Almost η-Ricci-Bourguignon solitons on submersions from Riemannian submersions

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Abstract. This research article attempts to explain the characteristics of Riemannian submersions in terms of almost η-Ricci-Bourguignon soliton, almost η-Ricci soliton, almost η-Einstein soliton, and almost η-Schouten soliton with the potential vector field. Also, we discuss the various conditions for which the target manifold of Riemannian submersion is η-Ricci-Bourguignon soliton, almost η-Ricci soliton, almost η-Einstein soliton, and almost η-Schouten soliton with the potential vector field and Killing vector field. Finally, we illustrate an example which verify some of our results.

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1 Introduction

Over the last two decades, the analysis of geometric flows are most significant geometrical tools to explain the geometric structures in Riemannian geometry. A certain section of solutions on which the metric evolves by dilations and diffeomorphisms plays an important part in the study of singularities of the flows as they appear as possible singularity models. They are often called soliton solutions.

Hamilton [14] first time introduced the concept of Ricci flow and Yamabe flow simultaneously in 1988. Ricci soliton and Yamabe soliton emerge as the limit of the solutions of the Ricci flow and Yamabe flow, respectively. In dimension $n = 2$ the Yamabe soliton is equivalent to Ricci soliton. However, in dimension $n > 2$, the Yamabe and Ricci solitons do not agree as the first preserves the conformal class of the metric but the Ricci soliton does not in general.

Ricci flow [14] and Yamabe flow [14] have been the focus of attraction of many geometers in over the years. Although Ricci solitons and Yamabe soliton are same in two dimensional study, they are essentially different in higher dimensions. An interpolation soliton between Ricci and Yamabe solitons is considered in [6] where the name Ricci-Bourguignon soliton corresponding to Ricci-Bourguignon flow but its

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depend on a single scalar [4, 5].

In 1981, the conception of Ricci-Bourguignon flow as a extension of Ricci flow [14] has been initiated by J. P. Bourguignon [4] based on some unprinted work of Lichnerowicz and a paper of Aubin [1]. Ricci-Bourguignon flows are intrinsic geometric flows on Riemannian manifolds, whose fixed points are solitons. Ricci-Bourguignon soliton, which generates self-similar solution to the Ricci-Bourguignon flow [4]

$$\frac{\partial g}{\partial t} = -2(Ric - \rho R g), \quad g(0) = g_0, \tag{1.1}$$

where Ric is the Ricci curvature of the Riemannian manifold, R is the scalar curvature with respect to the metric g and ρ is a non-zero real constant. It should be noticed that for special values of the constant ρ in equation (1.1) we obtain the following situations for the tensor $Ric - \rho Rg$ appearing in equation (1.1). The PDE (1.1) defines the evolution equation is of special interest, in particular [4, 26]

(i) $\rho = \frac{1}{2}$, the Einstein tensor $Ric - \frac{R}{2}g$, (for Einstein soliton) [24]

(ii) $\rho = \frac{1}{n}$, the traceless Ricci tensor $Ric - \frac{R}{n}g$,

(iii) $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $Ric - \frac{R}{2(n-1)}$, (for Schouten soliton)

(iv) $\rho = 0$, the Ricci tensor Ric (for Ricci soliton).

In dimension two, the first three tensors are zero, hence the flow is static and in higher dimension the value of ρ are strictly ordered as above in descending order.

Short time existence and uniqueness for the solution of this geometric flow has been proved in [4, 5]. In fact, for sufficiently small t the equation (1.1) has a unique solution for $\rho < \frac{1}{2(n-1)}$.

On the other hand, the quasi-Einstein metrics or Ricci solitons serve as a solution to Ricci flow equation [14]. This motivates a more general type of Ricci soliton by considering the Ricci-Bourguignon flow [27]. In fact, a Riemannian manifold M of dimension $n \geq 3$ is said to be a Ricci-Bourguignon soliton [1] if

$$\mathcal{L}_V g + 2Ric + 2(\Lambda - \rho R)g = 0, \tag{1.2}$$

where $\mathcal{L}_V$ denotes the Lie derivative operator along vector field V and Λ is an arbitrary real constant. Similar to Ricci solitons, a Ricci-Bourguignon soliton is called expanding if $\Lambda > 0$, steady if $\Lambda = 0$ and shrinking if $\Lambda < 0$.

According to Pigola et al. [21] if we replace the constant λ in (1.2) with a smooth function $\lambda \in C^\infty(M)$, called soliton function, then we say that (M, g) is an almost Ricci-Bourguignon soliton. It is worth to remark that they arise from the *Ricci-Bourguignon flow* [27] recently studied by G. Cantino and L. Mazzieri [6, 7].

Perturbing the equation that define (1.2) Ricci-Bourguignon soliton by multiple of a certain $(0, 2)$-tensor field $\eta \otimes \eta$, we obtain slightly more general notion, namely *almost η-Ricci-Bourguignon soliton* which is a generalization of almost η-Ricci soliton, almost η-Einstein soliton, and almost η-Schouten soliton (for more details see [3, 8, 9, 10, 11, 24, 26, 27, 28, 30, 31]. Therefore, from (1.2), we have

$$\mathcal{L}_V g + 2Ric + 2(\Lambda - \rho R)g + 2\omega\eta \otimes \eta = 0. \tag{1.3}$$

Here η is the 1-form and ω denotes some smooth function on M.

On the other end, the notion of Riemannian immersion has been intensively studied since the very beginning of Riemannian geometry. Indeed, initially the Riemannian manifolds to be studied were surfaces embedded in $\mathbb{R}^3$. In 1956, Nash [19] proved that a revolution for Riemannian manifold that every Riemannian manifold can be isometrically embedded in any small part of Euclidean space. As a consequences, the differential geometry of Riemannian immersions are well known.

