Coverings of Riemann surfaces and nonperturbative effects in String Theory

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Abstract. A connection between perturbative and nonperturbative effects in string theory is introduced. The Schottky covering of closed Riemann surfaces to a larger class which may be related to ideal boundaries with non-zero linear measure.

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1 Introduction

Closed string scattering amplitudes are evaluated by integrating the expectation values of products of vertex operators, after multiplying by picture-changing operators, over compact Riemann surfaces integrating over the moduli space of metrics at genus g and summing over the genus. The domain of string perturbation theory would be the closed surfaces of finite genus and defined such that effectively closed infinite-genus surfaces belong to the sum over histories. This class of surfaces is characterized by the absence of a connection with observations at large distances from the interactions regions.

Beginning with closed string theory described entirely by the diagrammatic expanion, the preservation of the conformal invariance of the two-dimensional sigma model is sufficient to yield conditions on the embedding space including the dimensionality and the metric. There is no relation, however, between the string states that are propagating within the Riemann surface and the dynamics in the target space. Nonperturbative effects in string scattering processes have been identified with the physical states that are represented by Dirichlet boundaries on finite-genus surfaces and the ideal boundaries at infinite genus. These include open string states and states on the ideal boundary that may comprise particles or strings.

A non-zero linear or harmonic measure of the ideal boundary requires a larger class of Riemann surfaces, A connection between surfaces in the perturbative expansion of the scattering matrix and these surfaces may be found by considering the coverings.

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The Schottky covering surface belongs to the class O_{AD} and can have non-zero capacity [17]. A transition from the class of surfaces in the perturbation series to the nonperturbative effects in string theory is found.

2 Surfaces in the class of Schottky coverings

The relation between the Dirichlet norm of an analytic function and the lengths of curves on a surface will be summarized. It is equal to $D = \int_{|t-t_0 \le d} \left|\frac{dw}{dt}\right| d\sigma_t$. Let

$$
\begin{aligned}
(2.1) \qquad w &= c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + ... + c_n(t-t_0)^n + ... \\
\frac{dw}{dt} &= c_1 + 2c_2(t-t_0) + ... + nc_n(t-t_0)^{n-1} + ...
\end{aligned}
$$

Then

$$
(2.2) \qquad D = \int_{|t-t_0|\le d} [c_1 + 2c_2(t-t_0) + ... + nc_n(t-t_0)^{n-1}]^{2\text{‘}} d\sigma_t.
$$

Terms with azimuthal dependence will not contribute because the integral vanishes

$$
\begin{aligned}
(2.3) \qquad D &= \int_0^{2\pi}\int_0^d r\ dr[|c_1|^2 + 2|c_2|^2 r^2 + ... + n|c_n|^2 r^{2(n-1)} + ...] \\
&= 2\pi\left[|c_1|^2\frac{d^2}{2} + 2^2|c_2|^2\frac{d^2}{4} + ... + n^2|c_n|^2\frac{d^{2n}}{2n} + ...\right] \\
&= \pi[|c_1|^2 + 2|c_2|c_2|^2 + ... + n|c_n|^2 + ...] \\
&= \pi\sum_{n=1}^{\infty} n|c_n|^2 d^{2n} \ge \pi|c_1|^2 d^2 = \pi^2 q^2 d^2.
\end{aligned}
$$

When the coefficients are bounded,

$$
(2.4) \qquad D \le \int_{|t-t_0|\le d}\left|\frac{dw}{dt}\right|^2 d\sigma_t = \pi d^2\left|\frac{dw}{dt}\right|^2 d\sigma_t = \pi d^2\left|\frac{dw}{dt}\right|^2_{max}.
$$

