About the projective Finsler metrizability: First steps in the non-isotropic case

T. Milkovszki and Z. Muzsnay

Abstract. We consider the projective Finsler metrizability problem: under what conditions the solutions of a given system of second-order ordinary differential equations (SODE) coincide with the geodesics of a Finsler metric, as oriented curves. SODEs with isotropic curvature have already been thoroughly studied in the literature and have proved to be projective Finsler metrizable. In this paper, we investigate the non-isotropic case and obtain new results by examining the integrability of the Rapcsák system extended with curvature conditions. We consider the n-dimensional generic case, where the eigenvalues of the Jacobi tensor are pairwise different. We identify the higher order compatibility condition of the system causing the non 2-acyclicity of the Spencer sequences and the Cartan's test to fail. We also consider the three-dimensional case, where we find a class of non-isotropic sprays for which the PDE system is integrable and, consequently, the corresponding SODEs are projective metrizable.

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1 Introduction

The projective metrizability problem can be formulated as follows: under what conditions the solutions of a given system of second-order ordinary differential equations coincide with the geodesics of some metric space, as oriented curves. This problem can be seen as a particular case of the inverse problem of the calculus of variations. One can consider the Riemannian [2, 10] and the more general Finslerian version of this problem [1, 4, 5, 8, 13, 15, 16]. In this paper we examine the latter: starting with a homogeneous system of second-order ordinary differential equations, which can be identified with a spray S, we seek for a Finsler metric F whose geodesics coincide with the geodesics of the spray S, up to an orientation preserving reparameterization. For flat sprays this problem was first studied by Hamel [12] and it is known as the

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Finslerian version of Hilberts fourth problem [1, 8]. Rapcsák obtained necessary and sufficient conditions in the general case [15].
There are different approaches to tackle the problem: In [9] the authors use the multiplier method which is, in the context of the inverse problem of the calculus of variations, probably the most used and studied approach. In [4] the projective metrizability problem was reformulated in terms of a first-order partial differential equation and a set of algebraic conditions on a semi-basic 1-form. Finally, [13] turns back to the original idea of Rapcsák by considering the so called Rapcsák system, which is the PDE system composed by the Euler-Lagrange partial differential equations and the homogeneity condition on the unknown Finsler function F. The different approaches are equivalent and can lead to effective results. For example sprays with isotropic curvature was investigated in all three different approaches and it was proved that this class of spray is projective metrizable. Unfortunately, beyond the isotropic case, there are practically no results about this problem in the literature.
In this paper we make a step in the direction to find new results in the non-isotropic case by investigating the integrability of the Rapcsák system. Here the difficulties come from the fact that the PDE system is largely overdetermined: on an n-dimensional manifold there are $n + 1$ equations on the unknown Finsler function, therefore many integrability conditions arise. That is why in the generic case there is no solution to the problem. In [13] the first compatibility conditions of the Rapcsák system were already determined: they can be expressed in terms of equations containing the associated nonlinear connection and its curvature tensor. When the spray is isotropic, these conditions are satisfied. However, when the spray is non-isotropic, the integrability conditions are not satisfied and further compatibility conditions appear. From that point, the analysis becomes quite difficult, because of two reasons. First: the curvature tensor has no canonical normal form, therefore each class of sprays having different curvature form must be considered separately. Second: as it has been shown in [13], the system containing the curvature condition may be not 2-acyclic (the Cartan's test fails), that is higher order compatibility conditions can arise.

The paper is organized as follows. In Section 2 we give a brief introduction to the canonical structures on the tangent bundle of a manifold and the main structures needed to discuss the geometry of a spray: connection, Jacobi endomorphism, curvature. We also recall the basic tools of Cartan-Kähler theory. In Section 3 the extended Rapcsák system with curvature condition is considered in the n-dimensional generic case, when the eigenvalues of the Jacobi curvature tensor Φ are pairwise different: in Subsection 3.1 we compute its first compatibility conditions and in Subsection 3.2, using the prolonged system, we find the higher order compatibility conditions. In Section 4 we consider the 3-dimensional case: We prove that the symbol of the prolonged system is 2-acyclic and discuss in detail the reducible case. Finally, we identify a class of non-isotropic sprays for which the second and the third order conditions are identically satisfied and the symbol of the prolonged system is 2-acyclic. Therefore we obtain the formal integrability of the system and, in the analytic case, the projective metrizability of this class of sprays.

2 Preliminaries

In this paper M denotes an n-dimensional smooth manifold, $C^\infty(M)$ is the ring of the smooth functions on M and $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of vector fields on M. The set of the skew-symmetric, symmetric and vector valued k-forms are $\Lambda^k(M)$, $S^k(M)$ and $\Psi^k(M)$, respectively. Furthermore, $\Lambda^k_v(TM)$ stands for the set of the semi-basic k-forms. The tangent bundle (TM, π, M) and the slashed tangent bundle $(\mathcal{T}M := TM \setminus \{0\}, \pi, M)$ of M will simply be denoted by TM and $\mathcal{T}M$. The tangent bundle of TM will be denoted by TTM or T. $VTM = \mathrm{Ker}\,(\pi_* : TTM \to TM)$ is the vertical sub-bundle of T.

We denote the coordinates on M by $x = (x^i)$ and the induced coordinates on TM by $(x, y) = (x^i, y^i)$. The local expressions of the Liouville vector field $C \in \mathfrak{X}(TM)$ corresponding to the infinitesimal dilatation in the fibres, and the vertical endomorphism $J \in \Psi^1(TM)$ are

$$C = y^i \frac{\partial}{\partial y^i}, \qquad J = dx^i \otimes \frac{\partial}{\partial y^i}.$$

Using Euler's theorem on homogeneous functions $f \in C^\infty(\mathcal{T}M)$ is positive homogeneous of degree k if $\mathcal{L}_C f = kf$.

A *spray* on M is a vector field $S \in \mathfrak{X}(\mathcal{T}M)$ satisfying the conditions $JS = C$ and $[C, S] = S$. The coordinate expression of a spray S takes the form

$$S = y^i \frac{\partial}{\partial x^i} + f^i(x, y) \frac{\partial}{\partial y^i},$$

where the spray coefficients $f^i = f^i(x, y)$ are 2-homogeneous functions. The *geodesics of a spray* S are curves $\gamma : I \to M$ such that $S \circ \dot\gamma = \ddot\gamma$. Locally, they are the solutions of the second order ordinary differential equations $\ddot x^i = f^i(x, \dot x)$, $i = 1, \dots, n$.

A *Finsler function* on a manifold M is a continuous function $F \colon TM \to \mathbb{R}$, smooth and positive away from the zero section, homogeneous of degree 1 and the matrix composed by the coefficients $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ of the metric tensor $g = g_{ij} dx^i \otimes dx^j$ is positive definite on $\mathcal{T}M$. Consequently, the Hessian of F is positive quasi-definite in the sense that $\frac{\partial^2 F}{\partial y^i \partial y^j} v^i v^j \geq 0$ with equality only if v is a scalar multiple of y (see [9]). The pair (M, F) is called Finsler manifold. To any Finsler function F there exits a unique canonical spray S_F, such that the geodesics of F are the geodesic of S_F. The canonical spray is characterized by the equation $i_{S_F} dd_J F^2 = -dF^2$. A spray S is called *Finsler metrizable* if there exist a Finsler metric F whose canonical spray is $S_F = S$. A spray S is called *projective Finsler metrizable* if there exist a Finsler metric F whose canonical spray is projective equivalent to S i.e. $S_F \sim S$. In that case the geodesics of the two sprays coincide up to an orientation preserving reparametrization. According to Rapcsák's result [15], the spray S is projective Finsler metrizable if and only if there exists a Finsler function F such that $i_S \Omega = 0$, where $\Omega := dd_J F$ $(\Omega \in \Lambda^2(TM))$. Consequently, the spray S is projective Finsler metrizable if and only if the *Rapcsák system*