On the contrary "dual" concepts of Riemannian submersions have been studied and its differential geometry was first exposed by O' Neill [20] in 1966 and Gray [13] in 1967.

We note that the Riemannian submersions have been studied widely not only in mathematics, but also in theoretical physics, because of their applications in the Yang-Mills theory, Kaluza Klein theory, super gravity, relativity and super-string theories (see [4], [5], [15], [16], [29]). Most of the studies related to Riemannian submersion can be found in the books ([12], [22]).

Recently, Meriç et al.[17, 18] introduced and studied Riemannian submersions whose total manifolds admits a Ricci soliton and an almost Yamabe soliton. In 2020, Siddiqi and Akyol [23] also have discussed η-Ricci-Yamabe solitons along Riemannian submersion. Recently, Siddiqi et al. [25] studied clairaut anti-invariant submersions from Lorentzian trans-Sasakian manifolds. In this paper, we will study Riemannian submersions whose total space admits an almost η-Ricci-Bourguignon soliton.

2 Riemannian submersions

In this section, we provide the necessary background for Riemannian submersions.

Let (M, g) and $(N, \check{g})$ be Riemannian manifolds, where $dim(M) > dim(N)$. A surjective map $\psi : (M, g) \to (N, g_N)$ is called a *Riemannian submersion* [20] if
(S1) $Rank(\psi) = dim(N)$.
In this case, for each $q \in N$, $\psi^{-1}(q) = \pi_q^{-1}$ is a k-dimensional submanifold of M and called a *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and ψ-related to a vector field X_* on N, i.e. , $\psi_*(X_p) = X_{*\psi(p)}$ for all $p \in M$, where ψ_* is derivative or differential map of ψ. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\psi_*$, and the horizontal distribution $ker\psi_*^{\perp}$, respectively. As usual, the manifold (M, g) is called *total manifold* and the manifold $(N, \check{g})$ is called *base manifold* of the submersion $\psi : (M, g) \to (N, \check{g})$.
(S2) ψ_* preserves the lengths of the horizontal vectors.
These conditions are equivalent to say that the derivative map ψ_* of ψ, restricted to $ker\psi_*^{\perp}$, is a linear isometry. If X and Y are the basic vector fields, ψ-related to $\check{X}, \check{Y}$, we have the following facts:

1. $g(X, Y) = \check{g}(\check{X}, \check{Y}) \circ \psi$,

2. $h[X, Y]$ is the basic vector field ψ-related to $[\check{X}, \check{Y}]$,

3. $h(\nabla_X Y)$ is the basic vector field ψ-related to $\check{\nabla}_{\check{X}} \check{Y}$.

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{2.1}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{2.2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection of g [20]. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We summarize the properties of tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let V, W be vertical and X, Y are horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \;\; \mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y].$$

On the other hand, from (2.1) and (2.2), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.3}$$

$$\nabla_V X = \mathcal{T}_V X + \mathcal{H}\nabla_V X, \tag{2.4}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.5}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.6}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. Moreover, if X is basic, then we have $\mathcal{H}\nabla_V X = \mathcal{A}_X V$. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution. For details on the Riemannian submersions, we refer to O'Neill's paper [20] and to the book [12].

Finally, we recall that the notion of second fundamental form of a map between Riemannian manifolds. Let (M, g) and $(N, \breve{g})$ be Riemannian manifolds and $f : (M, g) \to (N, \breve{g})$ is a smooth map. Then the second fundamental form of f is given by

$$(\nabla f_*)(E, F) = \nabla^f_E f_* F - f_*(\nabla_E F)$$

for $E, F \in \Gamma(TM)$, where ∇^f is the pull back connection and we denote for convenience by ∇ the Riemannian connection of the metrics g and $\breve{g}$. It is well-known that the second fundamental form is symmetric. Moreover, f is said to be *totally geodesic* (*harmonic* map) if $(\nabla f_*)(E, F) = 0$ ($trace(\nabla f_*) = 0$) for all $E, F \in \Gamma(TM)$ (see [2], page 73).

3 Curvature properties

In this section, we discuss some useful curvature properties of Riemannian submersions.

Proposition 3.1. *If the Riemannian curvature tensors of (M, g), $(N, \check{g})$ and any fiber of π are denoted by Riem, $\check{R}iem$ and $\hat{R}iem$, respectively. Then we have*

$$(i)\ \ Riem(E,F,G,H) = \hat{R}iem(E,F,G,H) - g(\mathcal{T}_E H, \mathcal{T}_F G) + g(\mathcal{T}_F H, \mathcal{T}_E G),$$

$$(ii)\ \ Riem(X,Y,Z,W) = \check{R}iem(\check{X},\check{Y},\check{Z},\check{W})\circ\psi + 2g(\mathcal{A}_X Y, \mathcal{A}_Z W) - g(\mathcal{A}_Y Z, \mathcal{A}_X W) + g(\mathcal{A}_X Z, \mathcal{A}_Y W).$$

for any $E, F, G, H \in \Gamma V(M)$ and $X, Y, Z, W \in \Gamma H(M)$.

