Slant curves and biharmonic Frenet curves in 3-dimensional para-Sasakian manifolds

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Abstract. In this paper, we study slant Frenet curves, slant curves with null normal and null slant curves in a 3-dimensional para-Sasakian manifold. A non-geodesic curve γ in a para-Sasakian 3-manifold M^3 is a slant curve if and only if $\eta(N) = 0$. Next, we find that for a slant Frenet curve in a 3-dimensional para-Sasakian manifold, the ratio of κ and $\tau + 1$ is constant (cf.[11]). There does not exist non-geodesic slant curves with null normals in a 3-dimensional para-Sasakian manifold for $\eta(\gamma')^2 = a^2 \neq 1$. Moreover, we construct *para-Bianchi-Cartan-Vrănceanu model* with 3-dimensional para-Sasakian structure and find the necessary and sufficient conditions for proper biharmonic Frenet curve. In particular, we prove that the biharmonic Frenet curves in Hyperbolic Heisenberg group are slant curves and find the parametric equations of them.

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1 Introduction

The harmonic maps $\phi : (M^m, g) \to (N^n, h)$ between two pseudo-Riemannian manifolds as critical points of the energy $E(\phi) = \int_M |d\phi|^2 dv$. The *tension field* τ_ϕ is defined by

$$\tau_\phi = trace\nabla^\phi d\phi = \Sigma_{i=1}^m \varepsilon_i(\nabla^\phi_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)),$$

where ∇^ϕ and $\{e_i\}$ denote the induced connection by ϕ on the bundle ϕ^*TN^n. A smooth map ϕ is called a *harmonic map* if its tension field vanishes.

Next, the bienergy $E_2(\phi)$ of a map ϕ is defined by $E_2(\phi) = \int_M |\tau_\phi|^2 dv$, and say that ϕ is biharmonic if it is a critical point of the bienergy. Harmonic maps are clearly biharmonic. Non-harmonic biharmonic maps are called *proper* biharmonic maps. We define the *bitension field* $\tau_2(\phi)$ by

$$\tau_2(\phi) := \Sigma_{i=1}^m \varepsilon_i((\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i} e_i})\tau_\phi - R^N(\tau_\phi, d\phi(e_i))d\phi(e_i)),$$

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where R^N is the curvature tensor of N^n and defined by $R^N(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. (see [2, 10])

We now restrict our attention to isometric immersions $\gamma : I \to (M, g)$ from an interval I to a pseudo-Riemannian manifold. The image $C = \gamma(I)$ is the trace of a curve in M and γ is a parametrization of C by arc length. In this case the tension field becomes $\tau_\gamma = \varepsilon_1 \nabla_{\gamma'}\gamma'$ and the biharmonic equation reduces to

$$\tau_2(\gamma) = \varepsilon_1(\nabla^2_{\gamma'}\tau_\gamma - R(\tau_\gamma, \gamma')\gamma') = 0. \tag{1.1}$$

Note that $C = \gamma(I)$ is part of a geodesic of M if and only if γ is harmonic. Moreover, from the biharmonic equation if γ is harmonic, geodesics are a subclass of biharmonic curves.

A one-dimensional integral submanifold of D in 3-dimensional contact manifold is called a *Legendre curve*, especially to avoid confusion with an integral curve of the vector field ξ. As a generalization of Legendre curve, the notion of slant curves was introduced in [3] for a contact Riemannian 3-manifold, that is, a curve in a contact 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field. Thus, we studied biharmonic curves in 3-dimensional Sasakian space form and proved it is slant curve in [4]. Also, the author studied biharmonic curve and slant curves in Lorentzian Sasakian space forms in [7, 8, 9].

As with the contact Riemannian 3-manifold, a curve in a para-contact 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field, that is, $g(\gamma', \xi)$ is a constant. In particular, if $g(\gamma', \xi) = 0$ then γ is a Legendre curve.

J. Welyczko([11]) studied slant curves in 3-dimensional normal almost paracontact metric manifolds. He found properties of Frenet slant curves, null slant curves and slant curves with null normals.

In this paper, we consider a 3-dimensional para-Sasakian manifold. In section 3, we study slant Frenet curves, slant curves with null normals and null slant curves in a 3-dimensional para-Sasakian manifold. A non-geodesic curve γ in a para-Sasakian 3-manifold M^3 is a slant curve if and only if $\eta(N) = 0$. Moreover, we find that for a slant Frenet curve in a 3-dimensional para-Sasakian manifold, the ratio of κ and $\tau + 1$ is constant (cf.[11]). There does not exist non-geodesic slant curves with null normals in a 3-dimensional para-Sasakian manifold for $\eta(\gamma')^2 = a^2 \neq 1$.

