On the geometry in the large of Lichnerowicz type Laplacians and its applications

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Abstract. In the paper, we study the geometry of Lichnerowicz type Laplacians, which generalize the ordinary Laplacian of Lichnerowicz. We are based on the analytical method, due to Bochner, of proving vanishing theorems for the null space of Laplace operator. We pay a special attention to the kernel of the Lichnerowicz type Laplacian on Riemannian symmetric spaces of compact and complete noncompact types. We also consider applications of our results to the theories of Codazzi and Killing tensors, infinitesimal Einstein deformations and stability of Einstein manifolds.

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1 Introduction

Let (M, g) be an n-dimensional ($n \geq 2$) connected Riemannian manifold. The vector bundle $\otimes^p T^*M$ of *covariant p-tensors* ($1 \leq p < \infty$) over M carries the well-known Lichnerowicz Laplacian defined by the Weitzenböck decomposition formula (see [2, p. 388–389], [3, p. 54] and [24, p. 27])

$$\Delta_L = \bar{\Delta} + \Re, \tag{1.1}$$

where $\bar{\Delta}$ is the connection or Bochner Laplacian (see [3, p. 54] and [24, p. 27]) and $\Re : \otimes^p T^*M \to \otimes^p T^*M$ is the symmetric *Weitzenböck curvature operator* (see [35, pp. 343–345]) that depends linearly in known way on the Riemannian curvature tensor and the Ricci tensor of the metric g. This situation can be generalized to the following setting. Consider a Riemannian bundle E over M with a scalar product $g_x = \langle\cdot,\cdot\rangle$ in each fiber E_x, smoothly depending on $x \in M$, and with a compatible connection. Further, the scalar products and connections of E, as well as of M will be denoted by the same symbols g and ∇. Define the L^2 global scalar product on C^∞-sections of E by the formula $\langle u, v\rangle = \int_M g_x(u, v)\, \mathrm{d}\,\mathrm{vol}_g$ and consider the associated Hilbert space $L^2(E)$. Using the L^2-structures on $C^\infty(E)$, we define the *connection Laplacian*

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by the formula $\bar{\Delta} = \nabla^*\nabla$, where ∇^* is the formal-adjoint operator of the compatible connection $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$. In the paper, we consider a one-parameter family of *Lichnerowicz type Laplacians* on E, defined by the formula

$$\Delta_L = \bar{\Delta} + c\,\Re \tag{1.2}$$

for any $c \in \mathbb{R}$ and a smooth symmetric endomorphism $\Re$ of E, related to the curvature R^∇ of ∇ (for particular cases, see [19]; [35, p. 344]). The subscript L for Δ will denote that it is the Lichnerowicz type Laplacian due to above definition. Any Δ_L is a second order elliptic linear differential operator on $C^\infty(E)$, which is symmetric with respect to $\langle\cdot,\cdot\rangle$. On a closed (i.e., compact without boundary) (M, g) we have an orthogonal (with respect to the global scalar product on $\otimes^p T^*M$) decomposition

$$C^\infty(\otimes^p T^*M) = \operatorname{Ker}\Delta_L \oplus \operatorname{Im}\Delta_L, \tag{1.3}$$

and Δ_L has a discrete spectrum and its eigenvalues have finite multiplicities, which in few cases have been computed. The first component $\operatorname{Ker}\Delta_L$ of (1.3) is the kernel of Δ_L. Its smooth sections are called Δ_L-*harmonic* (see [36, p. 104]). Examples below show how this construction works (see also its applications in Sections 4 and 5).

Example 1.1. (i) For $c = 0$ and $p = 0$, Δ_L is the *Bochner Laplacian* Δ_H on $C^\infty(M)$. The kernel of Δ_L consists of harmonic functions.

(ii) If $c = 1$ and $E = \Lambda^p M$ is the bundle of p-forms ($1 \le p \le n-1$) over M, then Δ_L is the Hodge-de Rham Laplacian Δ_H on p-forms. In this case, we have (1.3) for Δ_H, where $\Re$ is expressed in terms of the curvature and Ricci tensors (see [33, p. 347]). It is known that the p-th Betti number $b_p(M)$ of a closed (M, g) is equal to the dimension of the kernel of Δ_L on p-forms. Elements of $\ker\Delta_L$ are called harmonic p-forms.

(iii) If $c = -1$ and $E = S^p M$ for the bundle of *covariant symmetric p-tensors* ($1 \le p < \infty$) over M, then Δ_L is the *Sampson Laplacian* Δ_S (see [39, p. 147]). In this case, (1.3) for Δ_S has the form $\Delta_S = \bar{\Delta} - \Re$, where $\Re$ is expressed in terms of the curvature and Ricci tensors of (M, g) (see [28]; [39, p. 147]; [44, p. 55]). Elements of $\ker\Delta_S$ are called harmonic symmetric p-tensors, and they form a finite-dimensional vector space on a compact Riemannian manifold (see [39, pp. 148, 150]).

(iv) Let $c = 1/4$ and $E = \Sigma M$ be the spinor bundle on a compact (M, g). In this case, Δ_L is the *spinor Laplacian* Δ_D and (1.1) has the form $\Delta_D = \bar{\Delta} + (1/4)\,s$, see [25], where s is the scalar curvature and Δ_D is the square of the *Dirac operator* D on C^∞-sections of $E = \Sigma M$ (see [3, p. 55] and [25]). Elements of $\ker\Delta_D = \ker D$ are called harmonic spinors (see, e.g., [1]; [3, p. 170]), and they form a finite-dimensional vector space on a compact manifold M, as follows from standard elliptic theory applied to the strongly elliptic system $\Delta_D T = 0$ for smooth sections of ΣM.

In the present work we study Δ_L-harmonic sections of (E, g) using the analytical method, due to S. Bochner (e.g., monographs [35, Chapter 9] and [6, 36, 46]), of proving Liouville type vanishing theorems for the null space of a Laplace operator admitting (1.1).

Recall that by classical Liouville theorem, any bounded harmonic function in $\mathbb{R}^n$ is constant. There is a more general approach to vanishing theorems of Liouville type in Riemannian geometry (e.g., [28, 38, 41]). Namely, let A be an elliptic operator of some order acting on a functional class $\mathfrak{F}(M)$ over a complete (M, g). Then, in accordance with the general theory, we say that the Liouville type vanishing theorem on (M, g) is true if any solution of equation $Af = 0$ from class $\mathfrak{F}(M)$ is trivial. The "triviality property" can be understand in different ways, for example, identically zero or constant, or for linear equations triviality can be equivalent to having finite dimension. This theory is a part of Geometric Analysis based on the contributions of K. Uhlenbeck, C. Taubes, S.T. Yau, R. Shoen and R. Hamilton.

We pay a special attention to the kernel of the Laplacian (1.2) on Riemannian symmetric spaces of compact and noncompact types. We also give applications of our results to the theories of Codazzi and Killing tensors, infinitesimal Einstein deformations and stability of Einstein manifolds (see [2] and [3, Chapter 12]).

2 The Lichnerowicz type Laplacian on covariant p-tensors

There are few general theorems on the kernel of the Lichnerowicz type Laplacian $\Delta_L : C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$ for $p \geq 2$. In the section we fill this gap.

2.1. Let M be a connected manifold of dimension n and g be a Riemannian metric on M with its Levi-Civita connection ∇. One can associate to $(M,\, g)$ a number of natural elliptic differential operators. Usually these operators act in the space $C^\infty(E)$ of smooth sections of some Riemannian vector bundle $E \to M$ over $(M,\, g)$. As an example we consider the Laplacian (1.2). A section $\xi \in C^\infty(E)$ is called Δ_L-*harmonic* if $\Delta_L\, \xi = 0$ (see [36, p. 104]). Define the vector space of Δ_L-harmonic C^∞-sections of $E \to M$ by

$$\operatorname{Ker}\Delta_L = \{\xi \in C^\infty(E) : \Delta_L\, \xi = 0\},$$

and the vector space of Δ_L-harmonic $L^q(E)$-sections of $E \to M$ by the condition

$$L^q(\operatorname{Ker}\Delta_L) = \{\, \xi \in \operatorname{Ker}\Delta_L : \|\xi\| \in L^q(M)\, \}.$$

Furthermore, Δ_L-harmonic sections satisfy the (strong) unique continuation property. In coordinates, the condition $\Delta_L\, \xi = 0$ becomes a system of elliptic equations satisfying the structural assumptions of Aronszajn-Cordes (see [36, Appendix]). Consequently, the following proposition holds (see [36, p. 104]).