If the disk $|t-t_0| \le d$ is subdivided into n regions K_ν, then it is represented by $\{\cup_{\nu=1}^n t \mid d_{\nu-1} \le |t-t_0| \le d_\nu\}$, and

$$
\begin{aligned}
\int_{d_{\nu-1}\le|t-t_0|\le d_\nu}\left|\frac{dw}{dt}\right|^2_{max} d\sigma_t &= \pi(d_\nu^2 - d_{\nu-1}^2)\ max_{K_\nu}\ \left|\frac{dw}{dt_\nu}\right|^2 \\
&= \pi(d_\nu^2 - d_{\nu-1}^2)q_\nu,
\end{aligned}
$$

where $q_\nu = max_{K_\nu}\left|\frac{dw}{dt_\nu}\right|$. Summing over the regions,

$$
(2.5) \qquad D \le \pi\sum_{\nu=1}^{n}(d_\nu^2 - d_{\nu-1}^2)q_\nu^2 \le 2d\pi\sum_{\nu=1}^{n}(d_\nu^2 - d_{\nu-1}^2)q_\nu^2.
$$

Similarly,

$$\int_{d_{\nu-1}\leq|t-t_0|\leq d_\nu} \left|\frac{dw}{dt}\right| \; d\sigma_t \geq \pi(d_\nu^2 - d_{\nu-1}^2)\; min_{K_\nu} \left|\frac{dw}{dt_\nu}\right|^2 \tag{2.6}$$

and

$$D \geq \pi \sum_{\nu=1}^{n} (d_\nu^2 - d_{\nu-1}^2)\; min_{K_\nu} \left|\frac{dw}{dt_\nu}\right|^2 . \tag{2.7}$$

Let $d_\nu^2 - d_{\nu-1}^2 \geq \frac{d^2}{m_1}$ and

$$min_{K_\nu} \left|\frac{dw}{dt_\nu}\right|^2 \geq \frac{1}{m_2} \left|\frac{dw}{dt_\nu}\right|^2 = \frac{q_\nu^2}{m_2}. \tag{2.8}$$

Then $D \geq \frac{\pi}{m_1 m_2} d^2 \sum_{\nu=1}^{n} q_\nu^2$. Setting $m_1 m_2 = m$,

$$D \geq \frac{\pi d^2}{m} \sum_{\nu=1}^{n} q_\nu^2. \tag{2.9}$$

The length of a curve would be $D_\gamma = \int_\gamma dw$. Since

$$\int_{d_{\nu-1}\leq|t-t_0|\leq d_\nu} \left|\frac{dw}{dt}\right| \leq 2\pi(d_\nu - d_{\nu-1}) q_\nu \tag{2.10}$$

and

$$q_\nu \;\leq\; \sqrt{m_2}\; min_{K_\nu} \; \left|\frac{dw}{dt_\nu}\right| \leq \sqrt{m_2} q_\nu, \tag{2.11}$$

$$\int_{d_{\nu-1}\leq|t-t_0|\leq d_\nu} \left|\frac{dw}{dt}\right| dt \leq 2\pi(d_\nu - d_{\nu-1})\sqrt{m_2} \sum_{K_i \subset K_\nu} q_i. \tag{2.12}$$

The constant m_2 must be larger than one, and

$$m \geq \frac{d^2}{d_\nu^2 - d_{\nu-1}^2}. \tag{2.13}$$

Choosing m such that $2\pi(d_\nu - d_{\nu-1})\sqrt{m_2} < m$, $D_\gamma < m \sum_i q_i$ and

$$D_\gamma^2 < m^2 \left(\sum_i q_i\right) < m^2 \ell_\nu \sum_i q_i^2 < \frac{m^3 \ell_\nu}{\pi d^2} D_\nu. \tag{2.14}$$

Similarly, $\sum_{\nu=1}^{n} D_\nu \leq N D_\gamma$ for some constant N. An inequality for the reciprocals of the lengths

$$\sum_{\nu=1}^{n} \frac{1}{\ell_\nu} < \frac{m^2}{\pi d^2} \frac{1}{D_\gamma^2} \sum_{\nu=1}^{n} D_\nu \leq \frac{m^3 N}{\pi d^2} \frac{1}{D_\gamma^2} D_\gamma = \frac{m^3 N}{\pi d^2} \frac{1}{D_\gamma} \tag{2.15}$$

follows. When $D_\gamma \to 0$, $\sum_{\nu=1}^{n} \frac{1}{\ell_\nu} \to \infty$. Consequently, the nonexistence of a nonconstant analytic function on the surface with finite Dirichlet norm requires this condition on the lengths of curves in the regions K_ν.

