Projective vector fields on the tangent bundle with the deformed complete lift metrics

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Abstract. Let (M_n, g) be a Riemannian manifold and TM_n its tangent bundle. In this paper, firstly, we determine the infinitesimal fiber-preserving projective(IFP) transformations on TM_n with respect to the Riemannian connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n and g^C and g^V are the complete lift and the vertical lift of g on TM_n, respectively. Then, we prove that (M_n, g) is locally flat, if $(TM_n, \tilde{G}_f)$ admits a non-affine infinitesimal fiber-preserving projective transformation. Finally, the infinitesimal complete lift projective transformations on $(TM_n, \tilde{G}_f)$ are studied.

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1 Introduction

Let M_n be a connected n-dimension manifold and TM_n its tangent bundle. We assume that the all geometric objects, which will be considered in this paper, are differentiable of class C^∞. Also the set of all tensor fields of type (r, s) on M_n and TM_n are denoted by $\Im^r_s(M_n)$ and $\Im^r_s(TM_n)$, respectively.

Let ∇ be an affine connection on M_n. If a transformation on M_n preserves the geodesics as point sets, then it is called projective transformation. Also, a transformation on M_n which preserves the connection is called affine transformation. Therefore, an affine transformation is a projective transformation which preserves the geodesics with the affine parameter.

A vector field V on M_n with the local one-parameter group $\{\phi_t\}$ is called an infinitesimal projective (affine) transformation, if for every t, ϕ_t be a projective (affine) transformation on M_n.

It is well known that, a vector field V is an infinitesimal projective transformation if and only if for every $X, Y \in \Im^1_0(M_n)$, we have

$$(L_V\nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X,$$

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where Ω is an one form on M_n and L_V is the Lie derivation with respect to V. In this case Ω is called the associated one form of V. One can see that V is an infinitesimal affine transformation if and only if $\Omega = 0$[19].

Now let $\tilde{\phi}$ be a transformation on TM_n. If $\tilde{\phi}$ preserves the fibers, then it is called the fiber-preserving transformation. Let $\tilde{V}$ be a vector field on TM_n and $\{\tilde{\phi}_t\}$ the local one-parameter group generated by $\tilde{V}$. If $\tilde{\phi}_t$, for every t, be a fiber-preserving transformation, then $\tilde{V}$ is called an infinitesimal fiber-preserving transformation. Infinitesimal fiber-preserving transformations form a rich class of infinitesimal transformations on TM_n which include infinitesimal complete lift, horizontal lift and vertical lift transformations as special subclasses. For more detalis see [15].

One of the interesting and important problems in the context of Riemannian geometry is the classification of Riemannian manifolds, when the Riemannian manifold or its tangent bundle admits an infinitesimal projective transformation, see [3, 6, 7, 8, 9] and [11, 12, 14, 16, 17, 18]. For instance, in [11], it is proved that if a complete Riemannian manifold M_n, with the parallel Ricci tensor, admits a non-affine infinitesimal projective transformation, then M_n is a space of positive constant curvature. Also, it is proved that a simply contact Riemannian manifold M_n is isometric to a unit sphere if M_n admits a non-affine infinitesimal projective transformation[12].

It is well-known that, from a Riemannian metric g on M_n, several metrics can be defined on TM_n such as 1) the Sasaki metric g^S which was introduced by Sasaki in [13], 2) the complete lift metric g^C, 3) the vertical lift metric g^V, and etc. For more details, one can refer to [20].

In [8] and [14], the following theorem is proved.

Theorem A: Let (M_n, g) be a complete Riemannian manifold and TM_n its tangent bundle. If TM_n, with respect to the Riemannian connection 1) the Sasaki metric or 2) the complete lift metric, admits a non-affine infinitesimal projective transformation, then M_n is locally flat.

Gezer and Özkan in [4], have considered a pseudo-Riemannian metric on TM_n, which is of the form $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n. They called it the deformed complete lift metric. This new class of metrics is very interesting because for $f = 0$, the metric $\tilde{G}$ is the complete lift metric g^C, thus this is a generalization of the complete lift metric g^C. Also the deformed complete lift metric is not included in the calss of g-natural metrics, in fact $\tilde{G}_f$ is a g-natural metric if and only if f is constant. For g-natural metrics, one can see [2, 1]. On the other hand $\tilde{G}_f$ is a subclass of the synectic lift metric of g, which is defined in [5] and is of the form $\tilde{G} = g^C + a^V$, where $a \in \Im^0_2(M_n)$ is a symmetric tensor field.

