The theory of infinitesimal harmonic transformations and its applications to the global geometry of Riemann solitons

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Abstract. In the present paper we consider applications of the theory of infinitesimal harmonic transformations to the global Riemann solitons theory.

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1 Introduction

The idea of the well known Ricci flow was generalized to the concept of the Riemann flow. Riemann solitons were introduced in [1] as an analog of Ricci solitons. Namely, Riemann solitons correspond to self-similar solutions of Riemann flow (see [2] and [3]). They can be viewed as fixed points of the Riemann flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms.

In the present paper we consider applications of the theory of infinitesimal harmonic transformations (see, for example, [4]) to the global Riemann solitons theory. In the second section of our paper we give a brief survey of the basic facts of the theory of infinitesimal harmonic transformations. The results of the third section "Riemann solitons" are obtained as applications of the results of the second section of the present paper.

The results of the second section were announced in our reports at the conference "Differential Geometry" organized by the Banach Center from June 18 to June 24, 2017 at Będlewo (Poland).

2 Infinitesimal harmonic transformations

In the present paper we consider an n-dimensional ($n \geq 3$) manifold M with a Riemannian metric g and its Levi-Civita connection ∇. We also consider a flow on M

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which is a local one-parameter group of diffeomorphisms $\varphi_t(x) : M \to M$ that is generated by the smooth vector field ξ on M (see [5, p. 13-14]). In addition, a vector field ξ on a Riemannian manifold (M, g) is called an *infinitesimal harmonic transformation* of (M, g) if ξ generates a local one-parameter group of *harmonic diffeomorphisms* (see [6]). An analytic characteristic of such vector field has the form $\Box\theta = 0$ for the *Yano rough Laplacian* $\Box : T^*M \to T^*M$ and the 1-form θ corresponding to ξ under the duality defined by the metric g.

The *Yano rough Laplacian* $\Box : T^*M \to T^*M$ was defined in [7] by the formula $\Box = \delta\delta^* - \delta^*\delta$ where $\delta^* : T^*M \to S^2T^*M$ is the *symmetric derivation* defined by the formula $\delta^*\theta = L_\xi g$ for the Lie derivation L_ξ with respect to ξ and $\delta : S^2T^*M \to T^*M$ its formal adjoint operator, and it is called the *divergence*. The Yano rough Laplacian $\Box$ has another form of notation. Namely, we have proved that $\Box\theta = \Delta\theta - 2Ric(\xi, \cdot)$ where Δ is the Hodge-de Rham Laplacian and Ric is the Ricci tensor of (M, g) (see [7] and [8]).

Remark 2.1. Examples and properties of infinitesimal harmonic transformations can be found in our papers [4]; [8]; [9]; [10]. In particular, in [4], it was shown that $\xi + X$ is an infinitesimal harmonic transformation for an infinitesimal harmonic transformation ξ and any infinitesimal isometry transformation or Killing vector field X. We recall here that a vector field X on a Riemannian manifold (M, g) is called an infinitesimal isometry transformation or Killing vector field if it generates a local one-parameter group of local isometric transformations. This means, that $L_X g = 0$.

The following theorem on infinitesimal isometric transformations is well known (see, for example, [11, p. 44]).

Theorem 2.1. *Let ξ be a vector field on a Riemannian manifold (M, g) and θ be the $1-$form corresponding to ξ under the duality defined by the metric g. If ξ is an infinitesimal isometric transformation, it satisfies the following differential equations: $\Delta\theta = 2Ric(\xi, \cdot)$ and $\delta\theta = 0$. Conversely, if M is compact and ξ satisfies the above system of differential equations, then ξ is an infinitesimal isometry.*

The first equation of Theorem 2.1 $\Delta\theta = 2Ric(\xi, \cdot)$ means that ξ is an infinitesimal harmonic transformation and the second equation $\delta\theta = 0$ means that $div\xi = 0$, and it is too strict condition. We can formulate our alternative version of this theorem.

