Non-existence of non-trivial warped product lightlike submanifolds of semi-Riemannian product manifolds

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Abstract. In the present paper, we set out to examine warped product lightlike submanifolds of semi-Riemannian product manifolds. Significantly, considering the warped product GCR-lightlike submanifolds of the type $N_\perp \times_\lambda N_T$ and $N_T \times_\lambda N_\perp$, we obtain their non-existence in a semi-Riemannian product manifold $\bar{N}$, where N_T and $N_\perp$, respectively, denotes a holomorphic submanifold and a totally real submanifold of $\bar{N}$.

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1 Introduction

To construct a large variety of negatively curved manifolds, the notion of warped product manifolds was firstly brought up in 1969 by Bishop and O'Neill [2]. But the study of warped products attracted mathematicians and physicists in the beginning of 21st century, when CR-warped product submanifolds were introduced by Chen [3] in Kaehler manifolds and he proved that warped product CR-submanifolds of the type $N_\perp \times_\lambda N_T$ do not exist in Kaehler manifolds. Further, the author examined CR-warped product submanifolds of the type $N_T \times_\lambda N_\perp$ in a Kaehler manifold. Later, a lot of work came into sight on the existence (or non-existence) of warped product submanifolds in various ambient space settings (see [4]).

The warped product manifolds have several productive applications in differential geometry and mathematical physics, particularly, in theory of general relativity (for details see [1], [9] and [16] etc.). For instance, to investigate cosmological models and black holes, warped products are very useful. Many exact solutions like Robertson-Walker models and Schwarzschild solution of the Einstein field equations are warped product structures. In addition, the Schwarzschild solution is used to depict the outer space around the black holes or massive stars. It may be noted that in the cosmological models, there do exist some points, where the warping function becomes zero. Such points are known as singular points. Moreover, at singular points, the metric of

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the product manifold becomes degenerate. Therefore, in order to deal with degenerate metric, one possible solution is to utilize the tools of semi-Riemannian geometry (see [5] and [18]). In this manner, one can successfully employ geometry of warped product lightlike manifolds to study such models. Thus, considering the growing importance of lightlike geometry and extensive uses of warped products, Sahin [17] defined the notion of warped product lightlike submanifolds and proved several characterizations on warped product lightlike submanifolds of semi-Riemannian manifolds. In this continuation, Kumar investigated warped products of lightlike submanifolds in indefinite almost Hermitian manifolds (see [11]-[13]).
Moreover, due to significant geometric properties, semi-Riemannian product manifolds are very important. In recent years, studies have been conducted on lightlike submanifolds of semi-Riemannian product manifolds (see [8], [10] and [14]). But warped product lightlike submanifolds are still not examined in semi-Riemannian product manifolds. Therefore, in the present paper, we investigate warped product GCR-lightlike submanifolds of semi-Riemannian product manifolds. In sect. 2 and 3, we recall basic formulae and notations related to lightlike submanifolds, semi-Riemannian product manifolds and GCR-lightlike submanifolds of semi-Riemannian product manifolds. In sect. 4, besides other basic results, we prove that warped product GCR-lightlike submanifolds of the type $N_\perp \times_\lambda N_T$ and $N_T \times_\lambda N_\perp$ do not exist in semi-Riemannian product manifolds, where N_T is a holomorphic submanifold and $N_\perp$ is a totally real submanifold.

2 Preliminaries

Let (N_m, g) be an immersed submanifold in a semi-Riemannian manifold $(\bar{N}_{m+n}, \bar{g})$, where $\bar{g}$ denotes the metric with constant index q (provided, $1 \leq q \leq m+n-1$ *and* $m, n \geq 1$). If the metric $\bar{g}$ is degenerate on TN, then T_pN and $T_pN^\perp$ both become degenerate orthogonal subspaces and there exists a subspace $Rad(T_pN)$ such that $Rad(T_pN) = T_pN \cap T_pN^\perp$, which is called the radical distribution with rank r, $1 \leq r \leq m$. If $Rad(TN) : p \in N \to Rad(T_pN)$ is a smooth distribution on N with rank $r(>0)$, then N is called an r-lightlike submanifold of $\bar{N}$ (for details, see [6]). Further, let $S(TN)$ be a screen distribution in TN such that