The aim of this paper is to study of the infinitesimal fiber-preserving projective (IFP) transformations on TM_n with respect to the Riemannian connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n. Firstly, the necessary and sufficient conditions are obtained that under which an infinitesimal fiber-preserving transformation on $(TM_n, \tilde{G}_f)$ to be projective. Then it is shown that the theorem A is true about of the deformed complete lift metric $\tilde{G}_f$. Finally, as a special case, the infinitesimal complete lift projective transformations on $(TM_n, \tilde{G}_f)$ are studied.

2 Preliminaries

Here, we give some of the necessary definitions and theorems on M_n and TM_n, that are needed later. The details of them can be founded in [20, 21]. In this paper, indices $a, b, c, i, j, k, \dots$ have range in $\{1, \dots, n\}$.

Let M_n be a manifold and covered by local coordinate systems (U, x^i), where x^i are the coordinate functions on the coordinate neighborhood U. The tangent bundle of M_n is defined by $TM_n := \bigcup_{x\in M} T_x(M_n)$, where $T_x(M_n)$ is the tangent space of M_n at a point $x \in M_n$. The induced local coordinate system on TM_n, from (U, x^i), is denoted by $(\pi^{-1}(U), x^i, y^i)$, where $\pi : TM_n \to M_n$ is the natural projection and y^i are the Cartesian coordinates on each tangent space $T_x(M_n)$, $x \in U$.

Let (M_n, g) be a Riemannian manifold and ∇ the Riemannian connection related to g. The coefficients of ∇ with respect to frame field $\{\partial_i := \frac{\partial}{\partial x^i}\}$ are denoted by Γ^h_{ji}, i.e. $\nabla_{\partial_j}\partial_i = \Gamma^h_{ji}\partial_h$.

Now, using the Levi-Civita Connection ∇, we can define the local frame field $\{E_i, E_{\bar{i}}\}$ on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM_n, as follows

$$E_i := \partial_i - y^b\Gamma^h_{bi}\partial_{\bar{h}}, \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$. This frame field is called the adapted frame on TM_n. The dual frame of $\{E_i, E_{\bar{i}}\}$ is $\{dx^h, \delta y^h\}$, where $\delta y^h := dy^h + y^b\Gamma^h_{ab}dx^a$. The following lemma is proved by the straightforward calculations.

Lemma 2.1. *The Lie brackets of the adapted frame $\{E_i, E_{\bar{i}}\}$ satisfy the following identities:*

1. $[E_j, E_i] = y^b R^a_{ijb} E_{\bar{a}}$,
2. $[E_j, E_{\bar{i}}] = \Gamma^a_{ji} E_{\bar{a}}$,
3. $[E_{\bar{j}}, E_{\bar{j}}] = 0$,
   where R^a_{ijb} are the coefficients of the Riemannian curvature tensor of ∇.

Let X be a vector field on M_n and expressed by $X = X^i\partial_i$ on a local coordinate system (U, x^i). We can define vector fields horizontal lift X^H, vertical lift X^V and complete lift X^C of X on TM_n as follows

$$X^H := X^iE_i, \quad X^V := X^iE_{\bar{i}}, \quad X^C := X^iE_i + y^a\nabla_aX^iE_{\bar{i}}.$$

An important class of vector fields on TM_n is the fiber-preserving vector fields, which is determined in the following lemma.

Lemma 2.2. *[15] Let $\tilde{V} = \tilde{V}^hE_h + \tilde{V}^{\bar{h}}E_{\bar{h}}$ be a vector field on TM_n. Then $\tilde{V}$ is an infinitesimal fiber-preserving transformation if and only if $\tilde{V}^h$ are functions on M_n.*

Thus, the class of fiber-preserving vector fields is include horizontal lift, vertical lift and complete lift vector fields, and any fiber-preserving vector field $\tilde{V} = V^hE_h + \tilde{V}^{\bar{h}}E_{\bar{h}}$ on TM_n induces a vector field $V := V^h\partial_h$ on M_n. Using a simple calculation, we have the following lemma.

Lemma 2.3. *Let $\tilde{V} = V^hE_h + \tilde{V}^{\bar{h}}E_{\bar{h}}$ be a fiber-preserving vector field on TM_n. Then we have*

1. $[\tilde{V}, E_i] = -(\partial_i V^a)E_a + (V^c y^b R^a_{icb} - \tilde{V}^{\bar{b}}\Gamma^a_{bi} - E_i \tilde{V}^{\bar{a}})E_{\bar{a}}$,
2. $[\tilde{V}, E_{\bar{i}}] = (V^b\Gamma^a_{bi} - E_{\bar{i}}\tilde{V}^{\bar{a}})E_{\bar{a}}$.