If $\Sigma_t \subset \sigma$ is an increasing sequence of submanifolds of Σ, with $L(g; \Sigma_s, \Sigma_t)$ being the infimum of the length of curves connecting $\partial\Sigma_s$ with $\partial\Sigma_t$ and $A(g; \Sigma_s, \Sigma_t)$ equal to the area of $\Sigma_t \backslash \Sigma_s$ with respect to the metric g, the surface Σ is parabolic if

$$lim_{t\to\infty} \frac{L(g; \Sigma_s, \Sigma_t)}{A(g; \Sigma_s, \Sigma_t)} = \infty. \tag{2.16}$$

When the hyperbolic lengths $\ell(c_n)$ of the components $\{c_n\}$ of the boundary of the convex $\partial C \subset \Sigma$ satisfy $\sum_n \ell(c_n)^{\frac{1}{2}} < \infty$, the Hausdorff dimension of the limit set equals one. Then the Hausdorff dimension and the capacity of the ideal boundary would be zero, which requires that the surface belongs to O_G.

The linear measure of $\partial\Sigma$ vanishes if $\Sigma \in O_{AB}$. If U is the unit disk representing the universal covering of the surface and containing $\partial\Sigma$, and f is a bounded analytic function on $\bar{U} - \partial\Sigma$, then f has an analytic extention to $\bar{U}$. Let $\Sigma_n \subset \Sigma$, $lim_{n\to\infty}\Sigma_n = \Sigma$ and $lim_{n\to\infty} \int_{\partial\Sigma_n} |dz| \to 0$. Then $f(z_0) = -\frac{1}{2\pi i}\int_{\partial U - \partial\Sigma_n} \frac{f(z_0)}{z - z_0} dz$ from the boundary of the disk to the interior where $z_0 \in U \cup \Sigma_1$, $\left| \int_{\partial\Sigma_n} \frac{f(z)}{z-z_0} dz \right| \leq M d^{-1} \int_{\partial\Sigma_n} |dz| \to 0$ for $|f| < M$ and $d < min_{\partial\Sigma_1}|z - z_0|$ and $f(z_0) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(z)}{z-z_0} dz$ is analytic in U. Therefore, O_{AB} is a subset of the class of surfaces with boundaries of zero linear measure.

Type I surfaces are characterized by divergent Poincare series of the uniformizing Fuchsian group and no border arc of the fundamental domain in the unit dis,. The ideal boundaries of Type II surfaces have non-zero linear measure. Since $O_{AB} \subset O_{AD}$, the possibility of surfaces in the class O_{AD}, having boundaries with continua, exists. The Dirichlet norm D_R is infinite when $\sum_{n=1}^{\infty} \frac{1}{\sigma_n}$ diverges, where σ_n is the number of faces in a polyhedric representation of the surface. Since there are two sheets for every border in any A-B cut, $\sigma_n \leq 4k(1 + 2n)$ with k being an upper bound for the number of disks that are required to cover the cuts in $\tilde{\Sigma}_n$, where $lim_{n\to\infty}\tilde{\Sigma}_n = \tilde{\Sigma}$ is a planar covering of a closed Riemann surface Σ. Then $\sum_n \frac{1}{\sigma_n} = \infty$ and $\tilde{\Sigma}_\in O_{AD}$ [17]. The Schottky covering is intermediate between the surface and a planar covering with a finite number of relations [18].

The Schottky covering of a Riemann surface has no immediate role in string scattering. However, it might be viewed as the equivalent of the discrete action of a hyperbolic group in three dimensions. There, the covering space of the manifold, realized as a quotient, is a manifold that may be relevant to three-dimensional physics. From the classification of conformally flat hypersurfaces in higher-dimensional Euclidean spaces, which are relevant to the quantum gravitational path integral, and the conformal equivalence to spheres and Schottky manifolds for dimensions greater than or equal to four [13]. It is evident that these higher-dimensional space could represent the background geometries for dynamical theories.

Theorem 1. A summation of the ratios of the linear measure of the ideal boundary of the n^{th} approximation to a Schottky covering surface with q algebraic relations to a covering surface with 2 relations, from the index n_0, where the n_0^{th} approximation is the first connected space, equals $\mathcal{O}(\zeta(q, n_0))$ for
$q \geq 2$.