Theorem 2.2. *Let (M, g) be a compact Riemannian manifold and ξ be an infinitesimal harmonic transformation on (M, g). If ξ satisfies the condition $L_\xi div\xi \geq 0$, then ξ is an infinitesimal isometric transformation.*

Proof. Let's consider the vector field $X = (div\xi)\xi$ for an arbitrary smooth vector field ξ on a compact Riemannian manifold (M, g). The divergence of this vector field has the form

$$divX = L_\xi(div\xi) + (div\xi)^2. \tag{2.1}$$

Integrating over M and using the classic Green's theorem (see [5, p. 281])

$$\int_M (divX)dVol_g = 0$$

to the vector field $X = (div\xi)\xi$, we imply the integral formula

$$\int_M \left(L_\xi(div\xi) + (divX)^2\right) dVol_g = 0. \tag{2.2}$$

If the inequality $L_\xi(div\xi) \geq 0$ holds anywhere on M, then from (2.2) we conclude that $div\xi = 0$. Next, to complete the proof we can refer to Theorem 2.1. □

Remark 2.2. The divergence of a vector field ξ on (M, g) is a scalar function defined by (see [11, p. 4]; [5, p. 281]; [12, p. 195])

$$(div\xi)dVol_g = L_\xi(dVol_g)$$

for the canonical measure $dVol_g$ which is associated to the metric g. Due to this formula, the scalar function $div\xi$ is called the *logarithmic rate of volumetric expansion* along the flow generated by the vector field ξ (see [12, p. 195]). Therefore, the condition $L_\xi(div\xi) \geq 0$ means that $dVol_g$ is a nondecreasing scalar function along trajectories of this flow.

We have proved in [4] that on a compact Riemannian manifold of negative Ricci curvature, every infinitesimal harmonic transformation is identically zero. We shall prove it only assuming *quasi-negative Ricci curvature.* We recall the Ricci curvature is quasi-negative if it is everywhere non-positive and is in addition negative (in all directions) at a point (see [17]). In accordance with this definition we can formulate the following theorem.

Theorem 2.3. *A compact Riemannian manifold with quasi-negative Ricci curvature has no nonzero infinitesimal harmonic transformation.*

Proof. A standard calculation yields

$$\Delta\frac{1}{2}\|\xi\|^2 = -Ric(\xi, \xi) + g(\Delta\theta, \theta) - \|\nabla\xi\|^2 \tag{2.3}$$

for the Laplacian Δ. In particular, if ξ is an infinitesimal harmonic transformation, then (2.3) can be rewritten in the form

$$\Delta\frac{1}{2}\|\xi\|^2 = Ric(\xi, \xi) - \|\nabla\xi\|^2. \tag{2.4}$$

Integrating over M and using the Green's theorem, then we imply

$$\int_M \left(Ric(\xi, \xi) - \|\nabla\xi\|^2\right) dVol_g = 0.$$

If the Ricci curvature is quasi-negative then this condition contradicts the integral formula. This contradiction shows that $\xi = 0$. □

Finally, we recall that the *kinetic energy* $E(\xi)$ of the flow generated on (M, g) by a vector field ξ is determined by the following equation (see [13, p. 2])

$$E(\xi) = \int_M e(\xi) dVol_g$$

where $e(\xi) = 2^{-1}\|\xi\|^2$ is the *energy density* of the flow.

Remark 2.3. The energy $E(\xi)$ can be infinite and finite. For example, $E(\xi) < +\infty$ for a smooth complete vector field ξ on a compact Riemannian manifold (M, g).

Using the definition of the kinetic energy of a flow, we can formulate the following

Theorem 2.4. *Let (M, g) be a complete Riemannian manifold and ξ be an infinitesimal harmonic transformation. If $Ric(\xi, \xi) \leq 0$ and the flow generated by ξ has the finite kinetic energy $E(\xi)$, then ξ is a parallel vector field. Moreover, if the volume of (M, g) is infinite then this infinitesimal harmonic transformation $\xi \equiv 0$.*