From a Riemannian metric g, the Sasaki metric g^S, the complete lift g^C and the vertical lift g^V are defined as follows, respectively:

$$\begin{aligned} g^S(X^H, Y^H) &= g(X,Y),\\ g^S(X^H, Y^V) &= 0,\\ g^S(X^V, Y^V) &= g(X,Y), \end{aligned} \tag{2.1}$$

$$\begin{aligned} g^C(X^H, Y^H) &= 0,\\ g^C(X^H, Y^V) &= g(X,Y),\\ g^C(X^V, Y^V) &= 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} g^V(X^H, Y^H) &= g(X,Y),\\ g^V(X^H, Y^V) &= 0,\\ g^V(X^V, Y^V) &= 0, \end{aligned} \tag{2.3}$$

for every $X, Y \in \Im^1_0(M_n)$. It would be noted that g^S is a Riemannian metric, g^C is a pseudo-Riemannian metric and g^V is a degenerate quadratic form. For more details, see [20].

A new class of metrics on TM_n was introduced in [4], which is a generalization of the complete lift metric g^C and is of the form $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n. It is called the deformed complete lift metric. It is easy to see that $\tilde{G}_f$ is a pseudo-Riemannian metric on TM_n and it is defined by

$$\begin{aligned} \tilde{G}_f(X^H, Y^H) &= fg(X,Y),\\ \tilde{G}_f(X^H, Y^V) &= g(X,Y),\\ \tilde{G}_f(X^V, Y^V) &= 0, \end{aligned} \tag{2.4}$$

for every $X, Y \in \Im^1_0(M_n)$.

The coefficients of the Levi-Civita connection $\tilde{\nabla}$, of the deformed complete lift metric $\tilde{G}_f$, with respect to the adapted frame field $\{E_i, E_{\bar{i}}\}$ are computed in [4]. In fact, the following lemma is proved.

Lemma 2.4. *[4] Let $\tilde{\nabla}$ be the Riemannian connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n, then we have*

$$\tilde{\nabla}_{E_j} E_i = \Gamma^h_{ji} E_h + \left\{ y^a R^h_{aji} + \tfrac{1}{2}(f_i \delta^h_j + f_j \delta^h_i - g_{ji} f^h_{.}) \right\} E_{\bar{h}},$$

$$\tilde{\nabla}_{E_j} E_{\bar{i}} = \Gamma^h_{ji} E_{\bar{h}},$$

$$\tilde{\nabla}_{E_{\bar{j}}} E_i = 0,$$

$$\tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0,$$

where Γ^h_{ji} and R^h_{aji} are the coefficients of the Levi-Civita connection and the Riemannian curvature of $g := (g_{ji})$, respectively and $f_i := \partial_i f$, $f^h_{.} := g^{hi} f_i$.

3 Main Results

Theorem 3.1. *Let (M_n, g) be a Riemannian manifold and TM_n its tangent bundle with the Riemannian connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$ where f is a nonzero differentiable function on M_n. Then $\tilde{V}$ is an infinitesimal fiber-preserving projective(IFP) transformation on TM_n, with the associated one form $\tilde{\Omega}$, if and only if there exist $\psi \in \Im^0_0(M)$, $V = (V^h), D = (D^h) \in \Im^1_0(M)$ and $C = (C^h_i) \in \Im^1_1(M)$, satisfying*

1. $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (V^h, D^h + y^a C^h_a)$,
2. $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\Psi_i, 0)$,
3. $\Psi_i = \partial_i \psi,\ \nabla_j \Psi_i = 0$
4. $V^a \nabla_a R^h_{jbi} + R^h_{abi} \nabla_j V^a + R^h_{jba} \nabla_i V^a + R^h_{jai} C^a_b - R^a_{jbi} C^h_a = 0$
5. $\nabla_i C^h_j = V^a R^h_{iaj} + \Psi_i \delta^h_j$
6. $L_V \Gamma^h_{ji} = \nabla_j \nabla_i V^h + V^a R^h_{aji} = \Psi_i \delta^h_j + \Psi_j \delta^h_i$,
7. $L_D \Gamma^h_{ji} = \nabla_j \nabla_i D^h + D^a R^h_{aji} = C^h_a M^a_{ji} - V^a \nabla_a M^h_{ji} - M^h_{ja} \nabla_i V^a - M^h_{ia} \nabla_j V^a$

where $\tilde{V} = (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$, $\tilde{\Omega} = (\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = \tilde{\Omega}_i dx^i + \tilde{\Omega}_{\bar{i}} \delta y^i$, $f_i := \partial_i f$, $f^a_{.} := g^{ia} f_i$ and $M^h_{ji} := \frac{1}{2}(f_i \delta^h_j + f_j \delta^h_i - g_{ji} f^h_{.})$.