$$TN = Rad(TN) \perp S(TN). \tag{2.1}$$

Similarly, let $S(TN^\perp)$ is a screen transversal vector bundle in $TN^\perp$ provided, $TN^\perp = S(TN^\perp) \perp Rad(TN)$.
On the other hand, let $tr(TN)$ and $ltr(TN)$ be vector bundles in $T\bar{N}\mid_N$ and $S(TN^\perp)^\perp$, respectively, such that

$$tr(TN) = ltr(TN) \perp S(TN^\perp) \tag{2.2}$$

and

$$T\bar{N}\mid_N = TN \oplus tr(TN) = S(TN) \perp (Rad(TN) \oplus ltr(TN)) \perp S(TN^\perp). \tag{2.3}$$

Further, the Gauss and Weingarten formulae are

$$\bar{\nabla}_P Q = \nabla_P Q + h(P, Q), \quad \bar{\nabla}_P U = -A_U P + \nabla^t_P U, \tag{2.4}$$

for any $U \in \Gamma(tr(TN))$ and $P, Q \in \Gamma(TN)$, where $\bar{\nabla}$ and ∇, respectively, denote the Levi-Civita connection on $\bar{N}$ and the torsion-free linear connection on N. Here, the second fundamental form h is a symmetric bilinear form on $\Gamma(TN)$ and A_U is a linear operator on N and is called the shape operator.
In particular, one has

$$\bar{\nabla}_P Q = \nabla_P Q + h^l(P,Q) + h^s(P,Q), \tag{2.5}$$

$$\bar{\nabla}_P N' = -A_{N'}P + \nabla^l_P N' + D^s(P,N'), \tag{2.6}$$

$$\bar{\nabla}_P W = -A_W P + \nabla^s_P W + D^l(P,W), \tag{2.7}$$

for $P, Q \in \Gamma(TN), W \in \Gamma(S(TN^{\perp}))$ and $N' \in \Gamma(ltr(TN))$.
Employing Eqs. (2.5) - (2.7), we obtain

$$\bar{g}(h^s(P,Q),W) + \bar{g}(Q, D^l(P,W)) = g(A_W P, Q), \tag{2.8}$$

$$\bar{g}(D^s(P,N'),W) = \bar{g}(A_W P, N'), \tag{2.9}$$

for any $P, Q \in \Gamma(TN)$, $N' \in \Gamma(ltr(TN))$ and $W \in \Gamma(S(TN^{\perp}))$.

3 Some basic results

3.1 Semi-Riemannian product manifolds

Let (N_1, g_1) and (N_2, g_2) be two n_1 and n_2-dimensional semi-Riemannian manifolds with constant indices $q_1 > 0$ and $q_2 > 0$, respectively. Let $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$ be the projections given by $\pi_1(u_1, u_2) = u_1$ and $\pi_2(u_1, u_2) = u_2$, for any $(u_1, u_2) \in N_1 \times N_2$. Let us denote the product manifold as $(\bar{N}, \bar{g}) = (N_1 \times N_2, \bar{g})$, where

$$\bar{g}(Y,Z) = g_1(\pi_{1*}Y, \pi_{1*}Z) + g_2(\pi_{2*}Y, \pi_{2*}Z),$$

for any $Y, Z \in \Gamma(T\bar{N})$, where $*$ denotes the tangential mapping. Moreover, one has

$$\pi^2_{1*} = \pi_{1*}, \quad \pi^2_{2*} = \pi_{2*}, \quad \pi_{1*}\pi_{2*} = \pi_{2*}\pi_{1*} = 0, \quad \pi_{1*} + \pi_{2*} = I,$$