Proposition 2.1. *Let Δ_L be an elliptic operator on C^∞-sections of $E \to M$ over (M, g), satisfying* (1.2)*. Let $\xi \in \operatorname{Ker}\Delta_L$ be an Δ_L-harmonic section of $E \to M$. If ξ has a zero of infinite order at some point $x \in M$, then ξ vanishes on M.*

2.2. An illustration of the construction (1.2) is given by $\Delta_L : C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$. Namely, Δ_L is defined for any $T \in C^\infty(\otimes^p T^*M)$ by, e.g., [35, p. 344],

$$\Delta_L T = \bar{\Delta}\, T + c\, \Re(T). \tag{2.1}$$

We will use the standard definition of the Riemannian curvature tensor:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \quad X, Y \in TM.$$

In this case, the Weitzenböck curvature operator $\Re : \otimes^p T^*M \to \otimes^p T^*M$ is defined by the following equalities (see also [3, p. 54]):

$$(\Re(T))_{i_1 \ldots i_p} = \sum\nolimits_a R_{i_a j}\, T_{i_1 \ldots}{}^{j}{}_{\ldots i_p} - 2 \sum\nolimits_{a<b} R_{j\, i_a k\, i_b} T_{i_1 \ldots}{}^{j}{}_{\ldots}{}^{k}{}_{\ldots i_p}, \tag{2.2}$$

where $T_{i_1\dots i_p}$, R_{ij} and R_{ijkl} are components of the tensor $T\in C^\infty(\otimes^p T^*M)$, the Ricci tensor and the Riemannian curvature tensor, respectively,

$$T_{i_1\dots i_p}=T(e_{i_1},\dots,e_{i_p}),\quad R_{kl}=R^i_{kil},\quad R_{ijkl}=g_{im}\,R^m_{jkl}.$$

Here $R(e_j,e_l)\,e_k=R^i_{kjl}e_i$ and $g_{im}=g(e_i,e_m)$ for an orthonormal frame $\{e_1,\dots,e_n\}$ of T_xM at an arbitrary point $x\in M$ and for any $i,j,k,\dots\in\{1,2,\dots,n\}$. Moreover, the following identity holds for any $T,T'\in\otimes^p T^*M$, see [24, p. 27]:

$$g(\Re(T),T')=g(\Re(T'),T).$$

Remark 2.1. In [35], operator (2.2) is presented by

$$(\Re(T))(X_1,\dots,X_p)=\sum_{a,j}(R(e_j,X_a)T)(\underbrace{X_1,\dots,e_j}_{a},\dots,X_p). \tag{2.3}$$

From the well known formula for curvature on the (p,l)-tensor bundle, keeping in mind that $R(Y,Z)(T(X_1,\dots,X_p))=0$ for a p-tensor T, we obtain

$$(R(Y,Z)T)(X_1,\dots,X_p)=-\sum\nolimits_a T(X_1,\dots,R(Y,Z)X_a,\dots,X_p).$$

Thus, (2.3) can be rewritten in the form, which obviously coincides with (2.2),

$$\begin{aligned}(\Re(T))(X_1,\dots,X_p)&=-2\sum_{j,a;b<a}T(\underbrace{X_1,\dots,R(e_j,X_a)X_b}_{b},\underbrace{\dots,e_j}_{a-b},\dots,X_p)\\&-\sum_{j,a}T(\underbrace{X_1,\dots,R(e_j,X_a)e_j}_{a},\dots,X_p)\\&=-2\sum_{j,k,a;b<a}R(e_j,X_a,e_k,X_b)\,T(\underbrace{X_1,\dots,e_k}_{b},\underbrace{\dots,e_j}_{a-b},\dots,X_p)\\&+\sum_{j,a}\mathrm{Ric}(e_j,X_a)\,T(\underbrace{X_1,\dots,e_j}_{a},\dots,X_p).\end{aligned}$$

For the case $p=1$, we get $(\Re(T))(X)=T(\mathrm{Ric}(X))$, thus Δ_L has the form $\Delta_L=\bar\Delta+\mathrm{Ric}$ (with $c=1$). In this case, the operator Δ_L is the Hodge-de Rham Laplacian Δ_H on one-forms. Therefore, $\ker\Delta_L$ consists of *harmonic one-forms* on (M,g). Moreover, if M is closed then we have the orthogonal decomposition (1.3), where dimension of $L^2(\mathrm{Ker}\,\Delta_L)$ equals to the first Betti number of (M,g), according to de Rham's theorem. On the other hand, if (M,g) is complete with nonnegative Ricci curvature, then $L^2(\mathrm{Ker}\,\Delta_L)$ consists of parallel one-forms on (M,g). Furthermore, if the Ricci curvature is positive at some point of (M,g) or the holonomy of (M,g) is irreducible then $L^2(\mathrm{Ker}\,\Delta_L)$ is trivial (for the proof, see [47, p. 666]). In particular, for $c=-1$ and $p=1$ we obtain from (2.1) that $\Delta_L=\bar\Delta-\mathrm{Ric}$. The kernel of this Laplacian consists of *infinitesimal harmonic transformations* (see [27]). Recall that a vector field ξ, generating a local one-parameter group of local harmonic diffeomorphisms on (M,g), is an *infinitesimal harmonic transformation* of (M,g) (see [32]). In this case, we proved in [43] that if (M,g) is a complete Riemannian manifold with nonpositive Ricci curvature, then $L^2(\ker\Delta_L)$ consists of parallel one-forms on (M,g). Moreover, if the volume of (M,g) is infinite then any infinitesimal harmonic transformation is equal to zero. Thus, we will not consider these well-known cases, and assume $p\ge 2$.

We can formulate the following corollary of Proposition 2.1.

Corollary 2.2. *Let Δ_L be acting on $C^\infty(\otimes^p T^*M)$ for $(p \ge 2)$. If a Δ_L-harmonic section T of $\otimes^p T^*M$ has zero of infinite order at some point $x \in M$, then T vanishes.*

By direct calculations, from (2.1) we obtain the *Bochner-Weitzenböck formula*

$$\begin{aligned}\frac{1}{2}\,\Delta_B\,(\|T\,\|^2) &= -g(\bar\Delta\,T, T) + \|\,\nabla\,T\,\|^2\\ &= -g(\Delta_L\,T, T) + \|\nabla\,T\,\|^2 + c\,g(\Re(T), T),\end{aligned}\tag{2.4}$$

where $c \ne 0$ and $\Delta_B = \operatorname{div}\circ\operatorname{grad}$ is the *Beltrami Laplacian* on $C^\infty(M)$.