Proof. Firstly, we prove the necessary conditions. Let $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be an infinitesimal fiber-preserving projective transformation and $\tilde{\Omega} = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}} \delta y^h$ its the associated one form on TM_n, thus for any $\tilde{X}, \tilde{Y} \in \Im^1_0(TM_n)$, we have

$$(L_{\tilde{V}} \tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X}. \tag{3.1}$$

From

$$(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = \tilde{\Omega}_{\bar{j}} E_{\bar{i}} + \tilde{\Omega}_{\bar{i}} E_{\bar{j}},$$

we have

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} = \tilde{\Omega}_{\bar{j}} \delta^h_i + \tilde{\Omega}_{\bar{i}} \delta^h_j. \tag{3.2}$$

Form (3.2) we obtain that, there exist $\Phi = (\Phi_i) \in \Im^0_1(M)$, $D = (D^h) \in \Im^1_0(M)$ and $C = (C^h_i) \in \Im^1_1(M)$ which are satisfied

$$\tilde{\Omega}_{\bar{i}} = \Phi_i, \tag{3.3}$$

and

$$\tilde{V}^{\bar{h}} = D^h + y^a C^h_a + y^h y^a \Phi_a. \tag{3.4}$$

From

$$(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_i) = \tilde{\Omega}_{\bar{j}} E_i + \tilde{\Omega}_i E_{\bar{j}},$$

and (3.3) and (3.4) we have

$$\Big\{(\nabla_i C_j^h + V^a R_{aij}^h) + y^b\Big((\nabla_i \varPhi_j \delta_b^h + \nabla_i \varPhi_b \delta_j^h)\Big)\Big\} E_{\bar{h}} = \varPhi_j \delta_i^h E_h + \tilde{\Omega}_i \delta_j^h E_{\bar{h}}. \tag{3.5}$$

Comparing the both sides of the equation (3.5), we see that

$$\varPhi_i = 0, \tag{3.6}$$

$$\tilde{\Omega}_i = \Psi_i = \partial_i \psi, \tag{3.7}$$

$$\nabla_i C_j^h = V^a R_{iaj}^h + \Psi_i \delta_j^h, \tag{3.8}$$

where $\psi := \frac{1}{n} C_a^a$.
Lastly from

$$(L_{\tilde{V}} \tilde{\nabla})(E_j, E_i) = \tilde{\Omega}_i E_j + \tilde{\Omega}_j E_i,$$

and (3.6)-(3.8) we obtain

$$\begin{aligned}\Psi_i E_j + \Psi_j E_i = \{\nabla_j \nabla_i V^h + V^a R_{aji}^h\} E_h + \Big\{&\nabla_j \nabla_i D^h + D^a R_{aji}^h \\ &+ V^a \nabla_a M_{ji}^h + \nabla_i V^a M_{ja}^h + \nabla_j V^a M_{ia}^h - C_a^h M_{ij}^a \\ &+ y^b \big(V^a \nabla_a R_{jbi}^h + R_{abi}^h \nabla_j V^a + R_{jba}^h \nabla_i V^a + R_{jai}^h C_b^a \\ &- R_{jbi}^a C_a^h + \nabla_j \Psi_i \delta_b^h\big)\Big\} E_{\bar{h}}.\end{aligned} \tag{3.9}$$

From which we have

$$L_V \Gamma_{ji}^h = \nabla_j \nabla_i V^h + V^a R_{aji}^h = \Psi_i \delta_j^h + \Psi_j \delta_i^h, \tag{3.10}$$

(that is, $V := V^h \partial_h$ is an infinitesimal projective transformation on M_n),

$$\begin{aligned}L_D \Gamma_{ji}^h = \nabla_j \nabla_i D^h + D^a R_{aji}^h = C_a^h M_{ji}^a - V^a \nabla_a M_{ji}^h - M_{ja}^h \nabla_i V^a \\ - M_{ia}^h \nabla_j V^a,\end{aligned} \tag{3.11}$$

$$V^a \nabla_a R_{jbi}^h + R_{abi}^h \nabla_j V^a + R_{jba}^h \nabla_i V^a + R_{jai}^h C_b^a - R_{jbi}^a C_a^h = 0, \tag{3.12}$$

and

$$\nabla_j \Psi_i = 0. \tag{3.13}$$

This completes the necessary conditions. The proof of the sufficient conditions are easy. □