In section 4, we construct *para-Bianchi-Cartan-Vrănceanu model* with 3-dimensional para-Sasakian structure and find the necessary and sufficient conditions for proper biharmonic Frenet curve. In section 5, we prove that the biharmonic Frenet curves in Hyperbolic Heisenberg group are slant curves and find the parametric equations of them.

2 Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold. M has an almost paracontact structure (φ, ξ, η) if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi, \ \eta(\xi) = 1. \tag{2.1}$$

Suppose M has an almost paracontact structure (φ, ξ, η). Then $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, the endomorphism φ has rank $2n$.

If a $(2n+1)$-dimensional smooth manifold M with almost paracontact structure (φ, ξ, η) admits a compatible pseudo-Riemannian metric such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

then we say M has an almost paracontact structure (η, ξ, φ, g). Setting $Y = \xi$ we have

$$\eta(X) = g(X, \xi). \tag{2.3}$$

Next, if the compatible pseudo-Riemannian metric g satisfies

$$d\eta(X, Y) = g(X, \varphi Y), \tag{2.4}$$

then η is a contact form on M, ξ the associated Reeb vector field, g an associated metric and $(M, \varphi, \xi, \eta, g)$ is called a *paracontact metric manifold.*

For a paracontact metric manifold M, one may define naturally an almost paracomplex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{\mathrm{d}}{\mathrm{d}t}) = (\varphi X + f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost paracomplex structure J is integrable, then the paracontact metric manifold M is said to be *normal* or *para-Sasakian*. It is known that a paracontact metric manifold M is normal if and only if M satisfies

$$[\varphi, \varphi] - 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

Proposition 2.1 ([1], [11])**.** *An almost paracontact metric manifold $(M^{2n+1}, \eta, \xi, \varphi, g)$ is para-Sasakian if and only if*

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X. \tag{2.5}$$

Proposition 2.2 ([1], [11])**.** *Let $(M^{2n+1}, \eta, \xi, \varphi, g)$ be a paracontact metric manifold. Then*

$$\nabla_X \xi = -\varphi X + \varphi h X, \qquad for \quad h = \frac{1}{2} L_\xi \varphi. \tag{2.6}$$

If ξ is a Killing vector field with respect to the pseudo-Riemannian metric g, then we have

$$\nabla_X \xi = -\varphi X. \tag{2.7}$$

Proposition 2.3. *Let $\{T, N, B\}$ are orthonormal frame field in a Lorentzian 3-manifold. Then*

$$T \wedge_L N = \varepsilon_3 B, \quad N \wedge_L B = \varepsilon_1 T, \quad B \wedge_L T = \varepsilon_2 N.$$

3 Slant curves

A one-dimensional integral submanifold of D in 3-dimensional contact manifold is called a *Legendre curve*, especially to avoid confusion with an integral curve of the vector field ξ. As a generalization of Legendre curve, the notion of slant curves was introduced in [3] for a contact Riemannian 3-manifold, that is, a curve in a contact 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field.

Like in contact Riemannian 3-manifolds, a curve in a para-contact 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field, that is, $g(\gamma', \xi)$ is constant. In particular, if $g(\gamma', \xi) = 0$ then γ is a Legendre curve.

3.1 Slant Frenet curves

Let $\gamma : I \to M^3$ be a unit speed curve in Lorentzian 3-manifolds M^3 such that γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$. The constant ε_1 is called the *causal character* of γ. A unit speed curve γ is said to be a spacelike or timelike if its causal character is 1 or -1, respectively. A unit speed curve γ is said to be a *Frenet curve* if $g(\gamma'', \gamma'') \neq 0$. A Frenet curve γ admits a orthonormal frame field $\{T = \gamma', N, B\}$ along γ. Then the *Frenet-Serret* equations are following ([6], [10]):

$$\left\{ \begin{array}{l} \nabla_{\gamma'} T = \varepsilon_2 \kappa N, \\ \nabla_{\gamma'} N = -\varepsilon_1 \kappa T \quad - \varepsilon_3 \tau B, \\ \nabla_{\gamma'} B = \qquad \varepsilon_2 \tau N, \end{array} \right. \tag{3.1}$$

where $\kappa = |\nabla_{\gamma'} \gamma'|$ is the *geodesic curvature* of γ and τ its *geodesic torsion*. The vector fields T, N and B are called tangent vector field, principal normal vector field, and binormal vector field of γ, respectively.