$$\big\{\mathcal{L}_C F - F = 0, \quad i_S \Omega = 0\big\} \tag{2.1}$$

admits a positive quasi-definite solution F. The first compatibility conditions of the partial differential system (2.1) were determined in [13] in terms of geometric objects associated to the spray. To present them, let us consider the following notions. The *connection* associated to S is the vector valued one form $\Gamma := [J, S]$. One has $\Gamma^2 = \mathrm{Id}$ and the eigenspaces of Γ corresponding to the eigenvalue $+1$ and -1 are the vertical and the horizontal subspaces. The horizontal bundle is denoted by HTM. The horizontal and vertical projectors associated to the connection are $h := \frac{1}{2}(\mathrm{I} + \Gamma)$ and $v := \frac{1}{2}(\mathrm{I} - \Gamma)$. We have $hS = S$, that is the spray is a horizontal vector field. The *curvature* $R \in \Psi^2(TM)$ of the connection Γ is the Nijenhuis torsion of the horizontal projection $R = \frac{1}{2}[h, h]$. The *Jacobi endomorphism* Φ can be derived from the curvature by $\Phi := i_S R$. They are also related by the equation $\frac{1}{3}[J, \Phi] = R$. In [13] it was proved that the spray S is projective metrizable if and only if there exists a regular function F on $\mathcal{T}M$ such that it is a solution to the *extended Rapcsák system*:

$$\big\{\mathcal{L}_C F - F = 0, \quad i_\Gamma \Omega = 0\big\}. \tag{2.2}$$

The compatibility conditions of the system (2.2) can be expressed a coordinate free way in terms of the curvature tensor R of Γ by the equation $i_R\Omega = 0$. In the case when $\dim M = 2$ or S is flat or more of isotropic curvature that is the flag curvature does not depend on the direction (cf. [6, 7]), then this compatibility condition is satisfied and the system (2.2) is integrable. One obtains that in the analytic case the spray S is locally projective metrizable [13].

In this article we are investigating the projective metrizability when the curvature of the spray is non-isotropic. In that case the compatibility condition of (2.2) is not satisfied. This is why one has to consider an enlarged system by adding to (2.2) its compatibility condition. We remark that instead of the curvature tensor, one can express the compatibility condition in terms of the Jacobi tensor (see [9]) and consider the enlarged the system:

$$\mathcal{L}_C F - F = 0, \qquad i_\Gamma \Omega = 0, \qquad i_\Phi \Omega = 0. \tag{2.3}$$

Depending on the algebraic form of Φ there are many cases and sub cases to consider. Moreover, there is a new phenomenon appearing: the partial differential system (2.3) is not 2-acyclic [13, Section 5], which is the indication of the existence of higher order integrability condition. All these informations suggest that the non-isotropic case may be extremely complex. We consider the generic case, when Φ is diagonalizable with distinct eigenvalues in the following sense: A function $\lambda \in C^\infty(TM)$ is called an eigenfunction and $X \in \mathfrak{X}(TM)$ is an eigenvector field of Φ if X is a horizontal and $\Phi_u X_u = \lambda_u J X_u$ for any $u \in \mathcal{T}M$. It is easy to verify that the spray S is an eigenvector field of Φ and the corresponding Jacobi eigenfunction is $\lambda = 0$.

To investigate the integrability of the system (2.3) we use the Spencer-Goldschmidt's integrability theory. Here we just set the notation. For more details, we refer to [3, 11] and [13] in the context of projective metrizability. Let $E = (E, \pi, M)$ be a fibred bundle over the manifold M. $J_k(E)$ denotes the bundle of k-jets of sections of E. Then $J_{k+1}(E)$ become a fibred bundle over $J_k(E)$ with the projection $\pi_k : J_{k+1}(E) \to J_k(E)$. Let E and $\widetilde{E}$ be vector bundles over the manifold M and $P : Sec(E) \to Sec(\widetilde{E})$ be a k^{th} order differential operator. Then P can be identified with the map $p_k(P)\colon J_k E \to \widetilde{E}$ and a natural way the l^{th} prolongation can be introduced as $p_{k+l}(P)\colon J_{k+l}E \to J_l\widetilde{E}$. The elements of $Sol_{k+l} := \operatorname{Ker} p_{k+l}(P)$ is called the l^{th} order formal solutions. P is *formally integrable* if Sol_l is a vector bundle over M for all $l \geq k$, and the map $\overline{\pi}_l : Sol_{l+1} \to Sol_l$ is onto $\forall\ l \geq k$. In that case any k^{th} order solution can be lift into an infinite order formal solution. The highest order terms of P said to be the symbol of P. It can be interpreted as a map $\sigma_k \colon S^kT^*M \otimes E \to \widetilde{E}$. The l^{th} order prolongation of the symbol is denoted as $\sigma_{k+l} \colon S^{k+l}T^*M \otimes E \to S^lT^*M \otimes \widetilde{E}$.

The existence or the non-existence of higher order compatibility condition can be calculated classically with the Cartan's test but more information can be obtained about the higher order compatibility conditions from the Spencer cohomology groups. Classical version of the Cartan-Kähler integrability theorem uses the Cartan's test while its generalization, proved by Goldschmidt, uses the Spencer cohomology groups. Let us introduce the two concepts. A basis $(e_i)_{i=1}^n$ of T_xM is *quasi-regular* if $\dim g_{k+1,x} = \dim g_{k,x} + \sum_{j=1}^n \dim(g_{k,x})_{e_1,\dots,e_j}$, where one notes $g_{k+l} := \operatorname{Ker} \sigma_{k+l}$ and $(g_{k,x})_{e_1,\dots,e_j} := \left\{ A \in g_{k,x} \,|\, i_{e_1}A = 0, \dots i_{e_j}A = 0 \right\}$, $(1 \leq j \leq n)$. The symbol σ_k is *involutive* if there exists a quasi-regular basis of T_xM at any $x \in M$ (Cartan's test). Moreover, the symbol of P is called *2-acyclic* if for all $m \geq k$ the $(m, 2)$ Spencer cohomology groups $H^{m,2} = \operatorname{Ker} \delta_2^m / \operatorname{Im} \delta_1^m$ vanish where

$$g_{m+1} \otimes T^*M \xrightarrow{\delta_1^m} g_m \otimes \Lambda^2 T^*M, \qquad g_m \otimes \Lambda^2 T^*M \xrightarrow{\delta_2^m} g_{m-1} \otimes \Lambda^3 T^*M$$

are the natural skew-symmetrizations. We remark that if $\sigma_k(P)$ is *involutive* then all Spencer cohomology groups are zero.

Theorem 2.1 (Cartan-Kähler/Goldschmidt)**.** *Let $P \colon J_kE \to \widetilde{E}$ be a k^{th} order regular linear partial differential operator. If $\overline{\pi}_k : Sol_{k+1} \to Sol_k$ is surjective and σ_k is involutive/2-acyclic, then P is formally integrable.*

Remark 2.1 (Computation of the first compatibility condition)**.**
In the practice the surjectivity of $\overline{\pi}_k$, or in other words the first compatibility condition, can be computed as follows: Using the snake lemma of homological algebra one can show that there exits a morphism φ such that the sequence

$$Sol_{k+1} \xrightarrow{\overline{\pi}_k} Sol_k \xrightarrow{\varphi} \operatorname{Coker}(\sigma_{k+1}) \tag{2.4}$$

is exact. From the exactness of (2.4) we get that $\overline{\pi}_k$ is onto if and only if $\varphi = 0$. The partial differential equation $\varphi = 0$ is called the *first compatibility condition* of P. To compute φ we note that if $\tau : T^* \otimes \widetilde{E} \to K$ is a morphism such that $\operatorname{Ker} \tau = \operatorname{Im} \sigma_{k+1}$ then $\operatorname{Im} \tau$ is isomorphic to $\operatorname{Coker}(\sigma_{k+1})$ and

$$\varphi = \tau \circ (\nabla P)|_{Sol_k}, \tag{2.5}$$

where ∇ is an arbitrary linear connection on F (see [11, p. 28]). The meaning of formula (2.5) is the following: differentiate the kth order equations, take the convenient linear combination of the equation in order to eliminate the highest (that is the $k+1$st) order terms form the equation, and obtain this way the k order compatibility conditions.