where I represents the identity map of $T(N_1 \times N_2)$. It follows that $(\bar{N}, \bar{g})$ is an (n_1+n_2)-dimensional semi-Riemannian manifold with constant index (q_1+q_2). Next, if we take $F = \pi_{1*} - \pi_{2*}$ then it is easy to see that $F^2 = I$ and

$$\bar{g}(FY, Z) = \bar{g}(Y, FZ), \tag{3.1}$$

for $Y, Z \in \Gamma(T\bar{N})$, where F is called an almost product structure on $\bar{N}$. It is clear that if $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{N}$, then

$$(\bar{\nabla}_Y F)Z = 0, \tag{3.2}$$

for $Y, Z \in \Gamma(T\bar{N})$.

3.2 Generalized Cauchy-Riemann (GCR)-lightlike submanifolds

Definition 3.1. ([7]) A real lightlike submanifold $(N, g, S(TN))$ of a semi-Riemannian product manifold $(\bar{N}, \bar{g})$ is known as generalized Cauchy-Riemann (GCR)-lightlike submanifold, if

(I) There exist two sub-bundles D_1 and D_2 of $Rad(TN)$ satisfying

$$Rad(TN) = D_1 \oplus D_2, \quad F(D_1) = D_1, \quad F(D_2) \subset S(TN).$$

(II) There exist two sub-bundles D_0 and D' of $S(TN)$ satisfying

$$S(TN) = \{FD_2 \oplus D'\} \perp D_0, \quad F(D_0) = D_0, \quad F(D') = L_1 \perp L_2,$$

where L_1 and L_2 are vector subbundles of $ltr(TN)$ and $S(TN^\perp)$ respectively, and D_0 is a non-degenerate distribution on N.

Example 3.2. ([14]) Let $R_4^{12} = R_2^6 \times R_2^6$ be a semi-Riemannian product manifold along with the product structure $F(\partial x_i, \partial y_i) = (\partial y_i, \partial x_i)$, where (x^i, y^i) are the cartesian coordinates of R_4^{12}. Consider N be a submanifold of R_4^{12} with

$$x_1 = u_1, \quad x_2 = u_5, \quad x_3 = u_3, \quad x_4 = \sqrt{1 - u_4^2}, \quad x_5 = u_6, \quad x_6 = u_2,$$

$$y_1 = u_2, \quad y_2 = u_3, \quad y_3 = u_8, \quad y_4 = u_4, \quad y_5 = u_7, \quad y_6 = u_1.$$

Then TN is spanned by $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8$, such that

$$Z_1 = \partial x_1 + \partial y_6, \quad Z_2 = \partial y_1 + \partial x_6, \quad Z_3 = \partial x_3 + \partial y_2,$$

$$Z_4 = -y_4 \partial x_4 + x_4 \partial y_4, \quad Z_5 = \partial x_2, \quad Z_6 = \partial x_5, \quad Z_7 = \partial y_5, \quad Z_8 = \partial y_3.$$

Clearly, N is a 3-lightlike submanifold with $Rad(TN) = Span\{Z_1, Z_2, Z_3\}$ and $FZ_1 = Z_2$, therefore $D_1 = Span\{Z_1, Z_2\}$. Since $FZ_3 = \partial y_3 + \partial x_2 = Z_8 + Z_5 \in \Gamma(S(TN))$, therefore $D_2 = Span\{Z_3\}$. Also, $FZ_6 = Z_7$, therefore $D_0 = Span\{Z_6, Z_7\}$. Further, $ltr(TN)$ is spanned by

$$\{N_1 = \frac{1}{2}(-\partial x_1 + \partial y_6), N_2 = \frac{1}{2}(-\partial y_1 + \partial x_6), N_3 = \frac{1}{2}(-\partial x_3 + \partial y_2)\}.$$

Clearly, $Span\{N_1, N_2\}$ is invariant w.r.t. F and $FN_3 = -\frac{1}{2}Z_8 + \frac{1}{2}Z_5$. Hence $L_1 = Span\{N_3\}$. Moreover, we get $S(TN^\perp) = Span\{W = -y_4 \partial y_4 + x_4 \partial x_4\}$. Since $FZ_4 = W$, thus $L_2 = S(TN^\perp)$. Hence, $D' = Span\{FN_3, FW = Z_4\}$. Thus, N is a proper GCR-lightlike submanifold of a semi-Riemannian product manifold R_4^{12}.