Proof. The Schottky covering surface in two dimensions has a boundary in the planar surface given by the union of the isometric circles of the uniformizing group. However, the cycles in the surface can be cut and lifted to a covering surface such that $(C_{2i-1}^+, C_{2i-1}^-, C_{2i}^+, C_{2i}^-)$ comprises boundary curves in the planar surface. With a copey of this planar surface for each $(m_1, ...m_{2p}) \in \mathbb{Z}^{2p}$, C_{2i-1}^+ in $\Phi(m_1, ..., m_{2i-1}, m_{2i}, ...m_{2p})$ may be identified with C_{2i-1}^- in $\Phi(m_1, ..., m_{2i-1}, m_{2i}{+}1, ...m_{2p})$ and C_{2i}^+ in $\Phi(m_1, ..., m_{2i-1}, m_{2i}, ..., m_{2p})$ and
C_{2i}^- may be joined to $\Phi(m_1, ..., m_{2i-1}{+}1, m_{2i}, ..., m_{2p})$ to create an unramified covering surface.

If there are defining relations between these elements of the covering transformation group, $\sum_{j=1}^{2p} \gamma_{kj} C_j = 0$, $k = 1, ..., q$, $0 \leq q \leq 2p$, and $r = 2p - q$ is the rankof this group, the covering surface is closed when $r = 0$, the ideal boundary consists of two components when $r = 1$, it belongs to O_G when $r \leq 2$ and there is an infinite sequence of approximations along one end if $r \geq 2$. There are p relations $\gamma_{2i-1} C_{2i-1} + \gamma_{2i} C_{2i} = 0$, such that the rank is p, with the coefficients being integral and not all vanishing, for covering surfaces in O_{AB} [10]. In a coordinate system with a basis of r vectors from the standard basis of E^{2p} and the q vectors $(\gamma_{1j}, ..., \gamma_{qj})$, the coordinates $(x_1, ..., x_q.y_1, ..., y_r)$ may be defined by $(m_1, ..., m_{2p}) = (x_1, ..., x_q, y_1, ..., y_r)T$, where T is constructed fromm the new basis vectors. The length of the boundary of an n^{th} approximation to the covering surface within the region $Z(t_1, ..., t_r)$, defined by $Mt_\ell \leq y_\ell \leq M(t_\ell + 1)$ and $-n \leq t_\ell \leq n$, $\ell = 1, ..., r$, may be demonstrated to be $\mathcal{O}(n^{r-1})$ [11]. Defining the moduli of a component of the boundary to be $\mu_n^\kappa = \frac{2\pi}{\int_{\gamma_n^\kappa} dv_n^\kappa}$ and $\sigma_n = \frac{1}{\sum_{\kappa=1}^{k(n)} \frac{1}{\mu_\kappa^n}}$, the result on the relation between the class of the surface and the rank of the abelian group of covering transformations follows from $\sum_{n=1}^{\infty} \sigma_n = \infty$ for O_G [12], $\overline{lim_{N\to\infty}} \left\{ \sum_{n=1}^{N} \mu_n - \frac{1}{2} log\ K(N) \right\} = \infty$ for O_{AB}, where $K(N) = max_{n<N} k(n)$ [15], and $\sum_{n=1}^{\infty} \mu_n = \infty$ for O_{AD} [16].

The length of the ideal boundary can be estimated by the sum of the lengths of components modulo the equivalence relation. For O_{AB} surfaces, the rank of the abelian group on the component will not exceed one and it may be shown that L_n^κ which yields the result for the limit. However, under the equivalence of $(\gamma_1, \gamma_2, 0, ..., 0), ..., (0, ..., 0, \gamma_{2i-1},\ \gamma_{2i}, 0, ..., 0), ..., (0, ..., 0, \gamma_{2p-1}, \gamma_{2p})$, with
$(0, 0, ..., 0, 0)$, the linear measure of the ideal boundary may be set equal to zero in O_{AB}.