Theorem 3.2. *Let (M_n, g) be a complete n-dimensional Riemannian manifold and TM_n its tangent bundle with the Riemannian connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$ where f is a nonzero differentiable function on M_n. If $(TM_n, \tilde{G}_f)$ admits a non-affine infinitesimal fiber-preserving projective transformation then M_n is locally flat.*

Proof. Let $\tilde{V}$ be a non-affine infinitesimal fiber-preserving projective transformation on $(TM_n, \tilde{G}_f)$. It is easy to see that $\Psi := (\Psi_i)$ is a nonzero one form on M_n and $\|\Psi\|$ is a constant function.
We put $X := (\nabla_a V^h - C_a^h)\Psi^a$, where $\Psi^a := g^{ai}\Psi_i$. Using of (3.8), (3.10) and (3.13) one can see that

$$\begin{aligned} L_X g_{ji} &= \nabla_j X_i + \nabla_i X_j = (\nabla_j \nabla_a V_i - \nabla_j C_{ia})\Psi^a + (\nabla_i \nabla_a V_j - \nabla_i C_{ja})\Psi^a \\ &= 2(\Psi_a \Psi^a) g_{ji} = 2\|\Psi\| g_{ji}. \end{aligned} \tag{3.14}$$

This means that X is an infinitesimal non-isometric homothetic transformation on M_n. In [10] it is proved that if a complete Riemannian manifold (M_n, g) admits an infinitesimal non-isometric homothetic transformation then (M_n, g) is locally flat. Therefore M_n is locally flat. □

In [4], the Riemannian curvature of $(TM_n, \tilde{G}_f)$ is computed and the conditions are considered that under which $(TM_n, \tilde{G}_f)$ is locally flat(Theorem 4.2). In fact the following theorem is proved.

Theorem 3.3. *[4] Let (M_n, g) be a Riemannian manifold and TM_n be its tangent bundle equipped with the deformed complete lift metric $\tilde{G}_f$. Then TM_n is locally flat if and only if M_n is locally flat and the function f satisfies the condition*

$$\nabla_a(f_i\delta_j^h + f_j\delta_i^h - g_{ji}f^h_{\cdot}) - \nabla_i(f_a\delta_j^h + f_j\delta_a^h - g_{ja}f^h_{\cdot}) = 0. \tag{3.15}$$

where ∇ is the Levi-Civita connection of g .

From Theorems 3.2 and 3.3, the following thorem is proved.

Theorem 3.4. *Let (M_n, g) be a complete n-dimensional Riemannian manifold and TM_n its tangent bundle with the Riemannian connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$ where f is a nonzero differentiable function on M_n. Let $(TM_n, \tilde{G}_f)$ admits a non-affine infinitesimal fiber-preserving projective transformation then TM_n is locally flat if and only if the function f satisfies in the equation (3.15).*

As we said that, the complete lift vector fields are included in the class of fiber-preserving vector fields. Thus, as a special case, we consider the complete lift projective vector fields on $(TM_n, \tilde{G}_f)$.

Theorem 3.5. *Let (M_n, g) be a complete n-dimensional Riemannian manifold and TM_n its tangent bundle with the Riemannian connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$ where f is a nonzero differentiable function on M_n. Then every infinitesimal complete lift projective transformation on TM_n is an affine one and induced an infinitesimal affine trnsformation on (M_n, g).*

Proof. Let $V = V^h\partial_h$ be a vector field on M_n where V^C is a projective vector field on TM_n. From 5 and 6 of Theorem 3.1 we have that there exists a 1-form $\Psi := (\Psi_i)$ on M_n such that

$$L_V\Gamma_{ji}^h = \nabla_j\nabla_i V^h + V^a R_{aji}^h = \Psi_i\delta_j^h = \Psi_i\delta_j^h + \Psi_j\delta_i^h,$$

Therefore $\Psi_i = 0$ and thus V and V^C are affine vector fields. □

From the theorem 3.5, we have the following corollary.

Corollary 3.6. *Let* (M_n, g) *be a complete n-dimensional Riemannian manifold and* TM_n *its tangent bundle with the deformed complete lift metric* $\tilde{G}_f = g^C + (fg)^V$ *where* f *is a nonzero differentiable function on* M_n*. Then there is a one-to-one correspondence between complete lift projective vector fields on* $(TM_n, \tilde{G}_f)$ *and affine vector fields on* (M_n, g)*.*

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