Proposition 3.2. *Let Ric, $\check{R}ic$ and $\hat{R}ic$ denote the Ricci tensors of (M, g), $(N, \check{g})$ and any fiber of ψ, respectively. Then we have*

(3.1)

$$Ric(E,F) = \hat{R}ic(E,F) + g(N, \mathcal{T}_E F) - \sum_{i=1}^{n}\{g((\nabla_{X_i}\mathcal{T})(E,F), X_i) - g(\mathcal{A}_{X_i}E, \mathcal{A}_{X_i}F)\},$$

$$Ric(X,Y) = \check{R}ic(\check{X},\check{Y})\circ\psi - \frac{1}{2}\{g(\nabla_X N, Y) + g(\nabla_Y N, X)\} + 2\sum_{i=1}^{n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + \sum_{j=1}^{r} g(\mathcal{T}_{U_j}X, \mathcal{T}_{U_j}Y), \tag{3.2}$$

$$Ric(E,X) = -g(\nabla_E N, X) + \sum_{j=1}^{r} g((\nabla_{U_j}\mathcal{T})(E_j, E), X) - \sum_{i=1}^{n}\{g((\nabla_{X_i}\mathcal{A})(X_i, X), E) - 2g(\mathcal{A}_{X_i}X, \mathcal{T}_E X_i)\},$$

where $\{X_i\}$ and $\{E_i\}$ are the orthonormal basis of $\mathcal{H}$ (horizontal) and $\mathcal{V}$ (vertical), respectively, for any $E, F \in \Gamma V(M)$ and $X, Y \in \Gamma H(M)$ [13, 20].

On the other side, for any fiber of Riemannian submersion π, the mean curvature vector field H is given by $rH = N$ such that

$$N = \sum_{j=1}^{r} \mathcal{T}_{E_j}E_j. \tag{3.3}$$

Also, the dimension of any fiber of π is denoted by r and $\{E_1, E_2, ..., E_r\}$ represents an orthonormal basis on vertical distribution. We point that the horizontal vector field N vanishes if and only if any fiber of Riemannian submersion π is minimal.

Now, from (3.3) we find

$$g(\nabla_U N, X) = \sum_{j=1}^{r} g((\nabla_U \mathcal{T})(E_j, E_j), X)$$

for any $U \in \Gamma(TM)$ and $X \in \Gamma H(M)$.

Horizontal divergence of any vector field X on $\Gamma H(M)$ denoted by $div(X)$ and given by

$$div(X) = \sum_{i=1}^{n} g(\nabla_{X_i}X, X_i), \tag{3.4}$$

where $\{X_1, X_2, ..., X_n\}$ is an orthonormal basis of horizontal space $\Gamma H(M)$. Hence, after considering (3.4) we have

$$div(N) = \sum_{i=1}^{n}\sum_{j=1}^{r} g(\nabla_{X_i}\mathcal{T})(E_j, E_j), X_i).$$

We infer from equations (3.1) and (3.2) that the extrinsic vertical scalar curvature $\mathcal{R}_V$ and the extrinsic horizontal scalar $\mathcal{R}_H$ are given by

$$\mathcal{R}_V = \sum Ric(E_j, E_j) = -\sum\{g((\nabla_{X_i}\mathcal{T})E_j, E_j), X_i) - g(\mathcal{A}_{X_i}E_j, \mathcal{A}_{X_i}E_j)\} + \sum\{\hat{Ric}(E_j, E_j) + g(N, \mathcal{T}_{E_j}E_j)\} \quad (3.5)$$

$$\mathcal{R}_H = \sum Ric(X_i, X_i) = \sum \check{Ric}(\check{X}_i, \check{X}_i) \circ \psi + \sum g(\mathcal{T}_{E_j}X_i, \mathcal{T}_{E_j}X_i), +2\sum g(\mathcal{A}_{X_i}X_i, \mathcal{A}_{X_i}X_i) - \frac{1}{2}\{g(\nabla_{X_i}N, X_i) + g(\nabla_{X_i}N, X_i)\}, \quad (3.6)$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r$. In view of (3.5) and (3.6) we turn up

$$\mathcal{R}_V = \hat{\mathcal{R}} - ||\mathcal{A}||^2 + ||N||^2 - div(N), \quad (3.7)$$

$$\mathcal{R}_H = \hat{\mathcal{R}} \circ \psi + 2\|\mathcal{A}\|^2 + \|\mathcal{T}\|^2 - div(N), \quad (3.8)$$

where $\hat{\mathcal{R}}$ and $\mathcal{R}$ are the scalar curvatures of any fiber of ψ and N, respectively, such that

$$\|\mathcal{T}\|^2 = \sum_{i,j} g(\mathcal{T}_{E_j}X_i, \mathcal{T}_{E_j}X_i), \quad (3.9)$$

$$\|\mathcal{A}\|^2 = \sum_{i,j} g(\mathcal{A}_{X_i}E_j, \mathcal{A}_{X_i}E_j). \quad (3.10)$$

Finally from (3.7)-(3.10), the scalar curvature $\mathcal{R}$ on the base manifold M is given by

$$\mathcal{R} = \hat{\mathcal{R}} + (\mathcal{R}_N \circ \psi) + \|N\|^2 + \|\mathcal{T}\|^2 + \|\mathcal{A}\|^2 - 2div(N).$$

4 Almost η-Ricci-Bourguignon soliton along Riemannian submersions

This section deals with the study of almost η-Ricci-Bourguignon soliton on Riemannian submersion $\psi : (M, g) \longrightarrow (N, \check{g})$ from Riemannian manifold and discuss the nature of fiber of such submersion with target manifold $(N, \check{g})$:

As a consequences of equations (2.3)-(2.6) in case of Riemannian submersion, we obtain the following characteristics of $\mathcal{A}$ and $\mathcal{T}$.

