C_{bc}^a R_{ij}^b \sigma^c - R_{hj}^a \overset{\alpha}{D}{}_i^h - R_{ih}^a \overset{\alpha}{D}{}_j^h, \tag{3.50}$$

$$P_i^a|_k = \frac{\delta P_i^a}{\delta x^k} + C_{bc}^a P_i^b H_k^c - P_h^a \overset{\alpha}{F}{}_{ik}^h, \; P_i^a|_0 = \frac{\partial P_i^a}{\partial x^0} + C_{bc}^a P_i^b \sigma^c - P_h^a \overset{\alpha}{D}{}_i^h, \tag{3.51}$$

which are the components of some adapted tensor fields for the the characteristic foliation $\mathcal{F}_\xi$. For instance, $R_{ij}^a|_0$ satisfies

$$R_{ij}^a|_0 = \frac{\partial \widetilde{x}^k}{\partial x^i} \frac{\partial \widetilde{x}^l}{\partial x^j} \widetilde{R}_{kl}^a|_0,$$

and similar relations are satisfied by the other gauge covariant derivatives with respect to local changes of coordinates on M.

Also, the local gauge action of G on the above gauge covariant derivatives is given by the adjoint representation, that is we have

$$\overset{*}{\delta} (R_{ij}^a|_0) = \varepsilon^b C_{bc}^a R_{ij}^c|_0, \tag{3.52}$$

and similar relations for the others.

Now, using (2.9) and (2.12) we get that the Jacobi identity

$$\sum_{(i,j,k)} \left\{ \left[\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right], \frac{\delta}{\delta x^k} \right] \right\} = 0 \tag{3.53}$$

is equivalent to

$$\sum_{(i,j,k)} \frac{\partial \phi_{ij}}{\partial x^k} = 0, \tag{3.54}$$

where we have used $\partial \phi_{ij} / \partial x^0 = 0$. We also notice that (3.54) follows directly from $d(d\eta) = 0$.

Next, taking into account the local expression of the non- vanishing torsion field of an adapted connection $\overset{\alpha}{\nabla}$, that is

$$\overset{\alpha}{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) = 2\phi_{ji} \frac{\partial}{\partial x^0}, \; \overset{\alpha}{T} \left(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i} \right) = \overset{\alpha}{D}{}_i^k \frac{\delta}{\delta x^k}, \tag{3.55}$$

if we use (3.38), (3.49)-(3.51) and (3.54), we get the following *Bianchi identities* for the gauge covariant derivatives of strength fields:

$$\sum_{(i,j,k)} \left\{ R^a_{ij}|_k + 2P^a_i \phi_{kj} \right\} = 0, \tag{3.56}$$

$$P^a_i|_j - P^a_j|_i - R^a_{ij}|_0 + R^a_{jh} \overset{\alpha}{D^h_i} - R^a_{ih} \overset{\alpha}{D^h_j} = 0. \tag{3.57}$$

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