The constants ε_2 and ε_3 defined by $g(N, N) = \varepsilon_2$ and $g(B, B) = \varepsilon_3$, and called *second causal character* and *third causal character* of γ, respectively. Thus it satisfied $\varepsilon_1 \varepsilon_2 = -\varepsilon_3$.

A Frenet curve γ is a *geodesic* if and only if $\kappa = 0$. A Frenet curve γ with constant geodesic curvature and zero geodesic torsion is called a *pseudo-circle*. A *pseudo-helix* is a Frenet curve γ whose geodesic curvature and torsion are constant.

By differentiating $g(T, \xi) = a$ along γ in a para-Sasakian manifold, we get

$$a' = g(\varepsilon_2 \kappa N, \xi) + g(\gamma', -\varphi\gamma') = -\varepsilon_2 \kappa \eta(N).$$

This equation implies the following

Proposition 3.1. *A non-geodesic Frenet curve γ in a para-Sasakian 3-manifold M^3 is a slant curve if and only if $\eta(N) = 0$.*

Moreover, we have

Lemma 3.2 ([11])**.** *Let γ be a slant Frenet curve in 3-dimensional almost paracontact manifold M. Then we find an orthonormal frame field in M as follows:*

$$T = \gamma', \quad N = \frac{\varphi T}{\sqrt{|-\varepsilon_1 + a^2|}}, \quad B = \frac{\xi - \varepsilon_1 a T}{\sqrt{|-\varepsilon_1 + a^2|}},$$

also $\xi = \varepsilon_1 aT + \sqrt{|-\varepsilon_1 + a^2|}B$.

By using (2.2), (2.5), (2.7) and (3.1), we have the following

Theorem 3.3 (cf.[11])**.** *Let γ be a slant Frenet curve in a 3-dimensional para-Sasakian manifold M^3. Then we have the curvature and torsion as follows:*

$$\kappa = \sqrt{|-\varepsilon_1 + a^2|}|\delta|, \tag{3.2}$$

$$\tau = -1 - \varepsilon_1 a\delta, \tag{3.3}$$

where $\delta = \frac{1}{|-\varepsilon_1 + a^2|} g(\nabla_{\gamma'}\gamma', \varphi\gamma')$.

From this we find

Corollary 3.4. *Let γ be a slant Frenet curve in 3-dimensional para-Sasakian manifold M^3. Then the ratio of κ and $\tau + 1$ is constant.*

Moreover, we have

Corollary 3.5. *Let γ be a Legendre Frenet curve in 3-dimensional para-Sasakian manifold M^3. Then its torsion is $\tau = -1$.*

3.2 Slant curves with null normals

We further consider a curve with null normals; γ is called a curve with null normal if

$$g(\gamma', \gamma') = 1, \quad \nabla_{\gamma'}\gamma' \neq 0, \quad g(\nabla_{\gamma'}\gamma', \nabla_{\gamma'}\gamma') = 0.$$

Proposition 3.6 ([11])**.** *Let γ be a curve with null normal in a 3-dimensional para-Sasakian manifold. Then there exist the frame T, N, V and the curvature function κ along γ, which satisfy the following Cartan equations:*

$$\nabla_{\gamma'}T = N, \quad \nabla_{\gamma'}N = \kappa N, \quad \nabla_{\gamma'}V = -T - \kappa V.$$

Welyczko ([11]) studied slant curves in 3-dimensional normal almost paracontact metric manifolds in detail. In particular, we have for a 3-dimensional para-Sasakian manifold the following

Theorem 3.7. *There does not exist non-geodesic slant curves with null normals in a 3-dimensional para-Sasakian manifold for $\eta(\gamma')^2 = a^2 \neq 1$.*

3.3 Null slant curves

Now, we consider the case when γ is a null curve, i.e. it has a null tangent vector field $g(\gamma', \gamma') = 0$ and γ is not a geodesic, that is, $g(\nabla_{\gamma'}\gamma', \nabla_{\gamma'}\gamma') \neq 0$. We take a parametrization of γ such that $g(\nabla_{\gamma'}\gamma', \nabla_{\gamma'}\gamma') = 1$. Then it is in [5] proved that there exists only one Cartan frame $\{T, N, W\}$ and the function τ along γ whose Cartan equations are

$$\nabla_T T = N, \quad \nabla_T W = \tau N, \quad \nabla_T N = -\tau T - W,$$

where

$$T = \gamma', \quad N = \nabla_T T, \quad \tau = \frac{1}{2} g(\nabla_T N, \nabla_T N), \quad W = -\nabla_T N - \tau T. \tag{3.4}$$