Theorem 4.1. *Let $\psi : (M, g) \longrightarrow (N, \breve{g})$ be a Riemannian submersion between Riemannian manifolds. Then the vertical distribution $\mathcal{V}$ is parallel with respect to the connection ∇ if the horizontal parts $\mathcal{T}_F H$ and $\mathcal{A}_X F$ of (2.3) and (2.5) vanish identically. Similarly, the horizontal distribution $\mathcal{H}$ is parallel with respect to the connection ∇ if the vertical parts $\mathcal{T}_F X$ and $\mathcal{A}_X Y$ of (2.4) and (2.6) vanish identically for any $X, Y \in \Gamma H(M)$ and $F, H \in \Gamma V(M)$.*

Theorem 4.2. *Let $(M, g, V, \Lambda, \omega, \rho)$ be an almost η-Ricci-Bourguignon soliton with vertical potential field V and $\psi : (M, g) \longrightarrow (N, \breve{g})$ is a Riemannian submersion between Riemannian manifolds. If the vertical distribution $\mathcal{V}$ is parallel, then any fiber of Riemannian submersion ψ is an almost η-Ricci-Bourguignon soliton.*

Proof. Let (M, g) be an almost η-Ricci-Bourguignon soliton, then from (1.3) we have

$$\frac{1}{2}\mathcal{L}_V g + Ric + (\Lambda - \rho R)g + \omega\eta \otimes \eta = 0$$

for any $E, F \in \Gamma V(M)$. Using equation (3.1) in the above equation, we have

$$\frac{1}{2}\{g(\nabla_E V, F) + g(\nabla_F V, E)\} + \hat{Ric}(E, F) + g(N, \mathcal{T}_E F)$$

$$-\sum_{i=1}^{n}\{g((\nabla_{X_i}\mathcal{T})(E, F), X_i) - g(\mathcal{A}_{X_i}E, \mathcal{A}_{X_i}F)\} + (\Lambda - \rho\mathcal{R}_V)g(E, F) + \omega\eta(E)\eta(F) = 0,$$

where $\{X_i\}_{i=1}^n$ denotes an orthonormal basis of the horizontal distribution $\mathcal{H}$ and ∇ is the Levi-Civita connection on M. Then using Theorem 4.1, equations (2.2), (2.3) and (3.7) we find the following equation.

$$\begin{aligned}&\frac{1}{2}[\hat{g}(\hat{\nabla}_E V, F) + \hat{g}(\hat{\nabla}_F V, E)] + \alpha\hat{Ric}(E, F)\\ &+(\Lambda - \rho\left\{\hat{\mathcal{R}} - \|\mathcal{A}\|^2 + \|N\|^2 - div(N)\right\})\hat{g}(E, F) + \omega\eta(E)\eta(F) = 0.\end{aligned} \tag{4.1}$$

If we denote $\lambda = \Lambda - \rho(\|\mathcal{A}\|^2 + \|N\|^2 - div(N))$, then (4.1) follows

$$\frac{1}{2}[\hat{g}(\hat{\nabla}_E V, F) + \hat{g}(\hat{\nabla}_F V, E)] + \hat{Ric}(E, F) + (\lambda - \rho\hat{\mathcal{R}})\hat{g}(E, F) + \omega\eta(E)\eta(F) = 0 \tag{4.2}$$

for any $E, F \in \Gamma V(M)$, which means that a fiber of ψ is an almost η-Ricci-Bourguignon soliton. □

For particular values of ρ, easily, we can also obtain the similar results for other solitons.

Case I. If $\rho = 0$ and $\omega \neq 0$, then from equation (4.2) we turns up

$$\frac{1}{2}[\hat{g}(\hat{\nabla}_E V, F) + \hat{g}(\hat{\nabla}_F V, E)] + \hat{Ric}(E, F) + \lambda\hat{g}(E, F) + \omega\eta(E)\eta(F) = 0.$$

This entails the following result:

Corollary 4.3. *Let $(M, g, V, \lambda, \omega, \rho = 1)$ be an almost η-Ricci soliton with vertical potential field V and $\psi : (M, g) \longrightarrow (N, \breve{g})$ is a Riemannian submersion between Riemannian manifolds. If the vertical distribution $\mathcal{V}$ is parallel, then any fiber of Riemannian submersion ψ is an almost η-Ricci soliton.*

Case II. If $\rho = \frac{1}{2}$ and $\omega \neq 0$, then from equation (4.2) we get

$$\frac{1}{2}[\hat{g}(\hat{\nabla}_E V, F) + \hat{g}(\hat{\nabla}_F V, E)] + \hat{Ric}(E, F) + (\lambda - \frac{\hat{\mathcal{R}}}{2})\hat{g}(E, F) + \omega\eta(E)\eta(F) = 0.$$

Thus we have the following result:

Corollary 4.4. *Let $(M, g, V, \lambda, \omega, \rho = \frac{1}{2})$ be an almost η-Einstein soliton with vertical potential field V and $\psi : (M, g) \longrightarrow (N, \breve{g})$ is a Riemannian submersion between Riemannian manifolds. If the vertical distribution $\mathcal{V}$ is parallel, then any fiber of Riemannian submersion ψ is an almost η-Einstein soliton.*

Case III. If $\rho = \frac{1}{2n-1}$ and $\omega \neq 0$, then from equation (4.2) we find

$$\frac{1}{2}[\hat{g}(\hat{\nabla}_E V, F) + \hat{g}(\hat{\nabla}_F V, E)] + \hat{Ric}(E, F) + (\lambda - \frac{\hat{\mathcal{R}}}{(2n-1)})\hat{g}(E, F) + \omega\eta(E)\eta(F) = 0.$$

Thus we have the following:

Corollary 4.5. *Let $(M, g, V, \lambda, \omega, \rho = \frac{1}{2n-1})$ be an almost η-Schouten soliton with vertical potential field V and $\psi : (M, g) \longrightarrow (N, \breve{g})$ is a Riemannian submersion between Riemannian manifolds. If the vertical distribution $\mathcal{V}$ is parallel, then any fiber of Riemannian submersion ψ is an almost η-Schouten soliton.*

Since the total space (M, g) of Riemannian submersion $\psi : (M, g) \longrightarrow (N, g_N)$ admits an almost η-Ricci-Bourguignon soliton, therefore from equations (3.1) and (4.2) we find

$$\frac{1}{2}\{g(\nabla_E V, F) + g(\nabla_F V, E)\} + \hat{Ric}(E, F) + \sum_{j=1}^{r} g(\mathcal{T}_{E_j} E_j, \mathcal{T}_E F) \tag{4.3}$$

$$-\sum_{i=1}^{n}\{g((\nabla_{X_i}\mathcal{T})(E, F), X_i) - g(\mathcal{A}_{X_i}E, \mathcal{A}_{X_i}F)\} + (\Lambda - \rho\mathcal{R}_V)\hat{g}(E, F) + \omega\eta(E)\eta(F) = 0$$

for any $E, F \in \Gamma V(M)$. Also, the almost η-Ricci-Bourguignon soliton has totally umbilical fibers and using equation (2.3) in equation (4.3), we find

$$\frac{1}{2}\left\{g(\hat{\nabla}_E V, H) + g(\hat{\nabla}_H V, E)\right\} + \hat{Ric}(E, H) + \sum_{j=1}^{r} g(\mathcal{T}_{E_j} E_j, \mathcal{T}_E H)$$

$$-\sum_{i=1}^{n}\{(\nabla_{X_i} g)(E, H)g(W, X_i) - g(\nabla_{X_i} W, X_i)\hat{g}(E, H)\} - \sum_{i=1}^{n} g(\mathcal{A}_{X_i}E, \mathcal{A}_{X_i}H)$$
$$+\left(\Lambda - \rho\left\{\hat{\mathcal{R}} - \|\mathcal{A}\|^2 + \|N\|^2 - div(N)\right\}\right)\hat{g}(E, H) + \omega\eta(E)\eta(H) = 0.$$

Since the horizontal distribution $\mathcal{H}$ is integrable, we have

$$\frac{1}{2}(\mathcal{L}_V \hat{g})(E, H) + \hat{Ric}(E, H) - \sum_{i=1}^{n} g(\nabla_{X_i} W, X_i)\hat{g}(E, H) + r\|W\|^2\hat{g}(E, H)$$

$$+\left(\Lambda-\rho\left\{\hat{\mathcal{R}}-\|\mathcal{A}\|^2+\|N\|^2-div(N)\right\}\right)\hat{g}(E,H)+\omega\eta(E)\eta(H)=0,$$

where W is the mean curvature vector of any fiber of ψ. From (3.4), we find

$$\frac{1}{2}(\mathcal{L}_V\hat{g})(E,H)+\hat{Ric}(E,H)+[r\,\|W\|^2-div(W)+\lambda-\rho\hat{R}]\hat{g}(E,H)+\omega\eta(E)\eta(H)=0,$$

where $\lambda=\rho(\|\mathcal{A}\|^2-\|N\|^2+div(N))$, which shows that any fiber of ψ is an almost η-Ricci-Bourguignon soliton. Thus, we can state the following result:

Theorem 4.6. *Let $(M,g,\Lambda,\omega,\rho)$ be an almost η-Ricci-Bourguignon soliton with the vertical potential field V and $\psi:(M,g)\longrightarrow(N,\breve{g})$ is a Riemannian submersion between Riemannian manifolds with totally umbilical fibers. If the horizontal distribution $\mathcal{H}$ is integrable, then any fiber of the Riemannian submersion ψ is an almost η-Ricci-Bourguignon soliton.*

Note that, once again for specific values of ρ ($\rho=0,\frac{1}{2},\frac{1}{2n-1}$), one can also, obtain the similar kind of result for an almost η-Ricci soliton, almost η-Einstein soliton, and almost η-Schouten soliton, respectively.

Again, assuming the Theorem 4.6, we obtain the following:

Corollary 4.7. *Let $(M,g,\Lambda,\omega,\rho)$ be an almost η-Ricci-Bourguignon soliton and ψ be a Riemannian submersion between Riemannian manifolds, such that the horizontal distribution $\mathcal{H}$ is integrable. Then any fiber of Riemannian submersion ψ is an almost η-Ricci-Bourguignon soliton, provided any fiber of ψ is totally umbilical and has a constant mean curvature.*

Corollary 4.8. *Let $(M,g,\Lambda,\omega,\rho)$ be an almost η-Ricci-Bourguignon soliton and ψ is a Riemannian submersion between Riemannian manifolds, such that the horizontal distribution $\mathcal{H}$ is integrable. Then any fiber of Riemannian submersion ψ is an almost η-Ricci-Bourguignon soliton, if any fiber of π is totally geodesic.*

Now, we have the following:

Theorem 4.9. *Let $(M,g,\Lambda,\omega,\rho)$ be an almost η-Ricci-Bourguignon soliton with the potential field $U\in\Gamma(TM)$ and ψ is a Riemannian submersion between Riemannian manifolds. If the horizontal distribution $\mathcal{H}$ is parallel, then we have*

1. *If the vector field U is vertical, then $(N,\breve{g})$ is an η-Einstein manifold.*
2. *If the vector field U is horizontal, then $(N,\breve{g})$ is an almost η-Ricci-Bourguignon soliton with potential vector field U_N such that $\psi_*U=\check{U}$.*