The initial estimate of $L_n = \mathcal{O}(n^{r-1})$ in the n^{th}-order approximation to the covering surface may be compared with $\sum_{\kappa=1}^{k(n)} L_n^\kappa$ without equivalence relations. Each time there is a relation of the form $\gamma_{2i-1} C_{2i-1} + \gamma_{2i} C_{2i} = 0$, the factors of $2n$ and $2n+1$ in the formula $L_n = \sum_{\{\delta_i\}}[(2n_1)^r - (2n)^{r'}(2n+1)^{r-r'}]L(\delta_1,..,\delta_r)$, with the boundary arc with connecting regions $Z(t_1,...t_r)$ to $Z(t_1+\delta_1,...,t_r+\delta_r)$ and $r' = |\{\delta_\ell \neq 0\}|$ can be reduced to constants. When there are q relations with s additional terms, this estimate is replaced by

$$\sum_{\{\delta_\ell\}}[(2n+1)^{r-q} - (2n)^{r-q-s-r'}(2n+1)^{s+r'}]L(\delta_1,...,\delta_r) = (2n+1)^{s+r'}\mathcal{O}(n^{r-q-s-r'-1}) = \mathcal{O}(n^{r-q-1}). \tag{2.17}$$

Then

$$\frac{\sum_{\kappa=1}^{k(n)} L_n^\kappa}{L_n(no\ equivalence\ relations)} = \frac{\mathcal{O}(n^{r-q-1})}{\mathcal{O}(n^{r-1})} = \mathcal{O}\left(\frac{1}{n^q}\right). \tag{2.18}$$

A summation over n is consistent with the classification theory through the summation over moduli of subregions in the exhaustion of the covering surfaces. Summing over n from n_0, where the n_0^{th} approximation to the covering surface is the first to be a connected space, is

$$\mathcal{O}\left(\sum_{n=n_0}^{\infty}\frac{1}{n^q}\right) = \mathcal{O}(\zeta(q,n_0)). \tag{2.19}$$

relative to one for the linear measure of the ideal bounda of an O_{AD} surface with an abelian covering transformation group with no relations. when $q=1$, this sum diverges, and it sufficient to note that the ratio is $\mathcal{O}\left(\frac{1}{n}\right)$ for the n^{th} approximation. □

The covering surface with an ideal boundary of one or two components may be connected to the closed finite-genus surfaces and effectively closed infinite-genus surfaces through a supplementary series to the diagrammatic expansion to generate surfaces that would describe nonperturbative effects.

3 Deformations of Riemann surfaces

Worldsheet instantons represent the embedding of curves in the target space which are solutions to the effective field equations. These solutions often arise as instantons and solitons of the string theory that are backgrounds for a sigma model with a higher degree of worldsheet supersymmetry [4] [19]. Amongst the nonperturbative effects in Type I or heterotic string theory with $Spin(32)/\mathbb{Z}_2$ gauge group on a manifold $\mathbb{R}^4 \times X$ are the D-instanton path integral and contribution to the superpotential

$$\mathcal{Z}_{\mathcal{C}} = exp\left(-\frac{A(\mathcal{C})}{2\pi\alpha'} + i\int_{\mathcal{C}} B\right)\frac{Pfaff'(\mathcal{D}_F)}{\sqrt{det'\mathcal{D}_B}}$$
$$W_{\mathcal{C}} = exp\left(-\frac{A(\mathcal{C})}{2\pi\alpha'} + i\int_{\mathcal{C}} B\right)\frac{Pfaff'(\bar\partial_{V(-1)})}{(det\ \bar\partial_{\mathcal{O}}(-1))^2(det'\bar\partial_{\mathcal{O}})^2}. \tag{3.1}$$

where the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is the normal bundle of $\mathcal{C}$ in X and $\mathcal{O} \oplus \mathcal{O}$ is the normal bundle of $\mathcal{C}$ in $\mathbb{R}^4$ [19] where the operator on the fermion fields includes Lorentz indices.

This formula has been computed for genus-zero surfaces [19]. By constrast with the genus-zero contribution to the worldsheet path integral, there are higher-derivative interactions at higher genus that affect only the kinetic terms. Even though the superpotential is unchanged, it will be established whether the above mechanism is related to the transition of the surface to a class of covering surfaces with ideal boundaries of non-zero linear measure.