Hence

$$g(T, W) = g(N, N) = 1, \quad g(T, T) = g(T, N) = g(W, W) = g(W, N) = 0.$$

For a null Legendre curve γ, we easily prove that γ is geodesic. Hence we suppose that γ is non-geodesic, then we have

Theorem 3.8 (cf.[11])**.** *Let γ be a non-geodesic null slant curve in a para-Sasakian 3-manifold. Then we have*

$$N = \pm\frac{1}{a}\varphi\gamma', \quad \tau = -\frac{1}{2a^2} \pm 1, \quad W = -\frac{1}{2a^2}\gamma' + \frac{1}{a}\xi, \tag{3.5}$$

where $a = \eta(\gamma')$ is non-zero constant.

Proof. Let $\varphi\gamma' = p\gamma' + qN + rW$ for some functions p, q, r. From $r = g(\varphi\gamma', \gamma') = 0$ we have $\varphi\gamma' = p\gamma' + qN$. Moreover, we get $a^2 = g(\varphi\gamma', \varphi\gamma') = q^2$ and $pa = g(\varphi\gamma', \xi) = 0$. Since $a \neq 0$ we have $q = \pm a$ and $p = 0$. Hence we have $N = \pm\frac{1}{a}\varphi\gamma'$. Next, using (2.5) differentiating the normal vector field N then

$$\begin{aligned} \nabla_{\gamma'} N &= \nabla_{\gamma'}(\pm\frac{1}{a}\varphi\gamma') \\ &= \pm\frac{1}{a}\{(\nabla_{\gamma'}\varphi)\gamma' + \varphi\nabla_{\gamma'}\gamma'\} \\ &= (\mp 1 + \frac{1}{a^2})\gamma' - \frac{1}{a}\xi. \end{aligned} \tag{3.6}$$

Thus we have

$$\tau = \frac{1}{2} g(\nabla_{\gamma'} N, \nabla_{\gamma'} N) = -\frac{1}{2a^2} \pm 1, \tag{3.7}$$

and

$$W = -\nabla_T N - \tau T = -\frac{1}{2a^2}\gamma' + \frac{1}{a}\xi. \tag{3.8}$$

□

From Proposition 3.1 and Theorem 3.8, for all the slant curves γ in a para-Sasakian 3-manifold we have the following

Corollary 3.9. *A non-geodesic curve γ in a para-Sasakian 3-manifold M^3 is a slant curve if and only if $\eta(N) = 0$.*

4 Biharmonic curves

Now, we construct *para-Bianchi-Cartan-Vrănceanu model* with 3-dimensional para-Sasakian structure.

Let c be a real number and set

$$\mathcal{D} = \left\{ (x, y, z) \in \mathbb{R}^3(x, y, z) \mid 1 + \frac{c}{2}(x^2 + y^2) > 0 \right\}.$$

Note that $\mathcal{D}$ is the whole $\mathbb{R}^3(x, y, z)$ for $c \geq 0$. On the region $\mathcal{D}$, we take the contact form

$$\eta = dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)}.$$

Then the characteristic vector field of η is $\xi = \frac{\partial}{\partial z}$.

Consider the following Lorentzian metric:

$$g_c = \frac{-dx^2 + dy^2}{\{1 + \frac{c}{2}(x^2 + y^2)\}^2} + \left(dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)} \right)^2.$$

We take the following orthonormal frame field on $(\mathcal{D}, g_c)$:

$$u_1 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ u_2 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ u_3 = \frac{\partial}{\partial z}.$$