Proof. Since the total space (M,g) of Riemannian submersion ψ admits an almost η-Ricci-Bourguignon soliton with potential field $U\in\Gamma(TM)$, then using (3.1) and (1.3) we have

$$\frac{1}{2}[g(\nabla_XU,Y)+g(\nabla_YU,X)]+\check{Ric}(\check{X},\check{Y})\circ\psi-(g(\nabla_XN,Y)+g(\nabla_YN,X)) \tag{4.4}$$

$$+2\sum_{i=1}^{n}g(\mathcal{A}_XX_i,\mathcal{A}_YX_i)+\sum_{j=1}^{r}g(\mathcal{T}_{E_j}X,\mathcal{T}_{E_j}Y)+(2\Lambda-\rho R)g(X,Y)+\omega\eta(X)\eta(Y)=0,$$

where $\check{X}$ and $\check{Y}$ are ψ-related to X and Y respectively, for any $X, Y \in \Gamma H(M)$. Applying Theorem 4.1 to the above equation (4.4), we have

$$\frac{1}{2}[g(\nabla_X U, Y)+g(\nabla_Y U, X)]+\check{R}ic(\check{X}, \check{Y})\circ\psi+(\Lambda-\rho R)g(X,Y)+\omega\eta(X)\eta(Y)=0. \tag{4.5}$$

1. If vector field U is vertical, then from the equation (2.5) it follows

$$\frac{1}{2}[g(\mathcal{A}_X U, Y) + g(\mathcal{A}_Y U, X)]\check{R}ic(\check{X}, \check{Y}) \circ \psi + (\Lambda - \rho\mathcal{R}_V)\check{g}(X,Y) + \omega\eta(X)\eta(Y) = 0.$$

Since $\mathcal{H}$ is parallel, we get

$$\check{R}ic(\check{X}, \check{Y}) \circ \psi = \alpha g(X,Y) + \beta\eta(X)\eta(Y) = 0,$$

which shows that $(N, \check{g})$ is η-Einstein, where $\alpha = -(\Lambda - \rho\frac{\mathcal{R}_\mathcal{V}}{2})$ and $b = -\omega$.

2. If the vector field U is horizontal, then equation (4.5) becomes

$$\frac{1}{2}(\mathcal{L}_U g)(X,Y) + \check{R}ic(\check{X}, \check{Y}) \circ \psi + (\Lambda - \rho\frac{\mathcal{R}_\mathcal{H}}{2})g(X,Y) + \omega\eta(X)\eta(Y) = 0, \tag{4.6}$$

which shows that the Riemannian manifold $(N, \check{g})$ is an almost η-Ricci-Bourguignon soliton with horizontal potential field U. This completes the proof. □

Again using Theorem 4.1 and equation (3.2) together, we obtain the following one:

Lemma 4.10. *Let $(M, g, \xi, \Lambda, \omega, \rho)$ be an almost η-Ricci-Bourguignon soliton on Riemannian submersion ψ between Riemannian manifolds with horizontal potential field ξ such that $\mathcal{H}$ is parallel. Then the vector field N is Killing on the horizontal distribution $\mathcal{H}$.*

Since $(M, g, \xi, \Lambda, \rho)$ is an almost η-Ricci-Bourguignon soliton, therefore equations (3.2) and (1.3) give

$$\frac{1}{2}(\mathcal{L}_\xi g)(X,Y) + \check{R}ic(\check{X}, \check{Y}) \circ \psi - \{g(\nabla_X N, Y) + g(\nabla_Y N, X)\}$$

$$+2\sum_i^n g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + \sum_j^r g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y) + (\Lambda - \rho R)g(X,Y) + \mu\eta(X)\eta(Y) = 0,$$

where $\{X_i\}_{i=1}^n$ denotes an orthonormal basis of $\mathcal{H}$ for any $X, Y \in \Gamma H(M)$. In view of Theorem 4.1, the above equation reduces to

$$(\mathcal{L}_\xi g)(X,Y) + \check{R}ic(\check{X}, \check{Y}) \circ \psi + (\Lambda - \rho R)g(X,Y) + \omega\eta(X)\eta(Y) = 0.$$

Since the Riemannian manifold $(N, \check{g})$ is η-Einstein, therefore we can find that ξ is conformal Killing. Thus we can state the following result:

Theorem 4.11. *Let $(M, g, \xi, \Lambda, \omega, \rho)$ be an almost η-Ricci-Bourguignon soliton on Riemannian submersion ψ from Riemannian manifold to an η-Einstein manifold with horizontal potential field ξ such that horizontal distribution $\mathcal{H}$ is parallel. Then the vector field ξ is conformal Killing on the horizontal distribution $\mathcal{H}$.*

5 Examples

Example 5.1. Let $M^6 = \{(x_1, x_2, x_3, x_4, x_5, x_6)|x_6 \neq 0\}$ be a six-dimensional differentiable manifold, where $\{x_i,\ i = 1, 2, 3, 4, 5, 6\}$ denotes the standard coordinates of a point in $\mathbb{R}^6$. Suppose

$$E_1 = \partial x_1, \quad E_2 = \partial x_2, \quad E_3 = \partial x_3, \quad E_4 = \partial x_4, \quad E_5 = \partial x_5, \quad E_6 = \partial x_6$$

are linearly independent vector fields at each point of the manifold M^6, and therefore they form a basis for the tangent space $T(M^6)$. We define a positive definite metric g on M^6 as