Proof. Let's consider the well known *second Kato inequality* (see [14, p. 380])

$$\|\xi\|\Delta\|\xi\| \leq g(\overline{\Delta}\theta, \theta)$$

where $\overline{\Delta} := -trace_g \nabla \circ \nabla$ is the *rough Laplacian* and θ is the 1-form corresponding to ξ under the duality defined by the metric g. In turn, the rough Laplacian $\overline{\Delta}$ satisfies the Weitzenböck formula (see [14, p. 378])

$$\overline{\Delta}\theta = \Delta\theta - Ric(\xi, \cdot).$$

Then the second Kato inequality can be rewritten in the form

$$2\sqrt{e(\xi)}\Delta\sqrt{e(\xi)} \leq g(\Delta\theta, \theta) - Ric(\xi, \xi). \tag{2.5}$$

where $\|\xi\| = \sqrt{2e(\xi)}$. At the same time, we know that a vector field ξ is an infinitesimal harmonic transformation on (M, g) if and only if $\Delta\theta = 2Ric(\xi, \cdot)$. Using this equation, we can rewrite (2.5) in the form

$$2\sqrt{e(\xi)}\Delta\sqrt{e(\xi)} \leq Ric(\xi, \xi). \tag{2.6}$$

then from (2.6) we obtain $\sqrt{e(\xi)}\left(-\Delta\sqrt{e(\xi)}\right) \geq 0$. In [15, p. 664] and [16] was shown that every non-negative smooth function u defined on a complete Riemannian manifold (M, g) and satisfying the conditions $u(-\Delta u) \geq 0$ and $\int_M u^p dVol_g < +\infty$ for all $p \neq 1$, must be constant. In particular, if the volume of (M, g) is infinite, then $u = 0$. Therefore, if $Ric(\xi, \xi) \leq 0$ and

$$E(\xi) = \int_M e(\xi) dVol_g < +\infty, \tag{2.7}$$

then from (2.5) we conclude that the function $\sqrt{e(\xi)}$ is constant. At the same time, we obtain from (2.4) that the volume of (M, g) is finite unless ξ is identically equal to zero, i.e. $\xi \equiv 0$.

If $Ric(\xi, \xi) \leq 0$ and $e(\xi) = const$, then we obtain from (2.7) that $\nabla\xi = 0$ The proof is complete. □

3 Riemann solitons

Let g be a fixed Riemannian metric on a smooth manifold M and R be its Riemannian curvature tensor. Consider the family of diffeomorphisms $\varphi_t(x) : M \to M$ that is generated by the smooth vector field ξ on M. The evolutive metric $g(t) = \sigma(t)\varphi_t^*(x)g(0)$ for a positive scalar $\sigma(t)$ such that $\sigma(0) = 1$ and $g(0) = g$ is a *Riemann soliton* if the metric g is a solution of the nonlinear stationary PDF

$$2R + \lambda g \wedge g + g \wedge L_\xi g = 0 \tag{3.1}$$

where λ is a constant, " $\wedge$ " is the Kulkarni-Nomizu product (see [18, p. 47]). To simplify notation, we denote the Riemann soliton in the following way (M, g, ξ, λ). A Riemann soliton is called *shrinking* when $\lambda < 0$, *steady* when $\lambda = 0$ and *expanding* when $\lambda > 0$. If ξ is a gradient, i.e., $\xi = grad\ f$ for some smooth scalar function f, then we get the notion of *gradient Riemann soliton.* In [1] was shown that a Riemann soliton on a compact manifold M is gradient. We call the vector field ξ the *potential field* of the Riemann soliton. In particular, if the potential field of a Riemann soliton is identically zero, then we call this Riemann soliton a *trivial soliton.*

In terms of local coordinate system $x^1, x^2, \cdots, x^n$, the equation (3.1) has the form (see also [2])

$$\begin{aligned} -2R_{ijkl} = 2\lambda(g_{ik}g_{jl} - g_{il}g_{jk}) + (\nabla_i\xi_k + \nabla_k\xi_i)\, g_{jl} + (\nabla_j\xi_l + \nabla_l\xi_j)\, g_{ik} - \\ - (\nabla_i\xi_l + \nabla_l\xi_i)\, g_{jk} - (\nabla_j\xi_k + \nabla_k\xi_j)\, g_{il}. \end{aligned} \tag{3.2}$$

where R_{ijkl} and g_{ij} are local components of R and g, respectively. Moreover, $L_\xi g_{ij} = \nabla_i\xi_j + \nabla_j\xi_i$ where ∇_i is the covariant derivative with respect to $\dfrac{\partial}{\partial x^i}$ and $\xi_i = g_{ik}\xi^k$ for the potential field $\xi = \xi^k \dfrac{\partial}{\partial x^k}$. From (3.2) we obtain

$$-2R_{jl} = 2(n-1)\lambda g_{jl} + 2\nabla_k\xi^k g_{jl} + (n-2)(\nabla_j\xi_l + \nabla_l\xi_j); \tag{3.3}$$