Then the endomorphism field φ is defined by

$$\varphi u_1 = u_2,\ \varphi u_2 = u_1,\ \varphi u_3 = 0.$$

The Levi-Civita connection ∇ of this Lorentzian 3-manifold is described as

$$\nabla_{u_1} u_1 = -c\, y u_2, \quad \nabla_{u_1} u_2 = -c\, y u_1 + u_3, \quad \nabla_{u_1} u_3 = -u_2,$$

$$\nabla_{u_2} u_1 = -c\, x u_2 - u_3, \quad \nabla_{u_2} u_2 = -c\, x u_1, \quad \nabla_{u_2} u_3 = -u_1, \tag{4.1}$$

$$\nabla_{u_3} u_1 = -u_2, \quad \nabla_{u_3} u_2 = -u_1, \quad \nabla_{u_3} u_3 = 0.$$

$$[u_1, u_2] = -c\, y u_1 + c\, x u_2 + 2u_3, \quad [u_2, u_3] = [u_3, u_1] = 0.$$

The contact form η on $\mathcal{D}$ satisfies

$$d\eta(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(\mathcal{D}).$$

Moreover the structure $(\varphi, \xi, \eta, g_c)$ is para-Sasakian. The curvature tensor $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ on $(M^3, \eta, \xi, \varphi, g_c)$ is given by

$$R(u_1, u_2)u_2 = -\{3 + c^2(x^2 - y^2)\}u_1, \quad R(u_1, u_3)u_3 = u_1,$$

$$R(u_2, u_1)u_1 = \{3 + c^2(x^2 - y^2)\}u_2, \quad R(u_2, u_3)u_3 = u_2, \tag{4.2}$$

$$R(u_3, u_1)u_1 = -u_3, \quad R(u_3, u_2)u_2 = u_3.$$

The sectional curvature is given by

$$K(u_2,u_3) = -1 = -K(u_3,u_1),$$

and

$$K(u_1,u_2) = R(u_1,u_2,u_1,u_2) = -\{3+c^2(x^2-y^2)\}.$$

Hence $(\mathcal{D}, g_c)$ is of holomorphic sectional curvature $H = -\{3+c^2(x^2-y^2)\}$.

If $c = 0$ then it has constant holomorphic sectional curvature $H = -3$ and becomes para-Sasakian space forms. The tension field is $\tau_\gamma = \varepsilon_1 \nabla_{\gamma'}\gamma'$ and from the Frenet-Serret equation (3.1), $\nabla_{\gamma'}\gamma' = 0$ if and only if $\kappa = 0$, whence we have

Proposition 4.1. *Let $\gamma : I \to \mathcal{M}^3$ be a Frenet curve in 3-dimensional para-B-C-V space M^3. Then γ is harmonic if and only if γ is a geodesic.*

Next, using (3.1) we get

$$\nabla^3_T T = 3\varepsilon_3\kappa\kappa' T + \varepsilon_2(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2+\varepsilon_3\tau^2))N + \varepsilon_1(2\kappa'\tau+\kappa\tau')B.$$

Using the curvature tensor (4.2) we have

$$\begin{aligned}
&R(\kappa N, T)T\\
=&\kappa R(N_1e_1+N_2e_2+N_3e_3, T_1e_1+T_2e_2+T_3e_3)(T_1e_1+T_2e_2+T_3e_3)\\
=&-\varepsilon_2\kappa\big[\{\varepsilon_3-(4+c^2(x^2-y^2))B_3^2\}N+(4+c^2(x^2-y^2))N_3B_3B\big].
\end{aligned}$$

From the biharmonic equation (1.1) we have

$$\begin{aligned}
\tau_2(\gamma) &= \nabla^3{}_T T - \varepsilon_2 R(\kappa N, T)T\\
&= 3\varepsilon_3\kappa\kappa' T + \big[\varepsilon_2\{\kappa''-\varepsilon_2\kappa(\varepsilon_1\kappa^2+\varepsilon_3\tau^2)\} + \kappa\{\varepsilon_3-(4+c^2(x^2-y^2))B_3^2\}\big]N\\
&\quad + \big[\varepsilon_1(2\kappa'\tau+\kappa\tau') + \kappa(4+c^2(x^2-y^2))N_3B_3\big]B\\
&= 0
\end{aligned}$$

Hence we have

Theorem 4.2. *Let $\gamma : I \to \mathcal{M}^3$ be a Frenet curve parametrized by arc-length in the para-B-C-V space $\mathcal{M}^3$. Then γ is a proper biharmonic curve if and only if*

$$\begin{aligned}
\kappa &= \text{constant} \neq 0,\\
\varepsilon_1\kappa^2+\varepsilon_3\tau^2 &= \varepsilon_3 - \{4+c^2(x^2-y^2)\}\eta(B)^2,\\
\tau' &= -\varepsilon_1\{4+c^2(x^2-y^2)\}\eta(N)\eta(B).
\end{aligned} \tag{4.3}$$

5 The hyperbolic Heisenberg group

In this section, we construct the biharmonic slant Frenet curve in the Heisenberg group $(\mathbb{H}_3, g)$ with para-Sasakian structure, that is $c = 0$, especially. It has constant holomorphic sectional curvature $H = -3$ and becomes a para-Sasakian space form. From now on, we shall call the Heisenberg group $(\mathbb{H}_3, g)$ with para-Sasakian structure as *the hyperbolic Heisenberg group* (see [1]).