$$g_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases},$$

where $i, j = 1, 2, 3, 4, 5, 6$ and it is given by $g = \sum_{i.j=1}^{6} dx_i \otimes dx_j$. Let the 1-form η be defined by $\eta(X) = g(X, P)$, where $P = E_6$. Then it is obvious that (M^6, g) is a Riemannian manifold of dimension 6. Moreover, let $\bar{\nabla}$ be the Levi-Civita connection with respect to metric g. Then we have $[E_1, E_2] = 0$. Similarly, $[E_1, E_6] = E_1$, $[E_2, E_6] = E_2$, $[E_3, E_6] = E_3$, $[E_4, E_6] = E_4$, $[E_5, E_6] = E_6$ and $[E_i, E_j] = 0$, where $1 \leq i, j \leq 5$, $i \neq j$. The Riemannian connection $\hat{\nabla}$ of the metric $\hat{g}$ is given by

$$2g(\hat{\nabla}_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

where ∇ denotes the Levi-Civita connection corresponding to the metric g.
By using Koszul's formula and (2.3) together, we obtain the following equations

$$(5.1) \qquad \hat{\nabla}_{E_1} E_1 = E_6,\ \hat{\nabla}_{E_2} E_2 = E_6,\ \hat{\nabla}_{E_3} E_3 = E_6,\ \hat{\nabla}_{E_4} E_4 = E_6, \hat{\nabla}_{E_5} E_5 = E_6,$$
$$\hat{\nabla}_{E_6} E_6 = 0,\ \hat{\nabla}_{E_6} E_i = 0,\ \hat{\nabla}_{E_i} E_6 = E_i,\ 1 \leq i \leq 5$$

and $\hat{\nabla}_{E_i} E_i = 0$ for all $1 \leq i,\ j \leq 5$. Now, from equation (5.1) and Proposition 3.1 we can notice the non-vanishing components of Riemannian curvature tensor $\hat{Riem}$, Ricci tensor $\hat{Ric}$ and scalar curvature $\hat{R}$ of fiber are given by

$$\hat{Riem}(E_1, E_2)E_1 = E_2,\ \hat{Riem}(E_1, E_2)E_2 = -E_1,\ \hat{Riem}(E_1, E_3)E_1 = -E_3,$$
$$\hat{Riem}(E_1, E_3)E_3 = E_1,\ \hat{Riem}(E_1, E_4)E_1 = -E_4,\ \hat{Riem}(E_1, E_4)E_4 = E_1,$$
$$\hat{Riem}(E_1, E_5)E_1 = -E_5,\ \hat{Riem}(E_1, E_5)E_5 = E_1,\ \hat{Riem}(E_1, E_6)E_1 = -E_6,$$
$$\hat{Riem}(E_1, E_6)E_6 = -E_1,\ \hat{Riem}(E_2, E_3)E_2 = -E_3,\ \hat{Riem}(E_2, E_3)E_3 = E_2,$$
$$\hat{Riem}(E_2, E_4)E_2 = E_4,\ \hat{Riem}(E_2, E_4)E_4 = -E_2,\ \hat{Riem}(E_2, E_5)E_2 = E_5,$$
$$\hat{Riem}(E_2, E_5)E_5 = -E_2,\ \hat{Riem}(E_2, E_6)E_2 = E_6,\ \hat{Riem}(E_2, E_6)E_6 = -E_2,$$
$$\hat{Riem}(E_3, E_4)E_3 = E_4,\ \hat{Riem}(E_3, E_4)E_4 = E_5,\ \hat{Riem}(E_3, E_5)E_5 = -E_3,$$
$$\hat{Riem}(E_3, E_6)E_3 = -E_6,\ \hat{Riem}(E_3, E_6)E_3 = -E_6,\ \hat{Riem}(E_3, E_6)E_6 = -E_3,$$
$$\hat{Riem}(E_4, E_5)E_4 = E_5,\ \hat{Riem}(E_4, E_5)E_5 = -E_4,\ \hat{Riem}(E_4, E_6)E_4 = -E_6,$$

$$\hat{Riem}(E_4, E_6)E_6 = -E_4,\ \hat{Riem}(E_5, E_6)E_5 = -E_6,\ \hat{Riem}(E_5, E_6)E_6 = -E_5,$$

$$\hat{Ric}(E_i, E_j) = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix},$$

and
$$\hat{R} = Trace(\hat{Ric}) = -20.$$

From equation (1.3), we have

$$[\hat{g}(\hat{\nabla}_{E_i} E_6, E_i) + \hat{g}(\hat{\nabla}_{E_i} E_6, E_i)] + 2\hat{Ric}(E_i, E_i) + (2\Lambda - \rho\hat{R})\hat{g}(E_i, E_i) + 2\omega\delta^i_j = 0$$

for all $i \in \{1, 2, 3, 4, 5, 6\}$. Therefore for $\Lambda = 10\rho - 4$ and $\omega = 22 - 30\rho$, the data $(\hat{g}, E_6, \Lambda, \omega, \rho)$ is an almost η-Ricci-Bourguignon soliton, which verifies equation (3.1). Also the data $(V, \hat{g}, \Lambda, 0, \rho)$ is expanding. Also the data $(V, \hat{g}, \Lambda, \omega, \rho)$ is admitting the expanding, shrinking and steady almost η-Ricci-Bourguignon soliton according $10\rho > 4\beta$, $10\rho < 4$ or $10\rho = 4$ respectively.

Now, we have following three main cases for particular values of ρ.

Case 1. In an almost η-Ricci soliton, we find $\Lambda = -4$ and $\omega = 22$, then the data $(\hat{g}, E_6, \Lambda, \omega, \rho = 0)$ represents an almost η-Ricci soliton.This verifies Theorem 4.3.