$$-s = n(n-1)\lambda + 2(n-1)\nabla_k\xi^k \tag{3.4}$$

where R_{jl} are local components of the Ricci tensor Ric and s is the scalar curvature of (M, g). Next, we rewrite the equations (3.3) and (3.4) in the following forms

$$\delta^*\theta = -\frac{2}{n-2}\left(Ric + (n-1)\lambda g - \delta\theta g\right); \tag{3.5}$$

$$\delta\theta = \frac{1}{2(n-1)}\left(s + n(n-1)\lambda\right). \tag{3.6}$$

In turn, from the equations (3.5) and (3.6) we obtain

$$\delta^*(\delta\theta) = \frac{1}{2(n-1)} ds; \tag{3.7}$$

$$\delta(\delta^*\theta) = -\frac{1}{n-2}\left(2\delta Ric - 2\delta(\delta\theta g)\right) = \frac{1}{n-2}\left(ds - 2\delta^*(\delta\theta)\right) \tag{3.8}$$

where we used the following identities $\delta Ric = -\frac{1}{2}ds$ (see [18, p. 43]). Then using the equations (3.7) and (3.8), we have

$$\Box\theta = d(\delta\theta). \tag{3.9}$$

where $d(\delta\theta) = \frac{1}{2(n-1)}ds$. The following theorem is obvious.

Theorem 3.1. *A Riemann soliton (M, g, ξ, λ) has the constant scalar curvature s if and only if its potential field ξ is an infinitesimal harmonic transformation.*

Remark 3.1. We have proved that the potential field of a *Ricci soliton* is an infinitesimal harmonic transformation (see [7]).

From the above theorem we conclude that the following corollaries hold.

Corollary 3.2. *If the scalar curvature s of a compact Riemann soliton (M, g, ξ, λ) satisfies the inequality $s \geq n(n-1)\lambda$ (or $s \leq n(n-1)\lambda$), then this soliton is a Riemannian manifold of constant curvature $C = -\lambda$ and its potential field ξ is a zero vector field.*

Proof. Consider a compact Riemann soliton. We apply the Green's theorem to its vector field ξ, then we obtain from (3.4) the integral formula

$$\int_M (s + n(n-1)\lambda)\, dVol_g = 0.$$

If the scalar curvature s of our Riemann soliton satisfies the inequality $s \geq n(n-1)\lambda$ (*or* $s \leq n(n-1)\lambda$) then from above integral formula it follows that $s = -n(n-1)\lambda$. It means that $\Box\theta = 0$ and $div\xi = 0$. In this case, from Theorem 2.1 we know that the potential field ξ is an infinitesimal isometric transformation. On the other hand, we know that $\xi = \nabla f$ on compact manifold (see [1]). In this case, the equation $L_\xi g = 0$ can be rewritten in the form $\nabla\nabla f = 0$. In particular, from this equation we obtain that $\Delta f = 0$. Then $f = const$ because (M, g) is a compact Riemannian manifold. Then ξ is a zero vector field. Then from (3.2) we conclude that our Riemann soliton is a Riemannian manifold of constant curvature $C = -\lambda$. $\Box$

Remark 3.2. A compact Ricci soliton is trivial if the condition $L_\xi s \leq 0$ is satisfied (see [19]).

Corollary 3.3. *If the scalar curvature s of a compact Riemann soliton (M, g, ξ, λ) is a nonincreasing scalar function along trajectories of the flow that is generated by the potential field ξ then (M, g) is a Riemannian manifold of constant sectional curvature $C = -\lambda$ and the potential field ξ is a zero vector field.*

Proof. Let the scalar curvature s and the potential field ξ of a compact Riemann soliton satisfy the condition $L_\xi s \leq 0$. Using the equation (3.4), we can rewrite this condition in the form $L_\xi(div\xi) \geq 0$. Then from Theorem 2.2 we obtain that s is a constant and $div\xi = 0$. This means that ξ is an infinitesimal isometric transformation and therefore ξ is a zero vector field. In this case, from (3.2) we conclude that (M, g) is a Riemannian manifold of constant sectional curvature $C = -\lambda$. □