Corollary 5.1. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a Frenet curve parametrized by arc-length in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. Then γ is a proper biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \varepsilon_1\kappa^2 + \varepsilon_3\tau^2 &= \varepsilon_3 - 4\eta(B)^2, \\ \tau' &= -4\varepsilon_1\eta(N)\eta(B). \end{aligned} \tag{5.1}$$

Proposition 5.2. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a Frenet curve parametrized by arc-length in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. Then γ is a proper biharmonic curve if and only if γ is a pseudo-helix with*

$$\begin{aligned} \kappa &\neq 0, \\ \varepsilon_1\kappa^2 + \varepsilon_3\tau^2 &= \varepsilon_3 - 4\eta(B)^2, \\ \eta(N)\eta(B) &= 0. \end{aligned} \tag{5.2}$$

Proof. By using the Levi-Civita connection (4.1) and the Frenet-Serret equation (3.1), we have

$$\varepsilon_2\tau N_3 = -g(\nabla_T B, u_3) = B_3' + \varepsilon_2 N_3.$$

Therefore we have

$$\tau = \varepsilon_2\frac{B_3'}{N_3} + 1. \tag{5.3}$$

Now, we assume that γ is biharmonic and suppose that $\tau' = -4\varepsilon_1 N_3 B_3 \neq 0$. Then

$$\tau\tau' = (\varepsilon_2\frac{B_3'}{N_3} + 1)(-4\varepsilon_1 N_3 B_3) = 4\varepsilon_3 B_3 B_3' + \tau'.$$

Hence we have

$$(\tau - 1)^2 = 4\varepsilon_3 B_3^2 + b,$$

where b is a constant. Applying the second equation of (5.2) to the above equation, we see that τ is a constant, which yields a contradiction. $\square$

Therefore we have

Theorem 5.3. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a Frenet curve in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. Then γ is proper biharmonic if and only if γ is a pseudo-helix with*

$$\varepsilon_1\kappa^2 + \varepsilon_3\tau^2 = \varepsilon_3 - 4\eta(B)^2, \quad \eta(N) = 0, \quad \eta(B) = constant, \quad \kappa \neq 0. \tag{5.4}$$

Proof. We suppose that $B_3 = 0$. Then, from (5.3), we infer $\tau = 1$. Now, we assume that γ is biharmonic. Then, from the second equation of (5.2), we obtain that γ is a geodesic, which yields a contradiction. $\square$

From Proposition 3.1 and Theorem 5.3 we have

Corollary 5.4. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a Frenet curve in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. Then γ is proper biharmonic if and only if γ is a slant pseudo-helix with*

$$\varepsilon_1\kappa^2 + \varepsilon_3\tau^2 = \varepsilon_3 - 4\eta(B)^2, \quad \eta(B) = constant, \quad \kappa \neq 0. \tag{5.5}$$

We further consider a slant Frenet curve γ in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$, parametrized by arc-length.

5.1 Slant curves with spacelike normal vector field

First, we suppose that $\varepsilon_2 = 1$, that is $-\varepsilon_1 + a^2 > 0$. Then the tangent vector field has the form

$$T = \gamma' = \sqrt{-\varepsilon_1 + a^2}\cosh\beta u_1 + \sqrt{-\varepsilon_1 + a^2}\sinh\beta u_2 + au_3, \tag{5.6}$$

where $a = \textit{constant},\ \beta = \beta(s)$. Using (4.1), since γ is a non-geodesic, we may assume that $\kappa = \sqrt{-\varepsilon_1 + a^2}(\beta' - 2a) > 0$ without loss of generality. Then the normal vector field has the explicit form

$$N = \sinh\beta u_1 + \cosh\beta u_2, \tag{5.7}$$

and the binormal vector field is $\varepsilon_3 B = T \wedge_L N = a\cosh\beta u_1 + a\sinh\beta u_2 + \sqrt{-\varepsilon_1 + a^2}u_3$. By assumption, since $\varepsilon_2 = 1$, we have $\varepsilon_3 = -\varepsilon_1$. Hence

$$B = -\varepsilon_1(a\cosh\beta u_1 + a\sinh\beta u_2 + \sqrt{-\varepsilon_1 + a^2}u_3). \tag{5.8}$$