3 Extended Rapcsák system with curvature condition

Let S be a spray on M and $\mathcal{P} \colon C^\infty(TM) \longrightarrow C^\infty(TM) \times \Lambda^2_v(TM) \times \Lambda^2_v(TM)$ the differential operator corresponding to the second order linear partial differential system

(2.3):

$$\mathcal{P} := (P_\Gamma, P_C, P_\Phi) \tag{3.1}$$

where

$$P_C F := \mathcal{L}_C F - F, \qquad P_\Gamma F := i_\Gamma \Omega, \qquad P_\Phi F := i_\Phi \Omega,$$

with $\Omega := dd_J F$. We suppose that the Jacobi endomorphism Φ has n distinct eigenvalues. In this chapter we compute the first integrability conditions of order 2, and the the higher order compatibility condition of order 3 of system (3.1). It turns out that in some cases, even though the Cartan's test fails, these compatibility conditions give the complete set of obstructions to the integrability and in very specific situations the system becomes integrable (see Chapter 4).

3.1 First compatibility conditions

First we remark that $hS = S$ that is the spray is horizontal with respect to the connection associated. Moreover, from the definition of the Jacobi endomorphism we get

$$\Phi(S) = i_S R(S) = R(S,S) = 0, \tag{3.2}$$

that is S is an eigenvector of Φ and the corresponding eigenvalue is $\lambda = 0$. Let $\lambda_1, \dots, \lambda_n$ be the n distinct eigenfunctions of Φ and $h_1, \dots, h_n$ the corresponding eigenvector fields, where $\lambda_n = 0$ and $h_n = S$. For any $x \in \mathcal{T}M$ we consider the basis

$$\mathcal{B} := \{h_1, \dots, h_n, v_1, \dots, v_n\} \quad \subset T_x TM, \tag{3.3}$$

where $Jh_i = v_i$, $i = 1, \dots, n$. We have the following

Proposition 3.1. *A 2^{nd} order solution $s = j_2(F)_x$ of $\mathcal{P}$ at $x \in \mathcal{T}M$ can be lifted into a 3^{rd} order solution, if and only if*

$$i_{[\Phi,\Phi]} \Omega_x = 0, \tag{3.4a}$$

$$\sum_{ijk}^{cycl} (\Omega_x([v_i, h_j], h_k))_x = 0. \tag{3.4b}$$

To compute the compatibility conditions we use the method described in Remark 2.1. The symbol of the system $\mathcal{P}$ is composed by the symbol of the operator P_C, P_Γ and P_Φ:

$$\sigma_2(\mathcal{P}) = \big(\sigma_2(P_C), \sigma_2(P_\Gamma), \sigma_2(P_\Phi)\big) : \quad S^2T^* \longrightarrow T^* \times \Lambda^2 T_v^* \times \Lambda^2 T_v^*,$$
$$\sigma_3(\mathcal{P}) = \big(\sigma_3(P_\Gamma), \sigma_3(P_C), \sigma_3(P_\Phi)\big) : \quad S^3T^* \longrightarrow S^2T^* \times (T^* \otimes \Lambda^2 T_v^*) \times (T^* \otimes \Lambda^2 T_v^*),$$

where the symbol of the first order operator P_C is

$$\sigma_1(P_C) A_1 = A_1(C).$$

The prolongation of the symbol of P_C and the symbol of the second order P_Γ and P_Φ are

$$\begin{aligned}
&[\sigma_2(P_C)A_2](X) = A_2(X, C),\\
&[\sigma_2(P_\Gamma)A_2](X, Y) = 2[A_2(hX, JY) - A_2(hY, JX)],\\
&[\sigma_2(P_\Phi)A_2](X, Y) = A_2(\Phi X, JY) - A_2(\Phi Y, JX),
\end{aligned}$$

and their prolongations at third order level are

$$\begin{aligned}
&[\sigma_3(P_C)A_3](X, Y) = A_3(X, Y, C),\\
&[\sigma_3(P_\Gamma)A_3](X, Y, Z) = 2[A_3(X, hY, JZ) - A_3(X, hZ, JY)],\\
&[\sigma_3(P_\Phi)A_3](X, Y, Z) = A_3(X, \Phi Y, JZ) - A_3(X, \Phi Z, JY),
\end{aligned}$$

where $X, Y, Z \in T$, $A_i \in S^i T^*$, $i = 1, 2, 3$. According to Remark 2.1, to compute the compatibility condition we should construct a map τ such that $\operatorname{Ker}\tau = \operatorname{Im}\sigma_3(\mathcal{P})$. Let us consider the map

$$\tau := (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_{ijk}), \tag{3.5}$$

as follows: if $B = (B_C, B_\Gamma, B_\Phi)$ is an element of $S^2T^* \times (T^* \otimes \Lambda^2 T_v^*) \times (T^* \otimes \Lambda^2 T_v^*)$ then

$$\begin{aligned}
&\tau_1(B)(X, Y, Z) = B_\Gamma(hX, Y, Z) + B_\Gamma(hY, Z, X) + B_\Gamma(hZ, X, Y),\\
&\tau_2(B)(X, Y, Z) = B_\Gamma(JX, Y, Z) + B_\Gamma(JY, Z, X) + B_\Gamma(JZ, X, Y),\\
&\tau_3(B)(X, Y) = \tfrac{1}{2} B_\Gamma(C, X, Y) - B_C(hX, JY) + B_C(hY, JX),\\
&\tau_4(B)(X, Y, Z) = B_\Phi(\Phi X, Y, Z) + B_\Phi(\Phi Y, Z, X) + B_\Phi(\Phi Z, X, Y),\\
&\tau_5(B)(X, Y, Z) = B_\Phi(JX, Y, Z) + B_\Phi(JY, Z, X) + B_\Phi(JZ, X, Y),\\
&\tau_6(B)(Y, Z) = B_\Phi(C, Y, Z) - B_C(\Phi Y, JZ) + B_C(\Phi Z, JY),\\
&\tau_7(B)(X, Y) = B_\Phi(X, Y, S) - B_C(X, \Phi Y),\\
&\tau_{ijk}(B) = \tfrac{1}{2} B_\Gamma(v_i, h_j, h_k) + \tfrac{1}{\lambda_k - \lambda_i} B_\Phi(h_j, h_i, h_k) + \tfrac{1}{\lambda_i - \lambda_j} B_\Phi(h_k, h_i, h_j),
\end{aligned}$$

where i, j, k $(1 \leq i, j, k \leq n)$ are pairwise different indices.

Lemma 3.2. *With the above notation we have* $\operatorname{Ker}\tau = \operatorname{Im}\sigma_3(\mathcal{P})$.