Consider the projections S, Q_1 and Q_2 of TN on D, $F(L_1) = N_1$ and $F(L_2) = N_2$, respectively. Then for $Y \in \Gamma(TN)$, one has

$$Y = SY + Q_1 Y + Q_2 Y. \tag{3.3}$$

Applying F on both sides of Eq. (3.3), we get

$$FY = fY + \omega Q_1 Y + \omega Q_2 Y. \tag{3.4}$$

If we put $\omega Q_1 = \omega_1$ and $\omega Q_2 = \omega_2$, then Eq. (3.4) becomes

$$FY = fY + \omega_1 Y + \omega_2 Y, \tag{3.5}$$

where $fY \in \Gamma(D)$, $\omega_1 Y \in L_1 \subset \Gamma(ltr(TN))$ and $\omega_2 Y \in L_2 \subset \Gamma(S(TN^{\perp}))$ and we can rewrite Eq. (3.5) as

$$FY = fY + \omega Y, \tag{3.6}$$

where $fY \in \Gamma(TN)$ and $\omega Y \in \Gamma(tr(TN))$. Similarly,

$$FZ = BZ + CZ, \tag{3.7}$$

for $Z \in \Gamma(tr(TN))$, where $BZ \in \Gamma(TN)$ and $CZ \in \Gamma(tr(TN))$.

4 Warped product GCR-lightlike submanifolds of semi-Riemannian product manifolds

The geometry of warped product GCR-lightlike submanifolds was analyzed by Kumar [15], in indefinite nearly Kaehler manifolds. One of most perfect generalization of cartesian products are semi-Riemannian product manifolds and these manifolds have outstanding applications in a variety of fields in differential geometry and mathematical physics. Moreover, due to geometrical importance of semi-Riemannian product manifolds and warped product manifolds, it is obvious to investigate warped products of GCR-lightlike submanifolds in semi-Riemannian product manifolds. Therefore, we consider warped product GCR-lightlike submanifolds of the type $N_T \times_\lambda N_\perp$ and $N_\perp \times_\lambda N_T$ in semi-Riemannian product manifolds.
Firstly, we mention a fundamental result for later use.

Theorem 4.1. *([2]) Let $N = N_1 \times_\lambda N_2$ be a warped product manifold. Then, we have*

$$\nabla_P Q \in \Gamma(TN_1), \tag{4.1}$$

$$\nabla_P V = \nabla_V P = \left(\frac{P\lambda}{\lambda}\right) V, \tag{4.2}$$

$$\nabla_U V = -\frac{g(U,V)}{\lambda} \nabla\lambda. \tag{4.3}$$

for $P, Q \in \Gamma(TN_1)$ and $U, V \in \Gamma(TN_2)$.

Note: In the forthcoming part of the paper, we shall write **w. p.** for a warped product and $\bar{N}$ for a semi-Riemannian product manifold, unless otherwise stated.

Theorem 4.2. *Consider a GCR-lightlike submanifold N of $\bar{N}$. Then there exist no* **w. p.** *GCR-lightlike submanifold of the type $N_T \times_\lambda N_\perp$, where N_T and $N_\perp$, respectively, represent a holomorphic submanifold and a totally real submanifold of $\bar{N}$.*