The well-known *Bieberbach theorem* states that every compact flat Riemannian manifold (M, g) is finitely covered by a flat torus. More precisely, (M, g) has the form $(\Gamma \setminus G)/H$ where G is a group of translations of euclidian space, $\Gamma \subset G$ is a discrete subgroup, and H is a finite group of isometric of the space of right cosets $\Gamma \setminus G$. For a proof see [20]. Therefore we have

Corollary 3.4. *Let the scalar curvature s and the potential field ξ of a compact steady Riemann soliton (M, g, ξ, λ) satisfy the conditions $L_\xi s \leq 0$, then (M, g) is finitely covered by a flat torus.*

On the other hand, it was shown by Hopf that a compact, simply connected Riemannian manifold with positive constant sectional curvature $C > 0$ is isometric to a Euclidian sphere, equipped with its standard metric (see [21]; [22]). More generally, if (M, g) is a compact Riemannian manifold with constant sectional curvature $C > 0$, then (M, g) is a *spherical space form* (see [20, p. 69]). For the even dimensional these forms are the Euclidian $2k$-sphere $\mathbf{S}^{2k}$ and the real projective $2k$-space $\mathbb{R}\mathbf{P}^{2k}$. Therefore we have the following corollary.

Corollary 3.5. *Let the scalar curvature s and the potential field ξ of a compact shrinking n-dimensional Riemann soliton (M, g, ξ, λ) satisfies the conditions $L_\xi s \leq 0$, then (M, g) is a spherical space form. In particular, for the even dimensional $n = 2k$ it is the Euclidian sphere $\mathbf{S}^{2k}$ or the real projective space $\mathbb{R}\mathbf{P}^{2k}$.*

Using the definition of the kinetic energy of a flow, we can formulate the following theorem.

Theorem 3.6. *Let a nonzero potential field ξ of a complete, connected and nontrivial n-dimensional Riemann soliton (M, g, ξ, λ) generates a flow with finite kinetic energy. If the scalar curvature s of this soliton is a nonincreasing function along trajectories of the flow and $Ric(\xi, \xi) \leq 0$, then (M, g) is a Euclidian space form.*

Proof. Let's consider our variant of the second Kato inequality

$$2\sqrt{e(\xi)}\Delta\sqrt{e(\xi)} \leq g(\Delta\theta, \theta) - Ric(\xi, \xi). \tag{3.10}$$

where $\|\xi\| = \sqrt{2e(\xi)}$. On the other hand, we have proved that the potential field ξ of a Riemann soliton satisfies the equation $\Delta\theta = 2Ric(\xi, \cdot) + (n-1)^{-1}L_\xi s$. Therefore, we can rewrite the ine-quality (3.10) in the form

$$\sqrt{e(\xi)}\Delta\sqrt{e(\xi)} \leq \frac{1}{2}Ric(\xi, \xi) + \frac{1}{2(n-1)}L_\xi s. \tag{3.11}$$

If the Ricci tensor Ric is non-positive and $L_\xi s \le 0$, then from (3.13) we obtain $\sqrt{e(\xi)}\left(-\Delta\sqrt{e(\xi)}\right) \ge 0$. If, in addition, (M, g) is complete and

$$E(\xi) = \int_M e(\xi) dVol_g < +\infty,$$

then $\sqrt{e(\xi)}$ is a constant function (see [15, p. 664] and [16]). On the other hand, if ξ is a potential field of a Riemann soliton, then (2.6) can be rewritten in the form

$$\Delta e(\xi) = Ric(\xi, \xi) - \|\nabla\xi\|^2 + \frac{1}{n-1} L_\xi s. \tag{3.12}$$

If $Ric(\xi, \xi) \le 0$, $L_\xi s \le 0$ and $e(\xi)$ is a constant function, then we obtain from (3.12) that $\nabla\xi = 0$ In this case, from (3.2) we conclude that (M, g) is a Riemannian manifold of constant curvature $C = -\lambda$. At the same time, we have $0 = Ric(\xi, \cdot) = -\lambda(n-1)\theta$ for $\theta \ne 0$. This means that $\lambda = 0$. Therefore, the tensor curvature R of (M, g) is identically zero. If this (M, g) is complete, connected and simply connected Riemannian manifold, then it is a Euclidian space form by the well-known *Killing-Hopf theorem* (see [20, p. 69]). This completes the proof of the theorem. □

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