Recall that the Riemannian curvature tensor of (M, g) defines a symmetric algebraic operator $\bar R$: $\Lambda^2(T_xM) \to \Lambda^2(T_xM)$ on the vector space $\Lambda^2(T_xM)$ at an arbitrary point $x \in M$ (see [35, pp. 82–83]). This $\bar R$ is called the *curvature operator* of (M, g). The eigenvalues Λ_α of $\bar R$ are real numbers at each point $x \in M$. Then we can select the orthonormal frame $\{\Xi_\alpha\}$ for $\Lambda^2(T_x^*M)$ at each point $x \in M$, consisting of eigenvectors for $\bar R$, i.e., $\bar R(\Xi_\alpha) = \mathit{\Lambda}_\alpha \Xi_\alpha$. In this case, the quadratic form $g(\Re(T), T)$ can be represented in the following form at each point $x \in M$ (see [35, p. 346]):

$$g(\Re(T_x),\, T_x) = \sum\nolimits_\alpha \mathit{\Lambda}_\alpha \|\,\Xi_\alpha(T_x)\|^2.\tag{2.5}$$

Remark 2.2. There are many works devoted to the relationship between the curvature operator $\bar R$ of (M, g) and some global characterization of it, e.g., its homotopy type, topological type (see, e.g., [35, pp. 351, 353 and 390] and [7, 41]). In particular, if $\bar R \ge 0$ (respectively, $\bar R \le 0$) at a point $x \in M$, then all sectional curvatures $\sec(\pi) \ge 0$ (respectively, $\sec(\pi) \le 0$) for any 2-plane π in T_xM as well. The above statement is a consequence of the following theorem (see [35, p. 115]): if $\{e_1, \ldots, e_n\}$ is an orthonormal basis in T_xM such that $\{e_i \wedge e_j\}$ diagonalize the curvature operator: $\bar R(e_i \wedge e_j) = \lambda_{ij}\, e_i \wedge e_j$, then $\sec(\pi) \in \{\min \lambda_{ij}, \max \lambda_{ij}\}$ for any 2-plane π in T_xM.

Recall that the eigenvalues of $\bar R$ are real numbers at any $x \in M$. Thus, $\bar R$ is nonnegative, $\bar R \ge 0$ (or, strictly positive, $\bar R > 0$), if all eigenvalues of $\bar R$ are nonnegative (resp., strictly positive). For a Δ_L-harmonic section T of $\otimes^p T^*M$, by (2.4) we obtain

$$\frac{1}{2}\,\Delta_B\,(\|T\,\|^2) = \|\,\nabla\,T\,\|^2 + c\,g(\Re(T),\, T).\tag{2.6}$$

Therefore, if $c > 0$ and $g(\Re(T),\, T) \ge 0$, then $\Delta_B(\|T\|^2) \ge 0$, and hence, $\|T\|^2$ is a nonnegative subharmonic function. In this case, the following local theorem holds.

Theorem 2.3. *Let U be a connected open domain of a Riemannian manifold (M, g) with positive semi-definite curvature operator $\bar R$ at any point of U, and Δ_L be the Lichnerowicz type Laplacian acting on C^∞-sections of the bundle of covariant p-tensor fields $\otimes^p T^*M$ over (M, g) for $c > 0$ for $p \ge 2$. If $T \in \operatorname{Ker}\Delta_L$ at any point of U and the scalar function $\|T\|^2$ has a local maximum at some point of U, then $\|T\|^2$ is constant and T is parallel on U. In addition, if $\bar R \ge k > 0$ at a point $x \in U$ and $T \in C^\infty(\Lambda^p M)$ for any $p \in \{1, \ldots, n-1\}$, then $T \equiv 0$.*

Proof. From (2.5) we can conclude that the sign of the quadratic form $g(\Re(T), T)$ is opposite to the sign of the *curvature operator* $\bar R$ of a Riemannian manifold (M, g).

In particular, (2.5) shows that the quadratic form $g(\Re(T_x),\, T_x)$ is nonnegative (resp., positive) when the curvature operator is nonnegative (resp., positive) on M. From the above and (2.6) we conclude that if $\bar{R}$ is positive semi-definite on M, then $\|T\|^2$ is a subharmonic scalar function on U. Therefore, proceeding from (2.6) and using the *Hopf maximum principle* (see [6, p. 26] and [9]), we conclude that if the curvature operator is positive semi-definite on a connected domain U, then $\|T\|^2$ is a constant C and $\nabla T = g(\Re(T), T)) = 0$ on U. If $C > 0$, then T is nowhere zero. Now, at a point $x \in U$, where the curvature operator satisfies the inequality $\bar{R} \geq k > 0$, we have

$$g(\Re(T_x),\, T_x) = \sum\nolimits_{\alpha} \Lambda_\alpha \| \, \Xi_\alpha(T_x)\|^2 \geq k \sum\nolimits_{\alpha} \| \, \Xi_\alpha(T_x)\|^2 \geq 0. \tag{2.7}$$

In this case, the LHS of (2.7) is zero, while the RHS would be nonnegative. This contradiction shows that $\Xi_\alpha(T_x) = 0$ for all α. In particular, for $T \in C^\infty(\Lambda^p M)$ we have $T_x = 0$ (see [34, p. 351]). Then $C = 0$, hence $T \equiv 0$. $\square$

Remark 2.3. Recall that the Hopf maximum principle reads as follows: If a subharmonic function attains a local maximum value at some point of a connected domain U of (M, g) then it is a constant C in U (see [9, Theorem 1]).

Let (M, g) be closed, then (2.6) is globally defined and there exists a point $x \in M$, at which $\|T\,\|^2$ attains the global maximum. At the same time, let $\|T\,\|^2$ satisfies $\Delta_B(\|T\,\|^2) \geq 0$ on M. In this case, we use the *Bochner maximum principle*, which we deduce from the Hopf maximum principle. Namely, an arbitrary subharmonic function on a closed Riemannian manifold is constant (see [6, Theorem 2.2]). Thus, the following statement follows from Theorem 2.3.

Corollary 2.4. *Let (M, g) be closed with positive semi-definite curvature operator $\bar{R}$ and $\Delta_L : C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$ with $c > 0$ for $p \geq 2$. If $T \in \operatorname{Ker} \Delta_L$ on M, then $\|T\,\|^{\,2}$ is constant and T is parallel. If, in addition, $\bar{R} \geq k > 0$ at a point $x \in M$ and $T \in C^\infty(\Lambda^p M)$ where $p \in \{1, \ldots, n-1\}$ then $T \equiv 0$.*

Remark 2.4. The Hodge-de Rham Laplacian Δ_H on C^∞-sections of the bundle $\Lambda^p(M)$ is the most famous example of Δ_L. Thus we conclude from Theorem 2.3 that a closed Riemannian manifold with positive curvature operator has vanishing the p-th *Betti number* $\beta_p(M)$. The added benefit is that we also conclude from our theorem that if $\bar{R}$ is merely nonnegative, then the inequality $\beta_p(M) \leq \binom{n}{p}$ holds (see the Meyer-Gallot Theorem in [35, p. 351]).

As analogues of Theorem 2.3 and Corollary 2.4, we obtain the following theorem and corollary.

Theorem 2.5. *Let U be a connected open domain of a Riemannian manifold (M, g) with negative semi-definite curvature operator $\bar{R}$ at any point of U and Δ_L be the Lichnerowicz type Laplacian acting on C^∞-sections of the bundle of covariant p-tensor fields $\otimes^p T^*M$ with $c < 0$ for $p \geq 2$. If $T \in \operatorname{Ker} \Delta_L$ at any point of U and the scalar function $\|T\|^2$ has a local maximum at some point of U, then $\|T\|^2$ is constant and T is parallel on U. In addition, if $\bar{R} \leq k < 0$ at some point $x \in U$ and $T \in C^\infty(\Lambda^p M)$ for any $p \in \{1, \ldots, n-1\}$, then $T \equiv 0$.*

Corollary 2.6. *Let (M,g) be closed with negative semi-definite curvature operator $\bar{R}$ and $\Delta_L : C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$ with $c<0$ for $p \geq 2$. If $T \in \operatorname{Ker}\Delta_L$ on M, then $\|T\|^2$ is constant and T is parallel. If, in addition, $\bar{R} \leq k < 0$ at a point $x \in M$ and $T \in C^\infty(\Lambda^p M)$ where $p \in \{1,\dots,n-1\}$ then $T \equiv 0$.*

By direct calculation we find

$$\frac{1}{2}\,\Delta_B\,(\|T\|^2) = \|T\| \cdot \Delta_B(\|T\|) + \|\,d\,\|T\|\,\|^2.$$

Then (2.6) can be rewritten in the form

$$\|T\,\|\,\Delta_B(\|T\,\|) = c\,g(\Re(T),\,T) + \|\nabla\,T\,\|^2 - \|d\,\|T\,\|\,\|^2.$$