Proof. Using Eq. (3.2), for $Y_1 \in \Gamma(D)$ and $Z_1 \in \Gamma(D')$, one has $\bar{\nabla}_{Z_1} FY_1 = F\bar{\nabla}_{Z_1} Y_1$, further using Eqs. (2.4), (3.6), (3.7) and (4.2) and simplyfying, we obtain $fY_1(ln\lambda)Z_1 + h(Z_1, fY_1) = Y_1(ln\lambda)\omega Z_1 + Bh(Z_1, Y_1) + Ch(Z_1, Y_1)$. Then, on comparing the tangential and normal components, we get

$$fY_1(ln\lambda)Z_1 = Bh(Z_1, Y_1) \tag{4.4}$$

and

$$h(Z_1, fY_1) = Y_1(ln\lambda)\omega Z_1 + Ch(Z_1, Y_1). \tag{4.5}$$

Similarly, for $Z_1 \in \Gamma(D')$ and $Y_1 \in \Gamma(D)$, considering $(\bar{\nabla}_{Y_1} F)Z_1 = 0$, we obtain

$$-A_{\omega Z_1} Y_1 + \nabla^t_{Y_1} \omega Z_1 = Y_1(ln\lambda)\omega Z_1 + Bh(Y_1, Z_1) + Ch(Y_1, Z_1). \tag{4.6}$$

On comparing tangential components, Eq. (4.6) becomes

$$A_{\omega Z_1} Y_1 = -Bh(Y_1, Z_1). \tag{4.7}$$

Then, taking the inner product of Eq. (4.7) w. r. t. $Z_1 \in \Gamma(D')$, we have

$$g(A_{\omega Z_1} Y_1, Z_1) = -g(Bh(Y_1, Z_1), Z_1).$$

Further using Eqs. (2.5) and (3.7), we derive

$$\begin{aligned} \bar{g}(h^s(Y_1, Z_1), \omega Z_1) &= -\bar{g}(Fh(Y_1, Z_1) - Ch(Y_1, Z_1), Z_1) \\ &= -\bar{g}(h(Y_1, Z_1), FZ_1), \end{aligned}$$

which further gives

$$2\bar{g}(h^s(Y_1, Z_1), \omega_2 Z_1) = 0,$$

this implies that

$$\bar{g}(h^s(Y_1, Z_1), \omega_2 Z_1) = 0. \tag{4.8}$$

On the other hand, replacing Y_1 by fY_1 in Eq. (4.5), we get

$$fY_1(ln\lambda)\omega Z_1 = h(Z_1, Y_1) - Ch(Z_1, fY_1). \tag{4.9}$$

In view of Eq. (4.8), considering the inner product of Eq. (4.9) w.r.t. FZ_1, for $Z_1 \in \Gamma(D')$, we get

$$fY_1(ln\lambda)||\omega_2 Z_1||^2 = 0.$$

Thus, using the non-degeneracy of $S(TN^{\perp})$, we arrive at $fY_1(ln\lambda) = 0$, which implies that the warping function λ becomes constant on N_T. Hence, the desired result is accomplished. □

Theorem 4.3. *Consider a GCR-lightlike submanifold N of $\bar{N}$. Then, there exist no* **w. p.** *GCR-lightlike submanifold of the type $N = N_{\perp} \times_{\lambda} N_T$ in $\bar{N}$ where $N_{\perp}$ and N_T, respectively, represent a totally real and a holomorphic submanifold of $\bar{N}$.*

Proof. From Eq. (3.2), for $Z_1 \in \Gamma(D')$ and $Y_1 \in \Gamma(TN)$, we obtain $\bar{\nabla}_{Y_1} FZ_1 = F\bar{\nabla}_{Y_1} Z_1$. Then using Eqs. (2.4), (3.6), (3.7) and (4.2), we have $-A_{\omega Z_1} Y_1 + \nabla^t_{Y_1} \omega Z_1 = Z_1(ln\lambda) fSY_1 + \omega \nabla_{QY_1} Z_1 + Bh(Z_1, Y_1) + Ch(Z_1, Y_1)$. Then equating the tangential components, we derive

$$-A_{\omega Z_1} Y_1 = Z_1(ln\lambda) fSY_1 + Bh(Z_1, Y_1). \tag{4.10}$$

Similarly, for $Z_1 \in \Gamma(D')$ and $Y_1 \in \Gamma(TN)$, from Eq. (3.2), we get $h(Z_1, fSY_1) - A_{\omega QY_1} Z_1 + \nabla^t_{Z_1} \omega QY_1 = \omega \nabla_{Z_1} QY_1 + Bh(Z_1, Y_1) + Ch(Z_1, Y_1)$. Then, comparing the tangential components, we get