Using the Frenet-Serret equation (3.1), we have

Lemma 5.5. *Let γ be a slant Frenet curve in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$ with spacelike normal vector field parametrized by arc-length. Then γ has*

$$\begin{aligned} \kappa &= \sqrt{-\varepsilon_1 + a^2}|\beta' - 2a|, \\ \tau &= -1 - \varepsilon_1 a(\beta' - 2a). \end{aligned} \tag{5.9}$$

Moreover, the ratio of κ and $\tau + 1$ is a constant.

Remark 5.1. From the Proposition 3.1, we see that $\eta(N) = 0$ for a slant Frenet curve with spacelike normal vector field in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. By differentiating $\eta(N) = 0$ and by using (5.6), (5.7), (5.8) and $\varepsilon_2 = 1$, we infer

$$\begin{aligned} 0 &= g(\nabla_T N, \xi) + g(N, \nabla_T \xi) \\ &= g(-\varepsilon_1\kappa T - \varepsilon_3\tau B, \xi) + g(N, -\varphi T) \\ &= -\varepsilon_1\kappa a - \sqrt{-\varepsilon_1 + a^2}(\tau + 1). \end{aligned}$$

Therefore, for a non-geodesic slant Frenet curve with spacelike normal vector field we have

$$\frac{\tau + 1}{\kappa} = -\frac{\varepsilon_1 a}{\sqrt{-\varepsilon_1 + a^2}},$$

which is constant.

Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in $(\mathbb{H}_3, g)$. Then the tangent vector field T of γ is

$$T = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

Using these relations, we get:

$$\frac{\partial}{\partial x} = u_1 + yu_3,\ \frac{\partial}{\partial y} = u_2 - xu_3,\ \frac{\partial}{\partial z} = u_3,$$

If γ is a slant curve with spacelike normal vector field in $(\mathbb{H}_3, g)$, then from (5.6) the system of differential equations for γ is given by

$$\frac{dx}{ds}(s) = \sqrt{-\varepsilon_1 + a^2}\cosh\beta(s), \tag{5.10}$$

$$\frac{dy}{ds}(s) = \sqrt{-\varepsilon_1 + a^2}\sinh\beta(s), \tag{5.11}$$

$$\frac{dz}{ds}(s) = a + \sqrt{-\varepsilon_1 + a^2}(x(s)\sinh\beta(s) - y(s)\cosh\beta(s)). \tag{5.12}$$

Now, we construct a proper biharmonic Frenet curve γ. From (5.4) and (5.9) we have

Corollary 5.6. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a proper biharmonic Frenet curve parametrized by arc-length in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. Then*

$$\beta' = a \pm \sqrt{5a^2 - 4\varepsilon_1}. \tag{5.13}$$

Namely, β' is a constant, say A, hence $\beta(s) = As + b,\ b \in \mathbb{R}$. Thus, from (5.10) and (5.11) we have the following result:

Theorem 5.7. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a slant Frenet curve with spacelike normal vector field parametrized by arc-length s in hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. If it satisfies proper biharmonic equation, then the parametric equations are given by*

$$\begin{cases} x(s) = \frac{1}{A}\sqrt{-\varepsilon_1 + a^2}\sinh(As+b) + x_0, \\ y(s) = \frac{1}{A}\sqrt{-\varepsilon_1 + a^2}\cosh(As+b) + y_0, \\ z(s) = \{a + \frac{\varepsilon_1 - a^2}{A}\}s + \frac{\sqrt{-\varepsilon_1 + a^2}}{A}\{x_0\cosh(As+b) - y_0\sinh(As+b)\} + z_0. \end{cases}$$