Derivatives on the worldsheet can be decomposed into tangential and normal deformations. The path integral at genus g would have the form

$$\int D[X^\mu] D[\psi_\mu^\alpha] e^{-I_{NSR} + I_{D-inst.,g=0} + I_{D-inst.,g>0}}. \tag{3.2}$$

The tangential deformations of the metric are described by the quasiconformal transformations $dz \to dz + \mu^z{}_{\bar{z}} d\bar{z}$ and may be absorbed into the conventional moduli space integral. The normal bundle contribution again will be given by integration over the transverse oscillations and yields the factor $\frac{Pfaff'(D_F)}{\sqrt{D_B}}$ when $g = 0$. For $g > 0$, consider

$$\int D[X^\mu]\Big|_{norm.} e^{-\int d^2\xi \sqrt{h} (\partial X^\mu \partial X_\mu)^{g+1}} \int D[\psi_\mu^\alpha]\Big|_{norm.} e^{-\int d^2\xi \sqrt{h} (\bar{\psi} \bar{\partial} \psi)^{g+1}}. \tag{3.3}$$

The bosonic integral would follow from

$$\int_{-\infty}^{\infty} dp\ e^{-(ap^2)^{g+1}} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} dp\ e^{-p^{2(g+1)}} = \frac{1}{\sqrt{a}} I_{2(g+1)} \tag{3.4}$$

where $I_{2n} = \int_{-\infty}^{\infty} dp\ e^{-p^{2n}} = 2\Gamma\left(1 + \frac{1}{2n}\right)$, such that

$$\int D[X^\mu]\ e^{-\int d^2\xi \sqrt{h} (\partial X^\mu \partial X_\mu)^{g+1}} = (det\ \mathcal{D}_B)^{-4}\ 2^8\ \Gamma\left(1 + \frac{1}{2n}\right)^8. \tag{3.5}$$

The fermion integral, evaluated with Grassmann variables, would be

$$\int d\theta_1 d\theta_2 ... d\theta_7 d\theta_8\ [1 + (det\ \bar{\partial})\theta_1 ... \theta_8]^{g+1} = (g+1)(det\ \bar{\partial}). \tag{3.6}$$

Consequently, the determinant factors again would be

$$2^8 (g+1) \Gamma\left(1 + \frac{1}{2(g+1)}\right)^8 \frac{Pfaff'(\bar{\partial}_{V(-1)})}{(det' \bar{\partial}_{\mathcal{O}(-1)})^2 (det' \bar{\partial}_{\mathcal{O}})^2} \tag{3.7}$$

when the zero modes are not included in the determinants. The most general evaluation of the fermion integral, with either commuting or anticommuting spinors, is

(3.8)

$$\int D[\psi^\alpha_\mu]\Big|_{norm.} e^{-\int d^2\xi\sqrt{h}(\bar{\psi}\bar{\partial}\psi)^{g+1}} = \begin{cases} I^8_{2(g+1)} det'(\bar{\partial})^{-4} & \textit{commuting spinors} \\ (g+1)(det\ \bar{\partial}) & \textit{anticommuting spinors} \end{cases}$$

and the contribution of the two higher-derivative terms to the path integral is

$$(3.9) \qquad \int D[X^\mu]\Big|_{norm.} e^{-\int d^2\xi\sqrt{h}(\partial^\mu\partial X_\mu)^{g+1}} \int D[\psi^\alpha_\mu]\Big|_{norm.} e^{-\int d^2\xi(\bar{\psi}\bar{\partial}\psi)^{g+1}}$$

(3.10)

$$= \begin{cases} (2^{16}\Gamma\left(1+\frac{1}{2(g+1)}\right)^{16} \frac{det\ \bar{\partial}}{(det'\bar{\partial})^4(det'\bar{\partial}_{\mathcal{O}(-1)})^4} & \textit{commuting spinors} \\ 2^8(g+1)\Gamma\left(1+\frac{1}{2(g+1)}\right)^8 \frac{Pfaff'(\bar{\partial}_{V(-1)})}{(det'\bar{\partial}_{\mathcal{O}(-1)})^2(det'\bar{\partial}_{\mathcal{O}})^2} & \textit{anticommuting spinors} \end{cases}$$

It may be found that the normal bundle oscillations are consistent with the transition of the surface to a covering. The evaluation of the contribution of the k-fold covering to the D-instanton path integral has been calculated [3][5].