Using the *Kato inequality* $\|\nabla\,T\,\|^2 \geq \|\,d\,\|T\,\|\,\|^2$ (see [10]), we write the following inequality (with $c \neq 0$):

$$\|T\| \cdot \Delta_B(\|T\|) \geq c\,g(\Re(T),T). \tag{2.8}$$

Therefore, if $c>0$ and $g(\Re(T),T) \geq 0$ on (M,g), then we have $\Delta_B\,(\|T\,\|) \geq 0$ and hence, $\|T\|$ is a nonnegative subharmonic function on M. On the other hand, Greene and Wu proved in [16] the following: if (M,g) is complete noncompact with nonnegative sectional curvature and f is a nonnegative subharmonic function on M, then $\int_M f^q\,\mathrm{d}\,\mathrm{vol}_g = \infty$ for any $1 \leq q < \infty$ unless $f \equiv 0$. Based on (2.8), Theorem 2.3 and the Greene-Wu result, we conclude that if $T \in \operatorname{Ker}\Delta_L$ on M and $\int_M \|T\,\|^{\,q}\,\mathrm{d}\,\mathrm{vol}_g < \infty$ for some $1 \leq q < \infty$, then $T \equiv 0$ for the case of a complete noncompact (M,g) with nonnegative sectional curvature. Thus, we obtain the following.

Theorem 2.7. *Let (M,g) be complete noncompact with a positive semi-definite curvature operator $\bar{R}$ and $\Delta_L : C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$ with $c>0$ for $p \geq 2$. Then $L^q(\operatorname{Ker}\Delta_L)$ is trivial for any $1 \leq q < \infty$.*

As an analogue of Theorem 2.7 we obtain the following.

Theorem 2.8. *Let (M,g) be complete and simply connected with a negative semidefinite curvature operator $\bar{R}$ and $\Delta_L : C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$ with $c<0$ for $p \geq 2$. If $T \in L^q(\operatorname{Ker}\Delta_L)$ for some $q \in (0,\infty)$, then $\|T\|$ is constant and T is parallel. In particular, if $\operatorname{Vol}(M,g)=\infty$, then $T \equiv 0$. If, in addition, $\bar{R} \leq k < 0$ at some point $x \in M$ and $T \in C^\infty(\Lambda^p M)$ where $p \in \{1,\dots,n-1\}$ then $T \equiv 0$ as well.*

Proof. Note that if $\bar{R} \leq 0$, then $g(\Re(T_x),\,T_x) \leq 0$ at an arbitrary point $x \in M$ because $g(\Re(T_x),\,T_x) = \sum_\alpha \Lambda_\alpha \|\,\Xi_\alpha(T_x)\|^2 \leq 0$, where $\Lambda_\alpha \leq 0$ for all α. On the other hand, if $c<0$ and $g(\Re(T_x),\,T_x) \leq 0$ at any point of (M,g), then from (2.7) we obtain $\Delta_B(\|T\|) \geq 0$. In this case, $\|T\|$ is a nonnegative subharmonic function on (M,g). On the other hand, in [23, p. 288] was proved that every nonnegative subharmonic L^q-function for $q \in (0,\infty)$ on a complete simply connected (M,g) of nonpositive sectional curvature is constant. Therefore, if (M,g) is complete and simply connected and $T \in L^q(\operatorname{Ker}\Delta_L)$ for some $q \in (0,\infty)$, then $\|T\|$ is a constant $C \geq 0$ and $\nabla T = 0$.

In this case, the inequality $\int_M \|T\|^q\,\mathrm{d}\,\mathrm{vol}_g < \infty$ can be rewritten in the form $C^q \int_M \mathrm{d}\,\mathrm{vol}_g = C^q\,\operatorname{Vol}(M,g) < \infty$ Thus, if $\operatorname{Vol}(M,g)=\infty$ then $C=0$ and, hence,

$T \equiv 0$. On the other hand, at a point $x \in M$, where the curvature operator $\bar{R}$ satisfies the inequality $\bar{R} \leq k < 0$, we have

$$g(\Re(T_x),\, T_x) = \sum\nolimits_{\alpha} \Lambda_\alpha \| \,\Xi_\alpha(T_x)\|^2 \leq k \sum\nolimits_{\alpha} \| \,\Xi_\alpha(T_x)\|^2 \leq 0.$$

In this case, the LHS of (2.7) is zero, while the RHS would be nonpositive. This contradiction shows that $\Xi_\alpha(T_x) = 0$ for all α. In particular, for $T \in C^\infty(\Lambda^p M)$ we have $T_x = 0$ (see [35, p. 351]). Then $C = 0$ and hence $T \equiv 0$. □

2.3. Recall that the *Riemannian symmetric space* is a finite-dimensional (M, g), such that for every its point x there is an involutive geodesic symmetry s_x of a neighborhood of x, such that x is an isolated fixed point of s_x. (M, g) is said to be *Riemannian locally symmetric* if its geodesic symmetries are in fact isometries. Such (M, g) is a *Riemannian globally symmetric space* if, in addition, its geodesic symmetries are defined on all (M, g). A Riemannian globally symmetric space is complete (see [21, p. 244]). In addition, a complete and simply connected Riemannian locally symmetric space is a Riemannian globally symmetric space. These spaces can be classified in terms of their isometry groups, and the classification distinguishes three basic types: the spaces of *compact type*, the spaces of *noncompact type* and the spaces of *Euclidean type* (e.g., [21, p. 252]). If, in addition, (M, g) is a Riemannian globally symmetric space of compact type then it is a closed Riemannian manifold with non-negative sectional curvature and positive-definite Ricci tensor (see [21, p. 256]). Moreover, its $\bar{R}$ is nonnegative (see [13]). Using Remark 2.1 and Corollary 2.4, one can argue that the following statement holds.

Proposition 2.9. *Let (M, g) be an n-dimensional $(n \geq 2)$ simply connected Riemannian globally symmetric space of compact type and $\Delta_L : C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$ be the Lichnerowicz type Laplacian with $c > 0$, for $p \geq 1$. Then all Δ_L-harmonic sections of T^*M vanish and every Δ_L-harmonic section of $\otimes^p T^*M$ for $p \geq 2$ is parallel.*

Remark 2.5. If (M, g) is a simply connected Riemannian globally symmetric space of compact type then its Betti numbers satisfy the conditions $b_1(M) = b_{n-1}(M) = 0$ and $b_p(M) \leq \binom{n}{p}$ for $p = 2, \ldots, n-2$. This follows directly from Proposition 2.9.

Notice that a Riemannian symmetric space of noncompact type has nonpositive sectional curvature and negative-definite Ricci tensor, see [22, p. 256]. Also, a Riemannian symmetric space has the nonpositive curvature operator if and only if it has the nonpositive sectional curvature (see [13]). After the above remarks, the assertion of the following proposition becomes an obvious corollary of Theorem 2.8.

Proposition 2.10. *Let (M, g) be an n-dimensional $(n \geq 2)$ simply connected Riemannian globally symmetric space of noncompact type and $\Delta_L : C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$ be the Lichnerowicz type Laplacian with $c < 0$ for $p \geq 2$. Then $L^q(\operatorname{Ker} \Delta_L)$ for any $q \in (0, \infty)$ consists of parallel tensor fields. In particular, if $\operatorname{Vol}(M, g) = \infty$, then $L^q(\operatorname{Ker} \Delta_L)$ is trivial.*

3 The Lichnerowicz type Laplacian on symmetric bilinear forms

Here, we study the kernel $\operatorname{Ker} \Delta_L$ of Δ_L restricted to the space of C^∞-sections of $S^p M$ with $p \geq 2$ and, in particular, of C^∞-sections of $S^2 M$ on (M, g). In this section, we denote by φ an arbitrary element of $C^\infty(S^pM)$.