Proof. It is not difficult to verify that $\operatorname{Ker}\tau \subset \operatorname{Im}\sigma_3(\mathcal{P})$. Moreover, comparing the dimension of the two space, one can find that they are equal (see [14, Lemma 3.2]). Consequently on obtain the Lemma. □

Proof of Proposition 3.1. The following commutative diagram shows the maps introduced above:

$$\begin{array}{ccccccc}
g_3(\mathcal{P}) & \xrightarrow{\ i\ } & S^3T^* & \xrightarrow{\sigma_3(\mathcal{P})} & S^2T^* \times (T^* \otimes \Lambda^2 T_v^*) \times (T^* \otimes \Lambda^2 T_v^*) & \xrightarrow{\ \tau\ } & K \longrightarrow 0\\
\downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \\
Sol_3(\mathcal{P}) & \xrightarrow{\ i\ } & J_3\mathbb{R} & \xrightarrow{p_3(\mathcal{P})} & J_2(\mathbb{R}_{TM}) \times J_1(\Lambda^2 T_v^* \times \Lambda^2 T_v^*) & & \\
\downarrow{\scriptstyle \overline{\pi}_2} & & \downarrow{\scriptstyle \pi_2} & & \downarrow & & \\
Sol_2(\mathcal{P}) & \xrightarrow{\ i\ } & J_2\mathbb{R} & \xrightarrow{p_2(\mathcal{P})} & J_1(\mathbb{R}_{TM}) \times \Lambda^2 T_v^* \times \Lambda^2 T_v^* & &
\end{array}$$

To compute the first compatibility condition of the system represented by $\mathcal{P}$ we use the method described in Remark 2.1: differentiate the 2th order equations, take the convenient linear combination of the equations (given by the map τ) eliminate the highest (that is the 3st) order terms form the equation. What one gets this way is a new 2 order compatibility condition. Let F be a second order solution of $\mathcal{P}$ at a point x, that is $(\mathcal{P}F)_x = 0$:

$$(3.6)\qquad (\mathcal{L}_C F - F)_x = 0, \quad \big(\nabla(\mathcal{L}_C F - F)\big)_x = 0, \quad (i_\Gamma dd_J F)_x = 0, \quad (i_\Phi dd_J F)_x = 0.$$

Using the map τ given in (3.5) the integrability condition can be computed by the formula (2.5):

$$\begin{aligned}
\tau_1(\nabla\mathcal{P}F)_x &= d_h(i_\Gamma dd_J F)_x \\
&= (2d_h(i_h d - d i_h) d_J F)_x = (2 d_h d_h d_J F)_x \\
&= (d_R d_J F)_x = (i_R \Omega)_x. \\
\tau_2(\nabla\mathcal{P}F)_x &= d_J(i_\Gamma \Omega)_x = (d_J(i_{2h-I}\Omega))_x = (2d_J i_h dd_J F - 2d_J dd_J F)_x \\
&= (-2d_J i_h d_J dF - 2 i_J ddd_J F + 2d i_J dd_J F)_x = -(2d_J(d_J i_h dF + d_J dF))_x = 0,
\end{aligned}$$

where we used $[d, d_J] = 0$, $[i_h, d_J] = d_{Jh} - i_{[h,J]}$ and $[J, h] = 0$,

$$\begin{aligned}
\tau_3(\nabla\mathcal{P}F)_x(X,Y) &= \tfrac{1}{2}\nabla i_\Gamma \Omega(C,X,Y) - \nabla P_C F(hX, JY) + \nabla P_C F(hY, JX) \\
&= \tfrac{1}{2} d_C i_\Gamma \Omega(hX, hY) - \tfrac{1}{2} i_\Gamma d_C dd_J F(hX, hY) = \tfrac{1}{2} d_{[C,\Gamma]}\Omega(hX, hY) = 0. \\
\tau_4(\nabla\mathcal{P}F)_x &= (d_\Phi(i_\Phi\Omega))_x = (d_\Phi d_\Phi d_J F + d_\Phi d i_\Phi d_J F)_x = \tfrac{1}{2} d_{[\Phi,\Phi]} d_J F_x = \tfrac{1}{2} i_{[\Phi,\Phi]}\Omega_x. \\
\tau_5(\nabla\mathcal{P}F)_x &= (d_J(i_\Phi\Omega))_x = (-i_{[J,\Phi]}\Omega + i_\Phi d_J \Omega)_x = (-3 i_R \Omega)_x = 0
\end{aligned}$$

because of the identity $[i_\Phi, d_J] = d_{J\Phi} - i_{[\Phi,J]}$. Using the notation $F_c := CF - F$, we have

$$\begin{aligned}
\tau_6(\nabla\mathcal{P}F)_x &= (d_C(i_\Phi\Omega) - i_\Phi dd_J F_c)_x \\
&= (d_C d_\Phi d_J F - d_\Phi d_J d_C F)_x = (d_{[C,\Phi]} d_J F)_x = (i_\Phi \Omega)_x = 0. \\
\tau_7(\nabla\mathcal{P}F)_x(X,Y) &= X_x(i_\Phi dd_J F(Y,S)) - X_x(\Phi Y(F_c)) \\
&= -X_x(J[\Phi Y, S]F) + X_x(\Phi Y F) \\
&= -X_x([J,S](\Phi Y)F) + X_x(\Phi Y F) = -X_x((2h - I)\Phi Y F) - X_x(\Phi Y F) = 0.
\end{aligned}$$

Since $i_J\Omega(h_k, h_l) = 0$ we have $\Omega(v_k, h_l) = \Omega(v_l, h_k)$. Moreover,

$$i_\Phi\Omega(h_i, h_k) = \Omega(\Phi h_i, h_k) - \Omega(\Phi h_k, h_i) = (\lambda_i - \lambda_k)\Omega(v_i, h_k);$$

therefore, using the fact that $\lambda_i \neq \lambda_j$ and F is a 2^{nd} order solution at x, from the last equation of (3.6) we obtain $\Omega_x(v_i, h_k) = 0$. Consequently,

$$\nabla i_\Phi \Omega(h_j, h_i, h_k)_x = \big(h_j(i_\Phi\Omega(h_i, h_k))\big)_x = \big((\lambda_i - \lambda_k) h_j \Omega(v_i, h_k)\big)_x$$

and

$$\begin{aligned}
&\tau_{ijk}(\nabla\mathcal{P}F)_x(h_i, h_j, h_k) \\
&\quad = \tfrac{1}{2}\nabla i_\Gamma\Omega(v_i, h_j, h_k)_x + \tfrac{1}{\lambda_k - \lambda_i}\nabla i_\Phi\Omega(h_j, h_i, h_k)_x + \tfrac{1}{\lambda_i - \lambda_j}\nabla i_\Phi \Omega(h_k, h_i, h_j)_x \\
&\quad = v_i\Omega(h_j, h_k)_x + h_j\Omega(h_k, v_i)_x + h_k\Omega(v_i, h_j)_x.
\end{aligned}$$

From $d\Omega = 0$ we have $\sum_{ijl}^{cycl} v_i\Omega(h_j, h_k) - \Omega([v_i, h_j], h_k) = 0$, thus

$$\tau_{ijk}(\nabla\mathcal{P}F)_x = \Omega_x([v_i, h_j], h_k) + \Omega_x([h_j, h_k], v_i) + \Omega_x([h_k, v_i], h_j).$$

These calculations shows that the first compatibility condition (2.5) for $\mathcal{P}$ (representing the non-identically zero terms) is given by

$$\varphi(F)_x = \tau(\nabla\mathcal{P}F)_x = \big(i_R\Omega,\ \tfrac{1}{2}i_{[\Phi,\Phi]}\Omega,\ \sum_{ijk}^{cycl}\Omega([v_i, h_j], h_k)\big)_x$$

which proves the proposition.