Case 2. For an almost η-Einstein soliton, we find $\Lambda = 1$ and $\omega = 7$, then the data $(\hat{g}, E_6, \Lambda, \omega, \rho = \frac{1}{2})$ admits almost η-Einstein soliton. This case verifying Theorem 4.4.

Case 3. For an almost η-Schouten soliton, we find $\Lambda = \frac{10}{2n-1} - 4$ and $\omega = 22 - \frac{30}{2n-1}$. Therefore the data $(V, \hat{g}, \Lambda, \omega, \rho = \frac{1}{2n-1})$ is admitting the expanding, shrinking and steady almost η-Schoute soliton according $\frac{10}{2n-1} > 4\beta$, $\frac{10}{2n-1} < 4$ or $\frac{10}{2n-1} = 4$ respectively This case verifying Theorem 4.4.

Example 5.2. Let $\psi : \mathbb{R}^6 \to \mathbb{R}^3$ be a submersion defined by

$$\psi(x_1, x_2, ...x_6) = (y_1, y_2, y_3),$$

where
$$y_1 = \frac{x_1 + x_2}{\sqrt{2}},\ y_2 = \frac{x_3 + x_4}{\sqrt{2}} \text{ and } y_3 = \frac{x_5 + x_6}{\sqrt{2}}.$$

Then the Jacobian matrix of ψ has rank 3. That means ψ is a submersion. A straight computations yields

$$ker\psi_* = span\{V_1 = \frac{1}{\sqrt{2}}(-\partial x_1 + \partial x_2), V_2 = \frac{1}{\sqrt{2}}(-\partial x_3 + \partial x_4),$$
$$V_3 = \frac{1}{\sqrt{2}}(-\partial x_5 + \partial x_6)\}$$

and

$$(ker\psi_*)^{\perp} = span\{H_1 = \frac{1}{\sqrt{2}}(\partial x_1 + \partial x_2), H_2 = \frac{1}{\sqrt{2}}(\partial x_3 + \partial x_4),$$
$$H_3 = \frac{1}{\sqrt{2}}(\partial x_5 + \partial x_6)\}.$$

Also, by direct computations we find

$$\psi_*(H_1) = \partial y_1, \psi_*(H_2) = \partial y_2 \text{ and } \psi_*(H_3) = \partial y_3.$$

It is easy to see that

$$g_{\mathbb{R}^6}(H_i, H_i) = g_{\mathbb{R}^3}(\psi_*(H_i), \psi_*(H_i)),\ i = 1, 2, 3.$$

Thus ψ is a Riemannian submersion.

Now, we can compute the components of Riemannian curvature tensor $\hat{Riem}$, Ricci tensor $\hat{Ric}$ and scalar curvature $\hat{R}$ for $ker\psi_*$ (vertical space) and $ker\psi_*^{\perp}$ (horizontal space), respectively. For the vertical space, we have

$$\hat{Riem}(V_1, V_2)V_1 = -2V_2, \quad \hat{Riem}(V_1, V_2)V_2 = 2V_1, \quad \hat{Riem}(V_1, V_3)V_1 = -2V_3,$$
$$\hat{Riem}(V_1, V_2)V_3 = V_1, \quad \hat{Riem}(V_2, V_3)V_3 = V_2, \quad \hat{Riem}(V_2, V_3)V_2 = V_2,$$
$$\hat{Ric}(V_i, V_j) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\hat{R} = Trace(\hat{Ric}) = 5.$$

Using (1.3), we obtain $\Lambda = \frac{5\rho}{2} - 1$ and $\omega = 1$. Therefore, the data $(ker\psi_*, g, \Lambda, \omega)$ is an almost η-Ricci-Bourguignon soliton. Moreover, for particular values of ρ, we can also find the similar conclusions for almost η-Ricci soliton, almost η-Einstein soliton and almost η-Schouten soliton.

Next, for the horizontal space we have

$$\check{Riem}(\psi_*(H_1), \psi_*(H_2))\psi_*(H_1) = \frac{1}{2}(\partial x_3 + \partial x_4),$$
$$\check{Riem}(\psi_*(H_1), \psi_*(H_3))\psi_*(H_3) = \frac{1}{\sqrt{2}}(\partial x_6 - \partial x_5),$$
$$\check{Riem}(\psi_*(H_1), \psi_*(H_3))\psi_*(H_1) = \frac{1}{2}\partial x_6,$$
$$\check{Riem}(\psi_*(H_2), \psi_*(H_3))\psi_*(H_2) = (\frac{1}{\sqrt{2}} - 1)\partial x_6,$$
$$\check{Riem}(\psi_*(H_2), \psi_*(H_3))\psi_*(H_3) = -\frac{1}{2}(\partial x_3 + \partial x_4),$$
$$\check{Riem}(\psi_*(H_1), \psi_*(H_2))\psi_*(H_2) = \frac{1}{2\sqrt{2}}(\partial x_1 + \partial x_2),$$
$$\check{Ric}(\psi_* H_i, \psi_* H_j) = \begin{bmatrix} -\frac{3}{2\sqrt{2}} & 0 & 0 \\ 0 & -\frac{3}{2\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

and

$$\check{\mathcal{R}}_H = Trace(\check{Ric}) = -2\sqrt{2}.$$

Again using (1.3), we obtain $\Lambda = \frac{3}{2\sqrt{2}} - \sqrt{2}\rho$ and $\omega = -\frac{1}{2\sqrt{2}}$. Therefore $(ker\psi_*^{\perp}, \check{g}, \Lambda, \omega)$ is an almost η-Ricci-Bourguignon soliton.

In addition, for specific values of ρ, we can also find the conditions for almost η-Ricci soliton, almost η-Einstein soliton, and almost η-Schouten soliton.

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