5.2 Slant curves with timelike normal vector field

Further, for $\varepsilon_2 = -1$, that is $-\varepsilon_1 + a^2 < 0$, the tangent vector field has the form

$$T = \gamma' = \sqrt{\varepsilon_1 - a^2}\sinh\beta u_1 + \sqrt{\varepsilon_1 - a^2}\cosh\beta u_2 + au_3, \tag{5.14}$$

where $a = constant,\ \beta = \beta(s)$. Using (4.1), since γ is a non-geodesic, we may assume that $\kappa = \sqrt{\varepsilon_1 - a^2}(\beta' - 2a) > 0$ without loss of generality. Then the normal vector field is

$$N = \cosh\beta u_1 + \sinh\beta u_2. \tag{5.15}$$

The binormal vector field is $\varepsilon_3 B = T \wedge_L N = a\sinh\beta u_1 + a\cosh\beta u_2 - \sqrt{\varepsilon_1 - a^2}u_3$. By assumption, since $\varepsilon_2 = -1$, we have $\varepsilon_3 = \varepsilon_1$. Hence

$$B = \varepsilon_1(a\sinh\beta u_1 + a\cosh\beta u_2 - \sqrt{\varepsilon_1 - a^2}u_3). \tag{5.16}$$

Using the Frenet-Serret equation (3.1), we have

Lemma 5.8. *Let γ be a slant Frenet curve in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$ with timelike normal vector field parametrized by arc-length. Then γ has*

$$\begin{aligned} \kappa &= \sqrt{\varepsilon_1 - a^2}|\beta' - 2a|, \\ \tau &= -1 - \varepsilon_1 a(\beta' - 2a). \end{aligned} \tag{5.17}$$

Moreover, the ratio of κ and $\tau + 1$ is a constant.

Remark 5.2. From Proposition 3.1, we see that $\eta(N) = 0$ for a slant Frenet curve in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. By differentiating $\eta(N) = 0$ and by using (5.14), (5.15), (5.16) and $\varepsilon_2 = -1$, we have

$$\begin{aligned} 0 &= g(\nabla_T N, \xi) + g(N, \nabla_T \xi) \\ &= g(-\varepsilon_1 \kappa T - \varepsilon_3 \tau B, \xi) + g(N, -\varphi T) \\ &= -\varepsilon_1 \kappa a + \sqrt{\varepsilon_1 - a^2}(\tau + 1). \end{aligned}$$

Therefore for a non-geodesic slant Frenet curve with timelike normal vector field, we have

$$\frac{\tau + 1}{\kappa} = \frac{\varepsilon_1 a}{\sqrt{\varepsilon_1 - a^2}}$$

which proves to be constant.

Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in $(\mathbb{H}_3, g)$. Then the tangent vector field T of γ is

$$T = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

Using the relations:

$$\frac{\partial}{\partial x} = u_1 + y u_3, \ \frac{\partial}{\partial y} = u_2 - x u_3, \ \frac{\partial}{\partial z} = u_3,$$

If γ is a slant Frenet curve in $(\mathbb{H}_3, g)$, then from (5.14) the system of differential equations for γ is given by

$$\frac{dx}{ds}(s) = \sqrt{\varepsilon_1 - a^2}\sinh\beta(s), \tag{5.18}$$

$$\frac{dy}{ds}(s) = \sqrt{\varepsilon_1 - a^2}\cosh\beta(s), \tag{5.19}$$

$$\frac{dz}{ds}(s) = a + \sqrt{\varepsilon_1 - a^2}(x(s)\cosh\beta(s) - y(s)\sinh\beta(s)). \tag{5.20}$$

Now, we construct a proper biharmonic Frenet curve γ. From (5.4) and (5.17), we have

Corollary 5.9. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a proper biharmonic Frenet curve parametrized by arc-length in the hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. Then*

$$\beta' = a \pm \sqrt{5a^2 - 4\varepsilon_1}. \tag{5.21}$$

Namely, β' is a constant, say A, hence $\beta(s) = As + b, \ b \in \mathbb{R}$. Thus, from (5.18) and (5.19) we have the following result:

Theorem 5.10. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a slant Frenet curve with timelike normal vector field parametrized by arc-length s in hyperbolic Heisenberg group $(\mathbb{H}_3, g)$. If it satisfies proper biharmonic equation, then the parametric equations are given by*

$$\begin{cases} x(s) = \frac{1}{A}\sqrt{\varepsilon_1 - a^2}\cosh(As + b) + x_0, \\ y(s) = \frac{1}{A}\sqrt{\varepsilon_1 - a^2}\sinh(As + b) + y_0, \\ z(s) = \{a + \frac{\varepsilon_1 - a^2}{A}\}s + \frac{\sqrt{\varepsilon_1 - a^2}}{A}\{x_0 \sinh(As + b) - y_0 \cosh(As + b)\} + z_0. \end{cases}$$

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