∎

3.2 Higher order compatibility condition

The integrability theorem of Cartan-Kähler or Spencer-Goldschmidt states that if the first compatibility equations are satisfied and the symbol of the linear differential operator is surjective (resp. 2-acyclic), then there is no more (i.e. higher order) compatibility condition and the PDE operator is formally integrable. Unfortunately this is not the case with the PDE operator (3.1). As it was proved in [13, Theorem 5.1], the symbol of $\mathcal{P}$ is not 2-acyclic and therefore it is not involutive either. Since the first non trivial Spencer cohomology group is $H^{2,2}$, we can deduce that at the level of first prolongation of the PDE operator $\mathcal{P}$ there is an extra (third order) compatibility condition appearing. In this subsection we compute this extra condition and prove the following

Proposition 3.3. *If a 3^{rd} order solution F of $\mathcal{P}$ at $x \in \mathcal{T}M$ can be lifted into a 4^{th} order solution, if and only if for any $X, Y \in H_xTM$ we have*

$$\begin{aligned}
&[hX, \Phi X]\Omega(JY, Y) - [hY, \Phi Y]\Omega(X, JX) = \\
&\quad + \Phi X\Big(\sum_{\text{cyc}}\Omega([JY, hX], hY)\Big) - hX\Big(\sum_{\text{cyc}}\Omega([JY, \Phi X], hY) - \Omega([JY, \Phi Y], X)\Big) \\
&\quad - \Phi Y\Big(\sum_{\text{cyc}}\Omega([JX, hX], hY)\Big) + hY\Big(\sum_{\text{cyc}}\Omega([JX, \Phi X], Y) - \Omega([JX, \Phi Y], X)\Big)
\end{aligned} \tag{3.7}$$

Proof. If the Spencer cohomology groups $H^{2,2}$ is trivial, then the compatibility condition of the first prolongated system would be exactly the prolongation of the first compatibility condition of $\mathcal{P}$. However, from Theorem 5.1 in [13] we know that $\dim H^{2,2} = \frac{1}{2}(n-1)(n-2) = \mathcal{C}_{n,2}$. That indicates that there is $\mathcal{C}_{n,2}$ extra compatibility condition for the first prolongation of $\mathcal{P}$. To compute these conditions we will complete the prolongation of the map τ defined in (3.5) and apply the method described in Remark 2.1.

Let us consider the sequence

$$S^4T^* \xrightarrow{\sigma_4(\mathcal{P})} (S^3T^*)\times(S^2T^*\otimes\Lambda^2T^*)\times(S^2T^*\otimes\Lambda^2T^*) \xrightarrow{\tau^1} K^1 \longrightarrow 0 \tag{3.8}$$

where $\sigma_4(\mathcal{P})=id\otimes\sigma_3(\mathcal{P})$ is the prolongation of the symbol of $\mathcal{P}$, and the components of $\tau^1=(id\otimes\tau,\tau_h)$ are the prolongation $id\otimes\tau$ of (3.5) and τ_h with

$$\begin{aligned}\tau_h(B)(X,Y) :=& \frac{1}{2}\left(B_\Gamma(\Phi X,JY,X,Y)-B_\Gamma(\Phi Y,JX,X,Y)\right)\\ &+B_\Phi(hY,JX,X,Y)-B_\Phi(hX,JY,X,Y),\end{aligned} \tag{3.9}$$

where $B:=(B_C,B_\Gamma,B_\Phi)\in S^3T^*\times(S^2T^*\otimes\Lambda^2T_v^*)\times(S^2T^*\otimes\Lambda^2T_v^*)$, $X,Y\in T$. It is not difficult to show that the equation $\tau_h=0$ gives $\mathcal{C}_{n,2}$ extra independent equations with respect to the equations of the system $\tau^1=0$, thus the sequence (3.9) is exact.

Let us compute the compatibility conditions appearing from the new obstruction map τ_h. If F is a 3$^{\text{rd}}$ order solution of $\mathcal{P}$ at $x\in\mathcal{T}M$, then

$$\begin{aligned}&\tau_h(\nabla\nabla\mathcal{P}F)_x(X,Y)\\ &\quad=\tfrac{1}{2}\left(\nabla^2 i_\Gamma\Omega(\Phi X,JY,X,Y)-\nabla^2 i_\Gamma\Omega(\Phi Y,JX,X,Y)\right)\\ &\qquad+\nabla^2 i_\Phi\Omega(hY,JX,X,Y)-\nabla^2 i_\Phi\Omega(hX,JY,X,Y)\\ &\quad=\Phi X(JY i_\Gamma\Omega(X,Y))-\Phi Y(JX i_\Gamma\Omega(X,Y))\\ &\qquad+hY(JX i_\Phi\Omega(X,Y))-hX(JY i_\Phi\Omega(X,Y)).\end{aligned}$$

In the above formula the 2$^{\text{nd}}$ derivatives of Ω (and therefore 4$^{\text{th}}$ derivatives of F) appears. However, using the fact that $\Omega=dd_JF$ vanishes identically on the vertical sub-space, $\Omega(JX,hY)=\Omega(JY,hX)$ and $\Omega(Y,JX)=\Omega(JY,X)=0$, the 2$^{\text{nd}}$ derivatives of Ω can be expressed by 1$^{\text{st}}$ derivatives of Ω and find

$$\begin{aligned}\tau_h(\nabla\nabla\mathcal{P}F)_x(X,Y)&=[hY,\Phi Y]\Omega(X,JX)-[hX,\Phi X]\Omega(JY,Y)\\ &+\Phi X\Big(\sum_{\text{cyc}}\Omega([JY,hX],hY)\Big)-hX\Big(\sum_{\text{cyc}}\Omega([JY,\Phi X],hY)-\Omega([JY,\Phi Y],X)\Big)\\ &-\Phi Y\Big(\sum_{\text{cyc}}\Omega([JX,hX],hY)\Big)+hY\Big(\sum_{\text{cyc}}\Omega([JX,\Phi X],Y)-\Omega([JX,\Phi Y],X)\Big),\end{aligned}$$

containing only 3$^{\text{rd}}$ order derivatives of F. The compatibility condition is satisfied if and only if $\tau_h(\nabla\nabla\mathcal{P}F)_x=0$ which is equivalent to equation (3.7). □

Remark 3.1. In an adapted basis (3.3) we have

$$i_\Gamma\Omega=0 \iff \Omega(h_i,h_j)=0, \qquad 1\le i,j\le n, \tag{3.10a}$$

$$i_\Phi\Omega=0 \iff \Omega(v_i,h_j)=0, \qquad 1\le i,j\le n, \quad i\neq j, \tag{3.10b}$$

$$i_S\Omega=0 \iff \Omega(v_i,h_n)=\Omega(h_i,h_n)=0, \qquad 1\le i\le n, \tag{3.10c}$$

hence, using the notation $g_{ij}=\Omega(v_i,h_j)$ the only nonzero components of Ω are

$$g_{ii}=\Omega(v_i,h_i), \qquad i=1,\dots,n-1. \tag{3.11}$$

Moreover, because the Hessian of F must be positive quasi-definite, the terms in (3.11) are positives.

Corollary 3.4. *In an adapted basis* (3.3) *the compatibility condition* (3.7) *can be expressed as*

$$\beta^i_{ij}(\mathcal{L}_{v_i}g_{ii})+\beta^j_{ij}(\mathcal{L}_{v_j}g_{jj})+\gamma^i_{ij}(\mathcal{L}_{h_i}g_{ii})+\gamma^j_{ij}(\mathcal{L}_{h_j}g_{jj})+\sum_{k=1}^{n}\alpha^k_{ij}g_{kk}=0, \tag{3.12}$$

where $1\leq i,j\leq n$, *and the summation convention is not applied. The* α^k_{ij}, β^k_{ij}, γ^k_{ij} *are functions in a neighborhood of* $x\in\mathcal{T}M$ *determined by the Lie bracket of the elements of the local basis* (3.3).

If $\{\xi^i,\nu^i\}_{1\leq i\leq n}$ is the dual basis of (3.3), then $X=\xi^i_X h_i+\nu^i_X v_i$ and using $d\Omega=0$ and from (3.10) we get that the coefficients of the third order terms appearing in the Lie derivatives of g_{ii} and g_{jj} are:

$$\beta^i_{ij}=\lambda_j\nu^i_{[v_j,h_j]},\qquad\qquad \gamma^i_{ij}=\lambda_j\xi^i_{[v_j,h_j]}. \tag{3.13}$$

Definition 3.2. We say that the higher order compatibility condition of the spray S is reducible, if the coefficients β^i_{ij}, γ^i_{ij} are identically zero.