3.1. Consider the subbundle S^p_0M of $\otimes^p T^*M$ consisting of smooth traceless symmetric tensor fields. A section $\varphi \in C^\infty(S^p_0M)$ is defined by the condition

$$\operatorname{trace}_g \varphi := \sum\nolimits_{i=1}^n \varphi(e_i, e_i, X_3, \dots, X_p) = 0$$

for the orthonormal frame $\{e_i\}$ of T_xM at any point $x \in M$. It is known (see [2]) that $g(\Re(\varphi), \varphi) \ge 0$ for any $\varphi \in C^\infty(S^2_0M)$ if $\sec \ge 0$ for the sectional curvature of (M, g). This statement was generalized in [5, p. 8] in the following form: $g(\Re(\varphi), \varphi)$ is positive-semidefinite for $p \ge 2$ if $\sec \ge 0$. Moreover, for any $p \ge 2$, positive-semidefiniteness of $g(\Re(\varphi), \varphi)$ for $\varphi \in C^\infty(S^p_0M)$ and of $g(\Re(\varphi), \varphi)$ for $\varphi \in C^\infty(S^pM)$ are equivalent (see [5, p. 8]). Thus, we reformulate Theorem 2.7 in the following form.

Corollary 3.1. *Let (M, g) be complete noncompact with positive semi-definite sectional curvature and $\Delta_L : C^\infty(S^pM) \to C^\infty(S^pM)$ be the Lichnerowicz type Laplacian with $c > 0$ for $p \ge 2$. Then $L^q(\operatorname{Ker}\Delta_L)$ is trivial for an arbitrary $q \in [1, \infty)$.*

The fact that $\sec \le 0$ implies negative-semidefiniteness of $g(\Re(\varphi), \varphi)$ for $p \ge 2$ and $\varphi \in C^\infty(S^p_0M)$ was proved in [12, 18]. Thus, from Theorem 2.8 we get the following.

Corollary 3.2. *Let (M, g) be complete noncompact with negative semi-definite sectional curvature and $\Delta_L : C^\infty(S^p_0M) \to C^\infty(S^p_0M)$ be the Lichnerowicz type Laplacian with $c < 0$ for $p \ge 2$. Then $L^q(\operatorname{Ker}\Delta_L)$ for any $q \in (0, \infty)$ consists of parallel tensor fields. In particular, if $\operatorname{Vol}(M, g) = \infty$, then $L^q(\operatorname{Ker}\Delta_L)$ is trivial.*

3.2. Rewrite (2.1) for $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ in the form (with $c \ne 0$):

$$\Delta_L \varphi = \bar\Delta \varphi + c\,\Re(\varphi). \tag{3.1}$$

In this case, the Weitzenböck curvature operator (2.3) reduces to the form

$$\begin{aligned}(\Re(\varphi))(X_1, X_2) &= \sum\nolimits_j \big(\operatorname{Ric}(e_j, X_1)\varphi(e_j, X_2) + \operatorname{Ric}(e_j, X_2)\varphi(e_j, X_1)\big)\\ &- 2\sum\nolimits_{j,k} R(e_j, X_1, e_k, X_2)\,\varphi(e_j, e_k),\end{aligned}$$

or, equivalently, (2.2) has the following form (see [3, p. 64] and [2, 44]):

$$(\Re(\varphi))_{ij} = R_{ik}\varphi^k_j + R_{jk}\varphi^k_i - 2R_{ikjl}\varphi^{kl} \tag{3.2}$$

for components φ_{ij} of an arbitrary $\varphi \in C^\infty(S^2M)$. Directly from (3.1) and (3.2) we obtain $\operatorname{trace}_g(\Delta_L \varphi) = \bar\Delta(\operatorname{trace}_g \varphi)$ for an arbitrary $\varphi \in C^\infty(S^2M)$. Therefore, the following statement holds (see also [24]).

Proposition 3.3. *Let Δ_L acts on $C^\infty(S^2M)$, then* $\operatorname{trace}_g(\Delta_L \varphi) = \bar\Delta(\operatorname{trace}_g \varphi)$.

It is known that for any 2-tensor φ the following inequality holds:

$$\|\varphi\|^2 \ge (1/n)(\operatorname{trace}_g \varphi)^2$$

at an arbitrary point $x \in M$. Therefore, if $\varphi \in L^2(\operatorname{Ker}\Delta_L)$, then $\operatorname{trace}_g \varphi \in L^2(\ker \bar\Delta)$. On the other hand, if $\varphi \in \operatorname{Ker}\Delta_L$, then $\bar\Delta(\operatorname{trace}_g \varphi) = 0$. At the same time, Yau proved in [47] that any harmonic function f satisfying $f \in L^q(M)$ for some $q \in (1, \infty)$ is constant on a complete Riemannian manifold (M, g). In particular, if $\operatorname{Vol}(M, g) = \infty$, then $f \equiv 0$. From the above assumption we conclude the following.

Proposition 3.4. *Let $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ act on a complete noncompact (M, g). Then the trace of any smooth section of $L^2(\operatorname{Ker}\Delta_L)$ is a constant function. In particular, if $\operatorname{Vol}(M, g) = \infty$, then $L^2(\operatorname{Ker}\Delta_L)$ consists of traceless symmetric 2-tensors.*

On the other hand, it is known that there are no non-constant harmonic functions on a closed Riemannian manifold. Thus, we get the following corollary.

Corollary 3.5. *Let (M, g) be closed with Δ_L acting on $C^\infty(S^2M)$. Then $\operatorname{trace}_g \varphi$ is a constant for an arbitrary bilinear form $\varphi \in \operatorname{Ker}\Delta_L$.*

In our case, (2.4) can be rewritten in the following form (with $c \neq 0$):

$$\frac{1}{2}\,\Delta_B(\|\varphi\|^2) = -g(\Delta_L\,\varphi,\,\varphi) + \|\nabla\varphi\|^{\,2} + c\,g(\Re(\varphi),\,\varphi). \tag{3.3}$$

Next, for any point $x \in M$ there exists an orthonormal eigenframe $\{e_1, \ldots, e_n\}$ of T_xM such that $\varphi_x(e_i, e_j) = \mu_i\delta_{ij}$ for the Kronecker delta δ_{ij}. Then we have

$$g(\Re(\varphi_x), \varphi_x) = 2\sum\nolimits_{i<j} \sec(e_i \wedge e_j)(\mu_i - \mu_j)^2,$$

(see [2, p. 388] and [3, p. 436]), where $\sec(e_i \wedge e_j) = g(R(e_i, e_j)e_j, e_i)$ is the *sectional curvature* $\sec(\sigma_x)$ of (M, g) in the direction of the tangent two-plane section $\sigma_x = \operatorname{span}\{e_i, e_j\}$ at $x \in M$. Then we rewrite (3.3) in the following form:

$$\frac{1}{2}\,\Delta_B(\|\varphi\|^2) = -g(\Delta_L\,\varphi, \varphi) + \|\nabla\varphi\|^2 + 2\,c\sum\nolimits_{i<j} \sec(e_i \wedge e_j)(\mu_i - \mu_j)^2.$$

In particular, if φ is a covariant Δ_L-harmonic 2-tensor, then we have

$$\frac{1}{2}\,\Delta_B(\|\varphi\|^2) = \|\nabla\varphi\|^2 + 2\,c\sum\nolimits_{i<j} \sec(e_i \wedge e_j)(\mu_i - \mu_j)^2. \tag{3.4}$$

From (3.4) we conclude that $\|\varphi\|^2$ is a subharmonic function if $c > 0$ and the sectional curvature of (M, g) is non-negative. Therefore, proceeding from the above formula and using the Hopf maximum principle (see [6, p. 26] and [9]), we conclude the following: if the sectional curvature $\sec(\sigma_x)$ of (M, g) is non-negative at any point of a connected open domain $U \subset M$ and $\sec(\sigma_x)$ is positive (in all 2-dimensional directions σ_x) at some point $x \in U$, then $\|\varphi\,\|^2$ is a constant C and $\nabla\varphi = 0$ in U. If $C > 0$, then φ is nowhere zero. Now, at a point $x \in U$, where the sectional curvature $\sec(\sigma_x)$ is positive, the LHS of (5.5) is zero, while the RHS is nonpositive. This contradiction shows $\mu_1 = \ldots = \mu_n = \mu$ and hence $\varphi = \mu \cdot g$ for some constant μ on U. On the other hand, the fact that $\nabla\,\varphi = 0$ means that φ is parallel. In this case, if the holonomy of (M, g) is irreducible, then the tensor φ has a one eigenvalue, i.e., $\varphi = \mu \cdot g$ for some constant μ on U. As a result, we have the following local theorem.