The worldsheets represent elements of the second cohomology class of the embedding space, and the phase of $Pfaff\ \mathcal{D}_F$ would be constrained by a relation $C_1 + ... + C_s = 0$ if these homology classes are not independent in $H_2(\mathbb{R}^4 \times X; \mathbb{Z})$. Then a cross-section of this relation can be lifted to that between homology generators for curves defining an intermediate covering common to each of the surfaces $C_1, ..., C_s$. The nonperturbative effect resulting from the non-zero linear measure would be reduced by the factor in the theorem of §2 for this configuration.

4 Noncommutative boundaries of Riemann surfaces

The noncommutative generalization of the Riemann surface generates new terms in the integral over moduli space. It has been determined that the coefficient in the series expansion of the correlation function, derived from the functional derivatives of the partition function with sources, increases by a factor of $\left(1+\frac{1}{g-1}\right)$ for $g \geq 2$. Furthermore, noncommutative end theory [1] provides a Fock space consisting of multiparticle states in the quantization of the classical model on a Riemann surface with an ideal boundary [8].

The noncommutative end theory may be extended to a boundary continuum. The principle of induction of the C^* algebras on the union of ends $C_0(\cup_{k=1}E_k) \simeq C_0(E_k)$ is not necessarily valid for a continuum [2]. Nevertheless, when the ideal boundary of the surface has a non-zero linear measure, a Hilbert space can be constructed on a transformed boundary diffeomorphic to an interval. This Hilbert space would not include the states on the ends of the surface, because the basis functions in the latter space are not square integrable on the Cantor set [9]. It is a nonperturbative effect requiring quantization in an instanton Hilbert space.

Previous studies of noncommutative limits of the uniformization of Riemann surfaces have followed the extension to the boundary of the covering surface. For the

Schottky group Γ, the limit set Λ_Γ may be defined to have Hausdorff dimension less than or equal to 1. The Hilbert space of square integrable functions on this limit set is $\mathcal{H}_{\Lambda_\Gamma} = \mathcal{L}^2(\Lambda_\Gamma, d\mu)$, where $d\mu$ is the Patterson-Sullivan measure [7]. The ideal boundary of a surface uniformized by a Fuchsian group is a subset of the boundary of the hyperbolic disk which is the universal covering space. It may be surmised that the ideal boundary in the intermediate Schottky covering space is a subset of $\left[\left(\cup_{i=1}^g I_{T_\ell} \cup I_{T_\ell^{-1}}\right) \cup \cup_{\alpha \neq I} V_\alpha \left(\cup_{i=1}^g \left(I_{T_\ell} \cup I_{T_\ell^{-1}}\right)\right)\right] \backslash \Gamma = \left(\cup_{i=1}^g (I_{T_\ell} \cup I_{T_\ell^{-1}}\right)$, where $\{V_\alpha\}$ is the elements of the Schottky group not equal to the identity. For a closed finite-genus surface, every point on the isometric circles is the limit point of a sequence of points generated by the group. Therefore, the limit set coincides with the union of isometric circles, and there is no ideal boundary. By contrast, for an infinite-genus surface, there will be an accumulation of handles, and therefore, infinite sequences of points forming orbits of the uniformizing group, that have limit points exterior to the isometric circles, producing an ideal boundary.

Theorem 2. There exists a set representing the ideal boundary of a Riemann surface which is invariant under $PSL(2;\mathbb{Z})$ on which a Hilbert space can be defined.