In the reducible case the third order condition can be identically zero or may represent a second order compatibility condition.

Remark 3.3. From the expression (3.13) it is clear that the higher order compatibility condition is reducible if and only if the eigen-distributions $\mathcal{D}_i:=Span\{h_i,v_i\}$, $1\leq i\leq n$ are involutives. If in addition the the vector fields appearing in (3.3) are commuting, then (3.7) is identically zero that is the third order compatibility condition is identically satisfied.

4 The three dimensional case

In the previous chapter we investigated the integrability condition of the extended Rapcsák system completed with the curvature condition. We determined the compatibility condition to lift a second order solution into a third order solution (Proposition 3.1) and an extra higher order compatibility condition appearing to lift a second order solution into a third order solution (Proposition 3.3). In this chapter we focus on the case, when M is a $3-$dimensional manifold and the spray is non-isotropic. We consider the generic situation, when the eigenvalues of the Jacobi endomorphism are pairwise distinct. We identify the special cases when the integrability conditions are satisfied and by computing the higher order Spencer cohomology groups we prove that the system has no further compatibility condition.

4.1 Extended Rapcsák system with curvature condition

The compatibility condition to lift a second order solution of $\mathcal{P}$ into a third order solution is given by (3.4a) and (3.4b). Then, in the 3-dimensional case the can be simplified:

Lemma 4.1. *The compatibility condition* (3.4a) *is identically satisfied.*

Indeed, Φ is semi basic 1-1 tensor and from $i_S\Phi = 0$ we have $i_S[\Phi,\Phi] = 0$. Evaluating the semi basic 3-form $i_{[\Phi,\Phi]}\Omega$ on the 3-dimensional horizontal space by using the second equation of (2.2) we get that

$$i_{[\Phi,\Phi]}\Omega(h_1,h_2,h_3) = \sum_{\text{cyc}} \Omega\big([\Phi,\Phi](h_1,h_2),h_3\big) = i_S\Omega\big([\Phi,\Phi](h_1,h_2)\big) = 0.$$

Lemma 4.2. *The integrability condition* (3.4b) *can be written as*

$$i_{\Phi'}\Omega = 0, \tag{4.1}$$

where $\Phi' := v\circ[S,\Phi]\circ h$ *is the semi basic dynamical covariant derivative of* Φ *(see [5, 11]).*

Indeed, we have

$$\Phi'(S) = v[S,\Phi]S = v[S,\Phi S] - \Phi([S,S]) = 0,$$

thus (4.1) is satisfied if and only if $i_{\Phi'}\Omega(h_1,h_2) = 0$. Moreover, F being a second order solution, we have (3.6), and in particular $\Omega([v_1,h_2],S)=0$ and $\Omega(h_1,h_2)=0$ at x. Using

$$h_i = [J,S]h_i = [v_i,S] - J[h_i,S], \qquad i = 1,2,$$

and $h_3=S$ we can obtain that the condition (3.4b) is

$$\begin{aligned}(3.4b) &= \Omega([v_1,h_2],S) + \Omega([h_2,S],v_1) + \Omega([S,v_1],h_2) = \Omega([h_2,S],v_1) + \Omega([S,v_1],h_2)\\ &= \Omega([h_2,S],v_1) - \Omega(J[h_1,S],h_2) = \Omega([h_2,S],v_1) - \Omega(v_2,[h_1,S]).\end{aligned} \tag{4.2}$$

On the other hand, from to $i_\Phi\Omega = 0$ we have

$$\Omega(\Phi[h_i,S],h_j) = \Omega(\Phi h_j,[h_i,S]) = \lambda_j\Omega(v_j,[h_i,S]), \quad i,j = 1,2,\ i \neq j, \tag{4.3}$$

therefore,

$$\begin{aligned}i_{\Phi'}\Omega(h_1,h_2) &= \Omega\big([\Phi h_1,S] - \Phi[h_1,S],h_2\big) - \Omega\big([\Phi h_2,S] - \Phi[h_2,S],h_1\big) \overset{(4.3)}{=}\\ &= \lambda_1\Omega(v_2,[h_1,S]) - \lambda_2\Omega(v_2,[h_1,S]) - \lambda_2\Omega(v_1,[h_2,S]) + \lambda_1\Omega(v_1,[h_2,S])\\ &= (\lambda_1-\lambda_2)(\Omega(v_2,[h_1,S]) + \Omega(v_1,[h_2,S])).\end{aligned}$$

Comparing the result with (4.2) we obtain Lemma 4.2.

Considering further Spencer cohomology groups we have the

Lemma 4.3. *For any $m \geq 3$ the Spencer cohomology group $H^{m,2}$ is trivial.*

In order to show the Lemma, one considers the Spencer sequences

$$0 \to g_{m+2} \xrightarrow{i} T^* \otimes g_{m+1} \xrightarrow{\delta_1^m} \Lambda^2 T^* \otimes g_m \xrightarrow{\delta_2^m} \Lambda^3 T^* \otimes g_{m-1} \longrightarrow \dots \tag{4.4}$$

for $m \geq 3$, where i is the inclusion, δ_1^m and δ_2^m are the Spencer operators skew-symmetrizing the first two, resp. three variables. To prove the lemma we have to show that $H^{m,2} = \operatorname{Ker} \delta_2^m / \operatorname{Im} \delta_1^m = 0$, that is $\operatorname{Ker} \delta_2^m = \operatorname{Im} \delta_1^m$. Since $\operatorname{Im} \delta_1^m \subset \operatorname{Ker} \delta_2^m$, it is enough to show that the dimension of the two spaces are equal. Long and laborious linear algebraic computation shows that the dimension of $\operatorname{rank}(\delta_1^m)$ and $\dim \operatorname{Ker} \delta_2^m$ are equal, therefore the $\operatorname{Im} \delta_1^m = \operatorname{Ker} \delta_2^m$. For more detail we refer to [14, Lemma 4.3].

Proposition 4.4. *Let S be a non-isotropic spray on a 3-dimensional manifold with distinct Jacobi eigenvalues. Then the PDE operator $\mathcal{P}$ defined in* (3.1) *is formally integrable if and only if*

1. $\Phi' \in Span\{J, \Phi\}$
2. *The compatibility condition* (3.7) *is satisfied.*

Proof. Using Proposition 3.1 with Lemma 4.1 and Lemma 4.2 we get that the first condition of Proposition 4.4 guaranties that any 2^{nd} order solution of $\mathcal{P}$ can be prolonged into a 3^{rd} order solution. Using Proposition 3.3 we get that if the second condition of Proposition 4.4 holds then any 3^{nd} order solution of $\mathcal{P}$ can be prolonged into a 4^{th} order solution. Moreover, Lemma 4.3 shows that the Spencer cohomology groups $H^{m,2}$ are trivial and therefore there is no higher order compatibility condition for $\mathcal{P}$ that is, any m^{th} order solution can be prolonged into a $(m+1)^{\text{st}}$ order solution, $m \geq 4$. Therefore, we obtain the formal integrability of $\mathcal{P}$. □

Theorem 4.5. *Let S be a non-isotropic analytic spray on a 3-dimensional analytic manifold with distinct Jacobi eigenvalues. If conditions 1 and 2 of Proposition 4.4 are satisfied, then S is locally projective metrizable.*

Proof. Indeed, in the analytic case, for any initial data there exists a local analytic solution of $\mathcal{P}$. Choosing an initial data compatible with the positive quasi-definite criteria, one can obtain an analytic solution F such that F^2 is locally positive definite. The spray associated to the Finsler function F will be projective equivalent to S, that is S is locally projective metrizable. □

4.2 Reducible case: the complete system

In the previous subsection we investigated the integrability of the extended Rapcsák system completed with curvature condition. When the conditions 1. or 2. of Proposition 4.4 is not satisfied, then the PDE system $\mathcal{P}$ corresponding to the projective metrizability is not integrable. As usual, the necessary compatibility condition should be added to the system and restart the analysis of the enlarged system. In this section we consider the case where this extra compatibility condition is of second order.