Theorem 3.6. *Let U be a connected open domain of a Riemannian manifold (M, g) with nonnegative sectional curvature and Δ_L be the Lichnerowicz type Laplacian with $c > 0$ acting on C^∞-sections of the bundle S^2M over (M, g). If $\varphi \in \operatorname{Ker}\Delta_L$ at any point of U and the scalar function $\|\varphi\,\|^2$ has a local maximum at some point of U, then $\|\varphi\,\|^2$ is constant and φ is parallel on U. Moreover, if either $\sec(\sigma_x) > 0$ in all directions σ_x at some point $x \in U$ or the holonomy of (M, g) is irreducible, then φ is a constant multiple of g on U.*

Based on (3.3) and using Theorem 2.7 and the Greene-Wu result on subharmonic functions on a complete noncompact (M, g) with nonnegative sectional curvature (see [16]), we conclude that if $\varphi \in \operatorname{Ker} \Delta_L$ on (M, g) and $\int_M \|\varphi\|^2 \, d\operatorname{vol}_g < \infty$, then $\varphi \equiv 0$. Then we obtain the following.

Corollary 3.7. *Let (M, g) be complete noncompact with nonnegative sectional curvature and $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ with $c > 0$. Then $L^2(\operatorname{Ker} \Delta_L)$ is trivial.*

Consider now the case $n = 3$. We have the following equality:

$$\sec(\sigma_x) = (1/2)\, s - \operatorname{Ric}(X_x, X_x),$$

where $\sec(\sigma_x)$ is the sectional curvature in the direction of the plane $\sigma_x \subset T_xM$ for a point $x \in M$, X is a unit vector orthogonal to σ_x (see [45, Lemma 2.1]). Therefore, if $n = 3$ and $\operatorname{Ric} \le (1/2)s\, g$ on M, then the inequality $\sec(\sigma_x) \ge 0$ holds at each point $x \in M$. In this case, if $c > 0$ then from (3.4) we conclude that $\|\varphi\|^2$ is a subharmonic function, and using the Greene-Wu theorem on subharmonic functions, we get $\varphi \equiv 0$.

Thus, we obtain the following.

Corollary 3.8. *Let (M, g) be a three-dimensional complete noncompact Riemannian manifold and $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ be the Lichnerowicz type Laplacian. If the Ricci curvature* Ric *and the scalar curvature s of (M, g) satisfy* $\operatorname{Ric} \le (1/2)s\, g$, *then $L^2(\operatorname{Ker} \Delta_L)$ is trivial.*

3.3. Consider a closed (M, g) with nonnegative sectional curvature. Then, based on (3.4) and the Bochner maximum principle (see [6, p. 30]), we conclude that the kernel of $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ with $c > 0$ consists of parallel symmetric 2-tensor tensor fields, i.e., from the condition $\varphi \in \operatorname{Ker} \Delta_L$ we obtain $\nabla \varphi = 0$. It is known that every parallel symmetric tensor field $\varphi \in C^\infty(S^2M)$ on (M, g) with irreducible holonomy is trivial, i.e., $\varphi = \lambda\, g$ for some constant λ.

Therefore, we obtain the following.

Theorem 3.9. *Let (M, g) be closed with irreducible holonomy and nonnegative sectional curvature. Then the kernel of the Lichnerowicz type Laplacian $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ with $c > 0$ consists of trivial symmetric 2-tensors.*

Note that a *Riemannian symmetric space of compact type* is an example of a closed Riemannian manifold with non-negative sectional curvature and positive-definite Ricci tensor (see [21, p. 256]). Thus, the following corollary is valid.

Corollary 3.10. *Let (M, g) be a locally irreducible Riemannian symmetric space of compact type. Then the kernel of the Lichnerowicz type Laplacian $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ with $c > 0$ consists of trivial symmetric 2-tensors.*

Remark 3.1. A simple example of a Riemannian symmetric space of compact type is the n-dimensional round sphere (S^n, g_0) with standard metric g_0. Then an arbitrary Δ_L-harmonic tensor on (S^n, g_0) has the form $\varphi = \mu \cdot g_0$ for some real constant μ.

In conclusion, recall the definition of a *TT-tensor* (Transverse Traceless tensor), i.e., a divergence free and traceless covariant symmetric 2-tensor field. Such tensors are of fundamental importance in *stability analysis* in General Relativity (e.g., [15, 34, 37]) and in Riemannian geometry (see [20, 24]). In particular, Page and Pope have proved in [34] the following theorem on the kernel of Δ_L on TT-tensors.

Theorem 3.11. *Let $(M,\, g)$ be a Riemannian manifold and Δ_L be the Lichnerowicz type Laplacian with $c = 1$ acting on C^∞-sections of S^2M. If the holonomy of $(M,\, g)$ is reducible, then there exists a* TT*-tensor $\varphi \in C^\infty(S^2M)$ such that $\varphi \in \operatorname{Ker} \Delta_L$.*

Thus, Theorem 3.6 yields the following corollary for Δ_L-harmonic TT-tensors.

Corollary 3.12. *Let (M, g) be closed with positive sectional curvature and $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ with $c = 1$ be restricted to TT-tensors on (M, g). Then $L^2(\operatorname{Ker} \Delta_L)$ is trivial.*

It is known that in dimension three a metric g has positive sectional curvature if and only if its Ricci curvature and scalar curvature satisfy the inequality $\operatorname{Ric} < (1/2)s\, g$ (see [17, p. 277]). Therefore, we obtain the following.

Corollary 3.13. *Let (M, g) be a three-dimensional closed Riemannian manifold and the Lichnerowicz type Laplacian $\Delta_L\colon C^\infty(S^2M) \to C^\infty(S^2M)$ with $c = 1$ be restricted to* TT*-tensors on (M, g). If $\operatorname{Ric} < (1/2)\, s\, g$, then $L^2(\operatorname{Ker} \Delta_L)$ is trivial.*

Remark 3.2. Let (M, g) admit a spinor structure, see Example 1.1(iv). If $s > 0$, then the equation $\Delta_D T = 0$ for the spinor Laplacian Δ_D implies that the harmonic spinor T vanishes, since

$$0 = \langle \Delta_D T,\, T\rangle = \langle \nabla T,\, \nabla T\rangle + (1/4)\, s\, \langle T, T\rangle.$$

Therefore, there are no nonzero harmonic spinors on a closed Riemannian manifold with positive scalar curvature (see [25]). This proposition together with Atiyah-Singer index Theorem applied to the Dirac operator for $4k$-dimensional manifolds, gives a topological obstruction – namely, the vanishing of the $\hat{A}$-*genus of Hirzebruch* – for the existence of positive scalar curvature metrics on a compact spin manifold. The proof uses the classical Bochner technique (see [25]; [3, pp. 169–171]).

4 Applications to higher order Killing and Codazzi tensors

4.1. Here, we consider Δ_L, acting on $C^\infty(S^pM)$, where S^p_0M is the subbundle of trace-free covariant symmetric tensor fields for $p \geq 2$. In [18], it was proven that if (M, g) has non-negative sectional curvature, then for any $T \in S^p_0M$ the inequality $g(\Re(T), T) \leq 0$ holds on this manifold. Thus, we obtain Corollary 3.2 (from Theorem 2.8), which we use to study higher order symmetric Killing tensors.