Proof. The extension of the upper-half plane which is a universal cover of all surfaces of genus $g \geq 2$ to the real line introduces noncommutative tori including $\mathcal{A}_\theta = \langle U, V | UV =, e^{2\pi i\theta} VU,\ \theta \in \mathbb{R}\backslash\mathbb{Q}\rangle$ and the Morita equivalent $\mathcal{A}_{\theta'}$, where $\theta' = \frac{a\theta+b}{c\theta+d}$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which are subspaces of bounded operators in $\mathcal{L}^2(S^1)$ [6]. The uniformizing Fuchsian group G is a subgroup of $PSL(2;\mathbb{Z})$ and $PGL(2;\mathbb{Z})$, and the surface is $(\mathbb{H}^2 \times PSL(2;\mathbb{Z})/G)/(PSL(2;\mathbb{Z}))$ or $\mathbb{H}^2 \times (PGL(2;\mathbb{Z})/G)/PGL(2;\mathbb{Z})$, while the boundary of modular curves to be $C(\mathbb{RP}^1) \times (PGL(2;\mathbb{Z})/G) \rtimes PGL(2;\mathbb{Z})$. The orbits of $PGL(2;\mathbb{R})$ in $\mathbb{RP}^1$ under the equivalence relation $x \sim y$, $T^m x = T^m y$ for some $m,\ n \in \mathbb{Z}$ and $Tx = \frac{1}{x} - \left[\frac{1}{x}\right]$, and the Lyapounov spectrum for T will be defined by $L_c = \{\beta \in [0,1] | \lambda(\beta) = lim_{n\to\infty} \frac{1}{n} log|(T^n)'(\beta)| = c\}$ with Hausdorff dimensions between 0 and 1 such that $[0,1] = \cup_c L_c$ [14]. The ideal boundary of the Riemann surface again can have non-zero linear measure if the Hausdorff dimension of the limit set is less than 1, which occurs when $\Lambda_\Gamma \subset \cup_{c,\ d_H(L_c)<1} L_c$. It follows that the quantization on the boundary of modular curves may be formulated in a Hilbert space that is a subspace of $C([0,1]\backslash \cup_{c,\ d_H(L_c)<1} L_c) \times (PGL(2;\mathbb{Z})/G)$. Invariance under $PSL(2;\mathbb{Z})$ would increase the C^* algebras because the corresponding orbits of $PSL(2;\mathbb{Z})$ form a subset of $\cup_{c,\ d_H(L_c)<1} L_c$. Furthermore, this set can be transformed to an interval in $[0,1]$ and the Hilbert space with an orthonormal basis of square-integrable functions could be defined. □

The quantum theory will be given by operators representing coefficients in an expansion of classical fields.

5 Conclusions

The classification of Riemann surfaces has provided a delineation between those classes that belong to the perturbative expansion of the string scattering process and the categories resulting in nonperturbative effects. Since the closed Riemann surfaces define virtual amplitudes which can be factorized from any real scattering process, much of the effect on dynamics in the target space is related to the ends of the surfaces. External vertex operators representing physical states, for example, are described by semi-infinite cylindrical ends attached to a closed surface. The connection between infinite-genus surfaces with the non-perturbative formulation of string theory begins with the class characterized by ideal boundaries of non-zero harmonic measure. The extension to surfaces with ideal boundaries of non-zero linear measure generates physical string states in the target space.

The class of surfaces with ideal boundaries of vanishing linear measure includes O_{AB}, the set of surfaces that does not admit a bounded analytic function. The class O_{AD}, the category of surfaces with no analytic function with finite Dirichet norm, has a non-null intersection with the Type II surfaces with fundamental domains in the unit disk having border arcs on the boundary. The Schottky coverings of finite-genus surfaces are known to belong to O_{AD}. Therefore, the passage to the covering surface introduces a linear measure to the ideal boundary. The ratio of the linear measure of a Schottky covering surface with relations at finite genus to the linear measure of the ideal boundary for a covering with no relations has been calculated. It is found to be the order of $\zeta(q, n_0)$, where q is the number or relations and n_0 labels the first approximation that is a connected space. It decreases with q and would determine the fraction of the linear measure for the uniformization of surfaces by groups of Schottky type in the infinite-genus limit.

Surfaces of higher genus only contribute higher derivative terms and not tp the superpotential in the effective action. The normal bundle oscillations yield a contribution to the worldsheet instanton partition function that can be identified with a mechanism for inducing a transition to the covering of the surface. The projection of the relation between embeddings of different surfaces in the target space, affecting the phase of the superpotential, to curves in the first homology class, is lifted to a relation in an intermediate covering space that reduces the nonperturbative effects resulting from the linear measure of the ideal boundary.

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