We remark that the extended Rapcsák system is composed by equations on the components the 2-form $\Omega = dd_J F$ whose potentially nonzero components are listed in (3.11). When $\dim M = 3$, there are only two such components: $\Omega(v_1, h_1)$ and $\Omega(v_2, h_2)$. The extra compatibility condition gives a new relation between these two components. In the adapted basis this equation can be written as

$$\eta_1 \, \Omega(v_1, h_1) + \eta_2 \, \Omega(v_2, h_2) = 0, \tag{4.5}$$

where η_1 and η_2 are well defined function on $\mathcal{T}M$ determined by the spray S.

Remark 4.1. If (4.5) is nontrivial but one of the η_i vanishes, then one of the $g_{ii} = \Omega(v_i, h_i)$, $i = 1, 2$ must be zero. Consequently, $\mathcal{P}$ has no positive quasi-definite solution and the spray is not projective metrizable. Therefore further simplification is possible: introducing $\eta := \frac{\eta_2}{\eta_1}$ the equation can written as

$$\Omega(v_1, h_1) + \eta \, \Omega(v_2, h_2) = 0. \tag{4.6}$$

with $\eta < 0$. In the sequel we suppose that η satifies that condition.

The second order PDE operator corresponding to (4.5) will be denoted by

$$P_\Psi : C^\infty(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \qquad P_\Psi(F) = \eta_1 \Omega(v_1, h_1) + \eta_2 \Omega(v_2, h_2).$$

The PDE operator corresponding the completed system is $\widetilde{\mathcal{P}} := (\mathcal{P}, P_\Psi)$ i.e.:

$$\widetilde{\mathcal{P}} = (P_C, P_\Gamma, P_\Phi, P_\Psi). \tag{4.7}$$

Remark 4.2. If F is a solution of $\widetilde{\mathcal{P}}$ then for any $k \in \mathbb{N}$ the $\mathcal{L}^k \Omega$ (the k^{th} order Lie derivatives of Ω) can be calculated *algebraically* from Ω.

Proof. It is sufficient to check this property for the Lie derivatives with respect to the element of the adapted basis (3.3). According the Remark 3.1, the only nonzero components of Ω are $g_{11} = \Omega(v_1, h_1)$ and $g_{22} = \Omega(v_2, h_2)$. Using $d\Omega = 0$ and equation (3.10) with (4.5) we can find for the Lie derivatives of g_{11} the following:

$$\begin{aligned}
\mathcal{L}_{h_2} g_{11} &= h_2 \Omega(v_1, h_1) = \Omega([h_2, v_1], h_1) + \Omega([v_1, h_1], h_2) + \Omega([h_1, h_2], v_2), \\
\mathcal{L}_{v_2} g_{11} &= v_2 \Omega(v_1, h_1) = \Omega([v_2, v_1], h_1) + \Omega([v_1, h_1], v_2) + \Omega([h_1, v_2], v_1), \\
\mathcal{L}_{h_1} g_{11} &= h_1 \Omega(v_1, h_1) = (h_1 \eta) \Omega(v_2, h_2) + \eta \big(\Omega([h_1, v_2], h_2) + \Omega([v_2, h_2], h_1) + \Omega([h_2, h_1], v_2) \big), \\
\mathcal{L}_{v_1} g_{11} &= v_1 \Omega(v_1, h_1) = (v_1 \eta) \Omega(v_2, h_2) + \eta \big(\Omega([v_1, v_2], h_2) + \Omega([v_2, h_2], v_1) + \Omega([h_2, v_1], v_2) \big).
\end{aligned}$$

On the right hand side of the above expressions there are only terms containing Ω but not its derivatives. We remark that all these terms can be expressed as linear combinations of g_{11} and g_{22}. Any further derivatives can be computed the same way. On the other hand, one can use (4.6) to find the formulas for the derivatives of g_{22}. □

Corollary 4.6. *The system* (4.7) *is complete in the sense that either all compatibility conditions are satisfied or the spray is not projective metrizable.*

Proof. Indeed, according to Remark 4.2, if there is a non-trivial extra compatibility condition for $\widetilde{\mathcal{P}}$ then it can be expressed algebraically with Ω, that would give a new and independent linear (homogeneous) equation between g_{11} and g_{22}. Consequently, from Remark 3.1 we get that only the trivial solution ($g_{ij} = 0$) exists. □

Compatibility conditions

To compute the compatibility conditions of $\widetilde{\mathcal{P}}$, we can follow the methods presented in the previous chapter. The symbol of $\widetilde{\mathcal{P}}$ and its prolongations are

$$\begin{aligned}
\sigma_2(P_\Psi) :& S^2T^* \longrightarrow \mathbb{R}, && \big(\sigma_2(P_\Psi)A^2\big) = A^2(v_1,v_1) + \eta A^2(v_2,v_2),\\
\sigma_3(P_\Psi) :& S^3T^* \longrightarrow T^*, && \big(\sigma_3(P_\Psi)A^3\big)(X) = A^3(X,v_1,v_1) + \eta A^3(X,v_2,v_2),\\
\sigma_4(P_\Psi) :& S^4T^* \longrightarrow S^2T^*, && \big(\sigma_4(P_\Psi)A^4\big)(X,Y) = A^4(X,Y,v_1,v_1) + \eta A^4(X,Y,v_2,v_2),
\end{aligned}$$

$A^k \in S^kT^*$ and $X,Y \in T$. Using the map τ and τ_h defined in (3.5) and (3.9) respectively, we can consider the extended obstruction map:

$$\widetilde{\tau} := (\tau,\, \widetilde{\tau}_1,\, \widetilde{\tau}_2), \qquad \widetilde{\tau}^1 := (id\otimes\widetilde{\tau},\, \tau_h,\, \widetilde{\tau}_3,\, \widetilde{\tau}_4,\, \widetilde{\tau}_5,\, \widetilde{\tau}_6)$$

where

$$\begin{aligned}
\widetilde{\tau}_1(B) &= B_\Psi(C) - B_C(v_1,v_1) - \eta B_C(v_2,v_2),\\
\widetilde{\tau}_2(B) &= B_\Psi(S) - \tfrac{1}{2}B_\Gamma(v_1,S,h_1) - \tfrac{\eta}{2}B_\Gamma(v_2,S,h_2) - \tfrac{1}{\lambda_1}B_\Phi(h_1,h_1,S) - \tfrac{\eta}{\lambda_2}B_\Phi(h_2,h_2,S),\\
\widetilde{\tau}_3(\widetilde{B}) &= \widetilde{B}_\Psi(v_1,v_2) - \tfrac{1}{\lambda_1-\lambda_2}\widetilde{B}_\Phi(v_1,v_1,h_1,h_2) - \tfrac{\eta}{\lambda_1-\lambda_2}\widetilde{B}_\Phi(v_2,v_2,h_1,h_2),\\
\widetilde{\tau}_4(\widetilde{B}) &= \widetilde{B}_\Psi(h_1,h_2) - \tfrac{1}{2}\widetilde{B}_\Gamma(h_1,v_1,h_2,h_1) - \tfrac{\eta}{2}\widetilde{B}_\Gamma(h_2,v_2,h_1,h_2)\\
&\quad - \tfrac{1}{\lambda_1-\lambda_2}\widetilde{B}_\Phi(h_1,h_1,h_1,h_2) - \tfrac{\eta}{\lambda_1-\lambda_2}\widetilde{B}_\Phi(h_2,h_2,h_1,h_2),\\
\widetilde{\tau}_5(\widetilde{B}) &= \widetilde{B}_\Psi(h_1,v_2) - \tfrac{1}{\lambda_1-\lambda_2}\widetilde{B}_\Phi(h_1,v_1,h_1,h_2) - \tfrac{\eta}{2}\widetilde{B}_\Gamma(v_2,v_2,h_1,h_2) - \tfrac{\eta}{\lambda_1-\lambda_2}\widetilde{B}_\Phi(v_2,h_2,h_1,h_2),\\
\widetilde{\tau}_6(\widetilde{B}) &= \widetilde{B}_\Psi(v_1,h_2) + \tfrac{1}{2}\widetilde{B}_\Gamma(v_1,v_1,h_1,h_2) - \tfrac{1}{\lambda_1-\lambda_2}\widetilde{B}_\Phi(h_1,v_1,h_1,h_2) - \tfrac{\eta}{\lambda_1-\lambda_2}\widetilde{B}_\Phi(h_2,v_2,h_1,h_2),
\end{aligned}$$