If a tensor $T \in C^\infty(S^pM)$ satisfies $(\nabla_X T)(X, \ldots, X) = 0$ for any $X \in TM$, then it is called *symmetric Killing p-tensor*. In this case, $\delta^*T = 0$ for the operator $\delta^* : C^\infty(S^pM) \to C^\infty(S^{p+1}M)$ of degree one such that (see [27, 44])

$$(\delta^*T)(X_1, \ldots, X_{p+1}) = (\nabla_{X_1}T)(X_2, \ldots, X_{p+1}) + \ldots + (\nabla_{X_{p+1}}T)(X_1, \ldots, X_p)$$

for any $X_1, \ldots, X_{p+1} \in T_xM$ at a point $x \in M$. There exists its formal adjoint operator with respect to the L^2-product which is called the *divergence operator* (see [3, p. 356]). Notice that δ is the $\otimes^{p+1}T^*M$-restriction of ∇^* to $S^{p+1}M$. Using operators δ^* and δ, Sampson defined in [39, p. 147] the second order elliptic differential operator $\Delta_S : C^\infty(S^pM) \to C^\infty(S^pM)$ by the formula $\Delta_S = \delta\, \delta^* - \delta^*\delta$. In addition, we have the Weitzenböck decomposition formula $\Delta_S = \bar{\Delta} - \Re$ (see [18, 30, 39]).

Based on the foregoing, we conclude that any divergence-free (e.g., traceless) symmetric Killing p-tensor belongs to $\ker \Delta_S$ (see [18, 30]). In this case, the equation $\Delta_L T = (c+1)\Re(T)$ is valid. Therefore, any symmetric Killing p-tensor belongs to $\ker \Delta_L$ for the case $c = -1$. Thus, (2.4) can be rewritten as

$$(1/2)\Delta_B \|T\|^2 = \|\nabla T\|^2 - g(\Re(T), T).$$

Then the above reasoning shows that the following proposition is true.

Proposition 4.1. *Let Δ_L be the Lichnerowicz type Laplacian acting on $C^\infty(S^p M)$ for $p \geq 2$, then any symmetric traceless Killing p-tensor T belongs to* $\ker \Delta_L$ *for the case of $c = -1$. Moreover, if (M, g) is complete and simply connected with nonpositive sectional curvature, then T is parallel, and if* $\mathrm{Vol}(M, g) = \infty$ *then $T \equiv 0$.*

The above proposition completes the results from [12, 18, 30, 40], where symmetric Killing p-tensors were considered on compact Riemannian manifolds. On the other hand, in [5] they proved that for every $T \in S^p M$ on a Riemannian manifold with non-positive sectional curvature, the inequality $g(\Re(T), T) \geq 0$ holds at every point of this manifold. Then we obtain Corollary 3.1 (from Theorem 2.7), which we use to study higher order Codazzi tensors.

If a tensor $T \in C^\infty(S^p M)$ satisfies $\nabla T \in C^\infty(S^{p+1} M)$ then it is called *Codazzi p-tensor*, or, higher order Codazzi tensor (see [26, 29, 40]). Moreover, an arbitrary traceless or divergence-free Codazzi p-tensor T satisfies $\Delta_S T = (p+1)\bar{\Delta} T$ for the Sampson Laplacian $\Delta_L : C^\infty(S^p M) \to C^\infty(S^p M)$. In this case, the following equation $\Delta_L T = (c - 1/p)\Re(T)$ is valid. Therefore, any traceless Codazzi p-tensor belongs to $\ker \Delta_L$ for the case of $c = 1/p$. In its turn, (2.4) can be rewritten as

$$(1/2)\Delta_B \|T\|^2 = \|\nabla T\|^2 + (1/p) g(\Re(T), T).$$

Then the above reasoning shows that the following corollary is valid.

Proposition 4.2. *Let Δ_L be the Lichnerowicz type Laplacian acting on $C^\infty(S^p M)$ for $p \geq 2$, then any traceless Codazzi p-tensor belongs to* $\ker \Delta_L$ *for the case of $c = 1/p$. Moreover, if (M, g) is complete noncompact with non-negative sectional curvature, then any traceless higher order Codazzi L^q-tensor for an arbitrary $1 \leq q < \infty$ is parallel, and if* $\mathrm{Vol}(M, g) = \infty$ *then it is identically zero.*

Notice that Corollary 3.1 completes the results from [12, 40, 29], where Codazzi p-tensors are considered on complete and compact Riemannian manifolds.

4.2. In conclusion, consider Δ_L, which acts on the vector space of C^∞-sections of $\Lambda^p M$. If a tensor field $T \in C^\infty(\Lambda^p M)$ satisfies conditions $\nabla T = (1/(p+1)) dT$ and $\delta T = 0$ for the exterior differential $d : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1} M)$, then it is called a *Killing tensor* (see [6, pp. 65–66]). In this case, $\Delta_H T = (p+1)\bar{\Delta} T$ for the Hodge-de Rham Laplacian $\Delta_H = d\,\delta + \delta\, d$ (see [35, p. 335]). Then the equation $\Delta_L T = (c + 1/p)\Re(T)$ is valid. Therefore, any Killing p-tensor belongs to $\ker \Delta_L$ for the case of $c = -1/p$. In its turn, (2.4) can be rewritten in the form

$$(1/2)\Delta_B \|T\|^2 = \|\nabla T\|^2 - (1/p) g(\Re(T), T).$$

By the above, the following corollary from Theorem 2.8 is true.

Corollary 4.3. *Let Δ_L be acting on $C^\infty(\Lambda^p M)$ for $1 \le p \le \dim M - 1$. Then any Killing p-tensor T belongs to* $\ker \Delta_L$ *for the case of $c = -1/p$. Moreover, if (M, g) is complete and simply connected with negative semi-definite curvature operator $\bar{R}$, then T is parallel. If, in addition, $\bar{R} \le k < 0$ at some point $x \in M$, then $T \equiv 0$.*

Remark 4.1. More information on Codazzi and Killing tensors on complete Riemannian manifolds can be found in our articles [29] and [41], respectively.

5 Applications to the theories of infinitesimal Einstein deformations and the stability of Einstein manifolds

The Lichnerowicz Laplacian $\Delta_L : C^\infty(S^2M) \to C^\infty(S^2M)$ with $c = 1$ is of fundamental importance in the stability analysis in General Relativity (e.g., [5, 15, 37]) and appears in many problems of Riemannian geometry. For example, Δ_L acting on symmetric 2-tensor fields can be seen as infinitesimal deformations of metric g, and describes the change of the Ricci tensor in terms of these deformations (e.g., [2] and [3, Chapter 12]). Furthermore, Δ_L is a fundamental operator; when acting on covariant symmetric 2-tensor fields in context of *Ricci flow*, it seems to be more natural than the connection or Bochner Laplacian $\bar{\Delta}$. Examples of this naturalness are the appearance of Δ_L in the linearized Ricci flow equation (e.g., the evolution formula of the Ricci tensor under the Ricci flow in [11, p. 112]). Here, we complete some of these results.

5.1. Recall that an *Einstein manifold* is (M, g), whose Ricci tensor satisfies $\mathrm{Ric} = \kappa\, g$ for some $\kappa \in \mathbb{R}$. Taking trace of this, one can see that $\kappa = s/n$ for the scalar curvature s of $(M,\, g)$. We shall consider Einstein manifolds in this section.

Notice that the Riemannian curvature tensor of (M, g) defines a symmetric algebraic operator $\overset{\circ}{R} : S^2(T_xM) \to S^2(T_xM)$ on the vector space $S^2(T_xM)$ of symmetric bilinear forms over tangent space T_xM at an arbitrary point $x \in M$. The operator $\overset{\circ}{R}$ is called the *curvature operator of the second kind* of (M, g).

Remark 5.1. The definition, properties and applications of $\overset{\circ}{R}$ can be found in monographs [3, 4] and in articles from the following list: [8, 20, 31, 33, 40, 42].