$$A_{\omega QY_1} Z_1 = -Bh(Z_1, Y_1). \tag{4.11}$$

Since $N_\perp$ being a totally real and totally geodesic distribution in N, thus for $Z_1, Z_2 \in \Gamma(D')$, from Eq. (3.2), we get $\bar{\nabla}_{Z_2} FZ_1 = F\bar{\nabla}_{Z_2} Z_1$. Further, using Eqs. (2.4), (3.6) and (3.7), we obtain

$$-A_{\omega Z_1} Z_2 + \nabla^t_{Z_2} \omega Z_1 = F(\nabla_{Z_2} Z_1) + Bh(Z_1, Z_2) + Ch(Z_1, Z_2).$$

Then comparing the tangential components, we derive

$$A_{\omega Z_1} Z_2 = -Bh(Z_1, Z_2). \tag{4.12}$$

By interchanging the role of Z_1 and Z_2 in Eq. (4.12), we have

$$A_{\omega Z_2} Z_1 = -Bh(Z_2, Z_1), \tag{4.13}$$

which further gives

$$A_{\omega Z_1} Z_2 = A_{\omega Z_2} Z_1. \tag{4.14}$$

Now, for $Z_1, Z_2 \in \Gamma(D')$ and $Y_2 \in \Gamma(D_0)$, employing Eqs. (2.5) and (4.2), we derive

$$\begin{aligned} g(A_{\omega Z_1} Z_2, Y_2) &= -\bar{g}(\bar{\nabla}_{Z_2} \omega Z_1, Y_2) = \bar{g}(\omega Z_1, \bar{\nabla}_{Z_2} Y_2) \\ &= \bar{g}(\omega Z_1, h^s(Z_2, Y_2)) = \bar{g}(\omega Z_1, \bar{\nabla}_{Y_2} Z_2) \\ &= -\bar{g}(F\bar{\nabla}_{Y_2} Z_1, Z_2) = -\bar{g}(\bar{\nabla}_{Y_2} Z_1, FZ_2) \\ &= -\bar{g}([Y_2, Z_1] + \bar{\nabla}_{Z_1} Y_2, FZ_2) = -\bar{g}(\bar{\nabla}_{Z_1} Y_2, FZ_2) \\ &= \bar{g}(Y_2, \bar{\nabla}_{Z_1} FZ_2) = -g(Y_2, A_{\omega Z_2} Z_1), \end{aligned}$$

which implies

$$g(A_{\omega Z_1} Z_2 + A_{\omega Z_2} Z_1, Y_2) = 0. \tag{4.15}$$

Then using the non-degeneracy of D_0, we obtain

$$A_{\omega Z_1} Z_2 = -A_{\omega Z_2} Z_1, \tag{4.16}$$

for any $Z_1, Z_2 \in \Gamma(D')$. Adding Eqs. (4.14) and (4.16), we get $A_{\omega Z_1} Z_2 = 0$. Further from Eq. (4.12), we get $Bh(Z_1, Z_2) = 0$. Now for $Z_1 \in \Gamma(D')$, $Y_2 \in \Gamma(D_0)$ and $Y_1 \in \Gamma(D)$, we have

$$\begin{aligned} g(A_{\omega Z_1} Y_1, Y_2) &= -\bar{g}(\bar{\nabla}_{Y_1} \omega Z_1, Y_2) \\ &= \bar{g}(\omega Z_1, \bar{\nabla}_{Y_1} Y_2) \\ &= \bar{g}(\omega Z_1, h^s(Y_1, Y_2)). \end{aligned} \tag{4.17}$$