and $B=(B_C,B_\Gamma,B_\Phi,B_\Psi)$ denotes an element of $T^*\otimes\big(T^*\times\Lambda^2T_v^*\times\Lambda^2T_v^*\times\mathbb{R}\big)$ and $\widetilde{B}=(\widetilde{B}_C,\widetilde{B}_\Gamma,\widetilde{B}_\Phi,\widetilde{B}_\Psi)$ denotes an element of $S^2T^*\otimes\big(T^*\times\Lambda^2T_v^*\times\Lambda^2T_v^*\times\mathbb{R}\big)$. Then

$$\operatorname{Im}\sigma_3(\widetilde{\mathcal{P}}) = \operatorname{Ker}\widetilde{\tau}, \qquad \operatorname{Im}\sigma_4(\widetilde{\mathcal{P}}) = \operatorname{Ker}\widetilde{\tau}^1. \tag{4.8}$$

The compatibility conditions of the PDE operator $\widetilde{\mathcal{P}}$ can be calculated:

$$\begin{aligned}
\widetilde{\tau}_1(\nabla\widetilde{\mathcal{P}}F) &= \mathcal{L}_C\eta\,\Omega(v_2,h_2),\\
\widetilde{\tau}_2(\nabla\widetilde{\mathcal{P}}F) &= \mathcal{L}_S\eta\Omega(v_2,h_2) + 2\Omega([S,v_1],h_1) + 2\eta\Omega([S,v_2],h_2),\\
\widetilde{\tau}_3(\nabla^2\widetilde{\mathcal{P}}F) &= (v_1(v_2\eta))\Omega(v_2,h_2) + \eta[v_1,v_2]\Omega(v_2,h_2) + (v_1\eta)v_2\Omega(v_2,h_2)\\
&\quad + (v_2\eta)\Big(\sum_{\text{cyc}}\Omega([v_1,v_2],h_2)\Big) + v_1\Big(\sum_{\text{cyc}}\Omega([v_2,v_1],h_1)\Big) - \eta v_2\Big(\sum_{\text{cyc}}\Omega([v_2,v_1],h_2)\Big),\\
\widetilde{\tau}_4(\nabla^2\widetilde{\mathcal{P}}F) &= h_1(h_2\eta)\Omega(v_2,h_2) + (h_2\eta)h_1\Omega(v_2,h_2) + (h_1\eta)h_2\Omega(v_2,h_2) + \eta[h_1,h_2]\Omega(v_2,h_2)\\
&\quad + h_1\Big(\sum_{\text{cyc}}\Omega([h_2,v_1],h_1)\Big) - \eta h_2\Big(\sum_{\text{cyc}}\Omega([v_2,h_1],h_2)\Big),\\
\widetilde{\tau}_5(\nabla^2\widetilde{\mathcal{P}}F) &= (h_1(v_2\eta))\Omega(v_2,h_2) + (v_2\eta)h_1\Omega(v_2,h_2) + \eta[h_1,v_2]\Omega(v_2,h_2)\\
&\quad + h_1\Big(\sum_{\text{cyc}}\Omega([v_2,v_1],h_1)\Big) - \eta v_2\Big(\sum_{\text{cyc}}\Omega([v_2,h_1],h_2)\Big),\\
\widetilde{\tau}_6(\nabla^2\widetilde{\mathcal{P}}F) &= (v_1(h_2\eta))\Omega(v_2,h_2) + (h_2\eta)v_1\Omega(v_2,h_2) + (v_1\eta)h_2\Omega(v_2,h_2) + \eta[v_1,h_2]\Omega(v_2,h_2)\\
&\quad + v_1\Big(\sum_{\text{cyc}}\Omega([h_2,v_1],h_1)\Big) - \eta h_2\Big(\sum_{\text{cyc}}\Omega([v_2,v_1],h_2)\Big).
\end{aligned}$$

Then, using Remark 4.2, the above expressions can be written as

$$\begin{aligned} \widetilde{\tau}_i(\nabla\widetilde{\mathcal{P}}F) &= \eta_1^i\Omega(v_1,h_1)+\eta_2^i\Omega(v_2,h_2) \quad i=1,2, \\ \widetilde{\tau}_j(\nabla^2\widetilde{\mathcal{P}}F) &= \eta_1^j\Omega(v_1,h_1)+\eta_2^j\Omega(v_2,h_2) \quad j=3,4,5,6 \end{aligned} \tag{4.9}$$

where η_i^k are functions on $\mathcal{T}M$. If we consider the matrix

$$\Theta = \begin{pmatrix} 1 & \eta_1^1 & \dots & \eta_1^6 \\ \eta & \eta_2^1 & \dots & \eta_2^6 \end{pmatrix},$$

we can have the following

Proposition 4.7. *The operator $\widetilde{\mathcal{P}}$ is formally integrable if and only if* $\operatorname{rank}\Theta = 1$.

Proof. Applying the method used in the previous chapter we get that a 2^{nd} order solution F can be lifted into a third order solution iff $\tau_4(\nabla\widetilde{\mathcal{P}}F)=0$ and a 3^{rd} order solution can be lifted into a third order solution iff $\tau_4^1(\nabla^2\widetilde{\mathcal{P}}F)=0$. The compatibility conditions expressed in terms of Ω can be written as a system of (homogeneous) linear algebraic system equating the right hand side of (4.9) to zero. These equations are identically satisfied if and only if they are multiple of equation (4.5), that is the $\operatorname{rank}\Theta = 1$. Moreover, it is easy to show that (3.3) is a quasi-regular basis for the first prolongation of the symbol of $\widetilde{\mathcal{P}}$, that is the Cartan's test is satisfied. From the Cartan-Kähler theorem (Theorem 2.1) we get that $\widetilde{\mathcal{P}}$ is formally integrable. □

Based on Remark 4.1 and Proposition 4.7 we have the following

Theorem 4.8. *Let S be a non-isotropic analytic spray on a 3-dimensional analytic manifold with distinct Jacobi eigenvalues. If $\Phi' \notin Span\{J,\Phi\}$ or the compatibility condition* (3.7) *is reducible but not identically zero, then the spray is locally projective metrizable if and only if $\eta_1\cdot\eta_2<0$ and* $\operatorname{rank}\Theta = 1$.

Proof. Under the hypothesis of the theorem, the operator $\widetilde{\mathcal{P}}$ corresponding to the projective metrizability is integrable. Considering at any point $x\in\mathcal{T}M$ a second order solution $j_{2,x}$ having $g_{11}>0$ and $g_{22}>0$ one can extend it into an analytic positive quasi-definite solution. Therefore, the spray is locally projective metrizable. □

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Authors' address:

Tamás Milkovszki, Zoltán Muzsnay
Institute of Mathematics, University of Debrecen,
H-4032 Debrecen, Egyetem tér 1, Hungary.
E-mail addresses: milkovszki@science.unideb.hu , muzsnay@science.unideb.hu