We call the differential operator $\Delta_E = \bar{\Delta} - 2\,\overset{\circ}{R}$ acting on C^∞-sections of the bundle S^2M over an Einstein manifold $(M,\, g)$ the *Einstein operator*. This is a self-adjoint elliptic operator mapping from the vector space of TT-tensors to itself (see also [22]). If a TT-tensor φ belongs to $\mathrm{Ker}\, \Delta_E$ then it can be seen as an *infinitesimal Einstein deformation through g* (see [2] and [3, pp. 346–348]). Recall that a deformation of Einstein structures through g means a smooth curve $g(t)$ of Riemannian metrics, where t belongs to some open interval I containing 0 with $g(0) = g$ and such that for each $t \in I$ there exists a real number $\kappa(t)$ with the property $\mathrm{Ric}_{g(t)} = \kappa(t) \cdot g(t)$. The Einstein operator is closely related to Δ_L with $c = 1$; in fact, we have

$$\Delta_L = \Delta_E + 2\,(s/n)\,\mathrm{Id}. \tag{5.1}$$

Therefore, if $\varphi \in C^\infty(S^2M) \cap \mathrm{Ker}\, \Delta_L$ then $\Delta_E\, \varphi = -2(s/n)\, \varphi$, i.e., φ is an eigentensor of Δ_E with the eigenvalue $-2\, s/n$. The converse is also true. From (5.1) we find that the Einstein operator Δ_E is positive (resp., negative) for all nonzero TT-tensors belonging to $\operatorname{Ker}\Delta_L$, if (M,g) is an Einstein manifold with negative (resp., positive) scalar curvature. In particular, if $(M,\,g)$ is *Ricci-flat* (see [14]), then $\Delta_L = \Delta_E$. In this case, an arbitrary TT-tensor φ is an infinitesimal Einstein deformation of the metric g if φ belongs to $\operatorname{Ker}\Delta_L$. Therefore, we obtain the following.

Proposition 5.1. *Let (M,g) be an Einstein manifold, then* $\operatorname{Ker}\Delta_L$ *of Laplacian* Δ_L *acting on $C^\infty(S^2M)$ consists of eigentensors of the Einstein operator $\Delta_E = \bar\Delta - 2\,\overset{\circ}{R}$ with eigenvalues equal to $-2\,s/n$. The converse is also true. Furthermore, Δ_E is positive (resp., negative) on* TT*-tensors belonging to* $\operatorname{Ker}\Delta_L$*, if s is negative (resp., positive). In particular, if (M,g) is Ricci-flat, then a* TT*-tensor φ belongs to* $\operatorname{Ker}\Delta_L$ *if and only if it is an infinitesimal Einstein deformation of the metric g.*

Recall that (M,g) is *unstable*, if the Einstein operator admits negative eigenvalues on TT-tensors (see [22]). By Proposition 5.1, (M,g) is unstable with respect to a TT-tensor $\varphi\in\operatorname{Ker}\Delta_L$, if (M,g) is an Einstein manifold with positive scalar curvature. The following theorem from [3, p. 355] is well known: let g be an Einstein metric on M and a_0 – the largest eigenvalue of the zero order operator $\overset{\circ}{R}$ on the bundle of traceless symmetric 2-tensor fields, i.e., $a_0 = \sup\{\, g(\overset{\circ}{R}h,h)/\|h\|^2 :\ h\in C^\infty(S^2_0M)\,\}$. If $a_0 < \max\{-s/n; s/(2n)\}$, then g does not admit infinitesimal Einstein deformations. The following our theorem completes the above theorem.

Theorem 5.2. *Let (M,g) be a closed Einstein manifold with nonzero scalar curvature s and $K_{\min}$ – the minimum of its sectional curvature. If $K_{\min}\ge s/n^2$, then (M,g) is not unstable and does not admit infinitesimal Einstein deformations.*

Proof. Let (M,g) be an Einstein manifold with nonzero scalar curvature s and let φ be a TT-tensor on (M,g). Then (3.3) can be rewritten in the form

$$\begin{aligned}\frac12\,\Delta_B(\|\,\varphi\,\|^2) &= -g(\Delta_E\,\varphi,\varphi) - 2\,\frac{s}{n}\,\|\varphi\|^2 + \|\nabla\varphi\|^2\\ &\quad + 2\sum\nolimits_{i<j}\sec(e_i\wedge e_j)(\mu_i-\mu_j)^2,\end{aligned}\tag{5.2}$$

where $c=1$. If $\operatorname{trace}_g\varphi = \mu_1+\ldots+\mu_n = 0$, then the following equality holds: $\|\varphi\|^2 = \mu_1^2+\ldots+\mu_n^2 = \frac1n\sum_{i<j}(\mu_i-\mu_j)^2$. In this case, from (5.2) one can obtain

$$\frac12\,\Delta_B(\|\varphi\|^2) \ge -g(\Delta_E\,\varphi,\varphi) + \|\nabla\varphi\|^2 + 2\,\big(K_{\min} - \frac{s}{n^2}\big)\sum\nolimits_{i<j}(\mu_i-\mu_j)^2,\tag{5.3}$$

where we denoted by $K_{\min}$ the minimum of the sectional curvature of (M,g), i.e., $\sec(\sigma_x)\ge K_{\min}$ in all 2-dimensional directions σ_x at each point $x\in M$.

First, let (M,g) be unstable for φ, then $g(\Delta_E\,\varphi,\varphi) = -\lambda^2(\varphi)\|\varphi\,\|^2$ for some $\lambda(\varphi)\ne 0$. In this case, the inequality (5.3) takes the form

$$\frac12\,\Delta_B(\,\|\varphi\|^2) \ge \lambda^2(\varphi)\|\varphi\|^2 + \|\nabla\varphi\|^{\,2} + 2\,\big(K_{\min} - \frac{s}{n^2}\big)\sum\nolimits_{i<j}(\mu_i-\mu_j)^2.\tag{5.4}$$

If $K_{\min}\ge s/n^2$, then from (5.4) we conclude that $\|\varphi\|^{\,2}$ is a subharmonic function, i.e., $\Delta_B(\|\varphi\|^{\,2})\ge 0$. Furthermore, if (M,g) is a closed manifold, then using the Bochner maximum principle (see [6, p. 30]), we conclude that $\|\varphi\|^2$ is constant. In this case, from (5.4) we obtain that $\varphi \equiv 0$.

Second, if (M, g) is a stable manifold, then (5.3) can be rewritten in the form

$$\frac{1}{2}\,\Delta_B\,(\|\,\varphi\,\|^2) \geq \|\,\nabla\,\varphi\,\|^2 + 2\left(K_{\min} - \frac{s}{n^2}\right)\sum\nolimits_{i<j}(\mu_i - \mu_j)^2. \tag{5.5}$$

If $K_{\min} \geq s/n^2$, then from (5.5) we obtain $\Delta_B(\|\varphi\|^2) \geq 0$, i.e., $\Delta_B(\|\varphi\|^2)$ is a subharmonic function. Then proceeding from (5.5) and using the Bochner maximum principle (see [6, p. 30]), we conclude that $\|\varphi\| = \text{const}$ and hence $\nabla\,\varphi = 0$. In this case, by the Ricci identities, we have $\varphi_{ik}R^k_{jlm} + \varphi_{kj}R^k_{ilm} = 0$. Thus, $\mathring{R}(\varphi) = -(s/n)\varphi$. In this case, the equation $\Delta_E\,\varphi = \bar{\Delta}\,\varphi - 2\,\mathring{R}\,(\varphi) = 0$ can be rewritten in the form $\bar{\Delta}\,\varphi = -2(s/n)\varphi$. This implies

$$-2\,\frac{s}{n}\int_M \|\varphi\|^2\,\mathrm{d}\,\mathrm{vol}_g = \int_M g(\bar{\Delta}\varphi, \varphi)\,\mathrm{d}\,\mathrm{vol}_g = \int_M \|\nabla\varphi\|^2\,\mathrm{d}\,\mathrm{vol}_g = 0.$$

Hence, $\varphi \equiv 0$. By this, φ is a trivial infinitesimal Einstein deformation. □

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