As N_T is a holomorphic submanifold in $\bar{N}$, therefore it follows that

$$h(FZ_1, Z_2) = h(Z_1, FZ_2) = Fh(Z_1, Z_2). \tag{4.18}$$

Then, using Eq. (4.18) in Eq. (4.17), we obtain

$$g(A_{\omega Z_1} Y_1, Y_2) = \bar{g}(Z_1, Fh^s(Y_1, Y_2)) = \bar{g}(Z_1, h^s(FY_1, Y_2)) = 0.$$

Now, using the non-degeneracy of D_0, we derive

$$A_{\omega Z_1} Y_1 = 0. \tag{4.19}$$

Thus, we have

$$A_{\omega Z_1} Z_2 = 0, \tag{4.20}$$

for $Z_1, Z_2 \in \Gamma(TN)$.
Taking the inner product of Eq. (4.10) w.r.t. fY_1 for $Y_1 \in \Gamma(D_0)$ and using Eq. (4.20), we get

$$Z_1(ln\lambda)||fY_1||^2 = -g(A_{\omega Z_1} Y_1, fY_1) - g(Bh(Z_1, Y_1), fY_1) = 0.$$

Then, the non-degeneracy of D_0 gives that $Z_1(ln\lambda) = 0$. Thus, we conclude that λ becomes constant on $N_\perp$, which completes the proof. □

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References

[1] J. K. Beem, P. E. Ehrlich, T. G. Powell, *Warped product manifolds in relativity, In selected studies: Physics Astrophysics, Mathematics, History of Science*, North-Holland Publishing Co.-Amsterdam, New York (1982), 41-56.

[2] R. L. Bishop, B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. 145 (1969), 1-49.

[3] B. Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math. 133 (2001), 177-195.

[4] B. Y. Chen, *Geometry of Warped Product Manifolds and Submanifolds*, World Scientific, 2017.

[5] K. L. Duggal, *Constant scalar curvature and warped product globally null manifolds*, J. Geom. Phys. 4 (2002), 327-340.

[6] K. L. Duggal, A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Mathematics and its Applications, vol. 364, Kluwer Academic Publishers, 1996.

[7] K. L. Duggal, B. Sahin, *Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds*, Acta Math. Hungar. 112 (2006), 107-130.

[8] S. M. K. Haider, M. Thakur, A. Maseih, *Screen Cauchy-Riemann lightlike submanifolds of semi-Riemannian product manifolds*, Int. Electron. J. Geom. 4, 2 (2011), 141-154.

[9] S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, 1973.

[10] E. Kilic, B. Sahin, S. Keles, *Screen Cauchy-Riemann lightlike submanifolds of semi-Riemannian product manifolds*, Int. Electron. J. Geom. 4, 2 (2011), 120-135.

[11] S. Kumar, *Geometry of warped product lightlike submanifolds of indefinite nearly Kaehler manifolds*, J. Geom. 109 (2018), Article ID 21, 1-18.

[12] S. Kumar, *Warped product semi-transversal lightlike submanifolds of indefinite nearly Kaehler manifolds*, Differ. Geom. Dyn. Syst. 20 (2018), 106-118.

[13] S. Kumar, *Geometry of semi-transversal lightlike warped product submanifolds of indefinite Kaehler manifolds*, Asian-European J. Math. 21 (2021), Article ID 2150001, 1-12.

[14] S. Kumar, R. Kumar, R. K. Nagaich, *GCR-lightlike submanifolds of a semi-Riemannian product manifold*, Bull. Korean Math. Soc. 51 (2014), 883-899.

[15] S. Kumar, *Some characterization theorems on GCR-lightlike warped product submanifolds of indefinite nearly Kaehler manifolds*, Int. J. Geom. Methods Mod. Phys. 17, 3 (2020), Article ID 2050039, 1-17.

[16] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983.

[17] B. Sahin, *Warped product lightlike submanifolds*, Sarajevo J. Math. 14 (2005), 251-260.

[18] O. C. Stoica, *Warped products of singular semi-Riemannian manifolds*, preprint (2011), math.DG , arXiv: 1105.3404.

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