A Myers' Theorem for H-contact manifolds and some remarks on almost Ricci solitons

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Abstract. In this paper we give a generalization of Myers' Theorem for complete H-contact manifolds $(M, \eta, g, \xi, \varphi)$ with the metric g satisfying the critical point condition of the Chern-Hamilton functional. In this our result the role of the Ricci tensor Ric is replaced by Bakry-Emery Ricci tensor $Ric+\mathcal{L}_\xi g$. We also give sufficient conditions so that an almost Ricci soliton be trivial, with an application to the case of the geodesic flow as potential vector field of an almost contact Ricci soliton.

M.S.C. 2010: 53C25, 53C20, 53D10.
Key words: Myers' Theorem, H-contact manifolds, almost Ricci solitons, Bakry-Emery Ricci tensor, geodesic flow.

1 Introduction and statement of the results

Let (M, g) be a Riemannian manifold and let Ric be its Ricci tensor. If M admits a smooth vector field X_0 satisfying

$$Ric + \mathcal{L}_{X_0} g = \lambda g \tag{1.1}$$

some real constant λ, then (M, g, X_0, λ) is said to be a *Ricci soliton*, where $\mathcal{L}_{X_0}$ denotes the Lie derivative operator in the direction of the vector field X_0. Ricci solitons are natural generalizations of Einstein metrics and special solutions of the Ricci flow (see [5]). A Ricci soliton is said to be shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Non trivial compact Ricci solitons only exist in dimension ≥ 4 (cf. [12, 14]) and these manifolds must have positive scalar curvature [9], which is constant if and only if the soliton is trivial [8].

Since Ricci solitons (g, X_0, λ) generalize Einstein metrics, it is natural to ask whether Myers' Theorem remains valid replacing the Ricci tensor Ric by the tensor

$$Ric_{X_0} := Ric + \mathcal{L}_{X_0} g$$

Balkan Journal of Geometry and Its Applications, Vol.27, No.1, 2022, pp. 106-117.

also called the *Bakry-Emery Ricci tensor* (cf., for example, [24] and references therein).

Recall that the classic Myers' Theorem [15] states that a complete Riemannian manifold (M, g) satisfying $Ric \geq cg$, $c = const. > 0$, is compact. As noted in [8], if we replace the condition $Ric \geq cg > 0$ in Myers' Theorem by $Ric_{X_0} \geq cg$, the conclusion fails; an example is the Euclidean space $(\mathbb{R}^n, g_0)$ with X_0 the radial vector field. Then, M. Fernández-López and E. García-Río proved the following result under the condition that $\|X_0\|$ is bounded.

Theorem A ([8], Theorems 1,2) *Let (M, g) be a complete Riemannian n-manifold satisfying*

$$Ric_{X_0} \geq \lambda g$$

for some positive constant λ, and $\|X_0\| \leq a$ for some constant $a \geq 0$. Then, M is compact and the fundamental group $\pi_1(M)$ is finite.

More recently, with the same hypothesis of Theorem A, Wu [24] has proved that the diameter of (M, g) satisfies $diam(M) \leq (2a/\lambda) + \big(\pi/\sqrt{\lambda/(n-1)}\big)$. Moreover, Derdzinski [10] proved that if (M, g) is a compact Riemannian manifold such that $Ric_{X_0} > 0$ for some smooth vector field X_0, then $\pi_1(M)$ has only finitely many conjugacy classes.

In this paper, by using Theorem A and the tensor $Ric_\xi = Ric + \mathcal{L}_\xi g$, we give a generalization of Myers' Theorem for complete H-contact manifolds $(M, \eta, g, \xi, \varphi)$ with the metric g satisfying the critical point condition (2.3) of the Chern-Hamilton functional ([6], [23]). More precisely, our result is the following.

Theorem 1.1. *Let $(M, \eta, g, \xi, \varphi)$ be a complete H-contact $(2n+1)$-manifold with the metric g satisfying the critical point condition of the Chen-Hamilton functional. If*

$$Ric_\xi \geq \lambda g, \quad \lambda > -2 + \mu \geq -2 + \frac{\|\tau\|}{\sqrt{2n}}, \quad \tau = \mathcal{L}_\xi g, \tag{1.2}$$

where λ, μ are constant, then M is compact, the fundamental group $\pi_1(M)$ is finite and the first Betti number $b_1(M) = 0$.

The main idea of the proof of Theorem 1.1 is to use the so called *D-homothetic deformation.* This technique was introduced by Tanno [22] and used by Hasegawa and Seino [13], Goldberg and Toth [11] and by Blair and Sharma [4].

As a consequence of our Theorem 1.1 we have the following

Corollary 1.2. *([22],[13]) Let (M, η, g) be a complete K-contact $(2n+1)$-manifold. If $Ric \geq \lambda\, g > -2\, g$, then M is compact, the fundamental group $\pi_1(M)$ is finite and the first Betti number $b_1(M) = 0$.*

More precisely, about the Corollary 1.2, *Tanno [22] proved that: if a compact K-contact manifold satisfies $Ric + 2\, g > 0$, then $b_1(M) = 0$; Hasegawa and Seino [13] proved that: a complete Sasakian manifold for which $Ric \geq \lambda\, g$, $\lambda > -2$, is compact.*

Pigola et al. [17] defined a *Ricci almost soliton* as a Riemannian manifold (M, g) satisfying the equation (1.1) where λ is an arbitrary smooth function. Barros et al.

[2] proved that any compact non-trivial almost Ricci soliton with constant scalar curvature is isometric to a Euclidean sphere S^n. On the other hand, another interesting question is to find conditions under which an almost Ricci soliton is a trivial Ricci soliton [17].

If a unit vector field is Killing, then it is geodesic and divergence-free, but the converse in general is not true (the Reeb vector field of an arbitrary contact metric manifold satisfies these conditions). The next result is given by the following

Theorem 1.3. *Let (M, g, X_0, λ) be an almost Ricci soliton, $dimM = m > 2$. Suppose that the potential vector field X_0 is unit, geodesic and divergence-free. Then, the following are equivalent.*

1) The almost Ricci soliton is trivial (i.e. X_0 is Killing and g is Einstein);
2) X_0 is an infinitesimal harmonic transformation;
3) X_0 is an eigenvector of the Hodge-Laplacian with eigenvalue $\mu \leq 2\lambda$.

As a consequence of this Theorem we get the following.

Corollary 1.4. *Let (M, η, g, ξ) be a contact metric manifold. If (g, ξ, λ) is an almost Ricci soliton, then M is an Einstein K-contact manifold.*

Corollary 1.5. *Let (M, θ, J) be a strictly pseudo-convex CR manifold. If $(g_\theta, -T, \lambda)$ is an almost Ricci soliton, where T is the Reeb vector field of the contact from θ and g_θ is the corresponding Webster metric, then $(\eta = -\theta, g_\theta)$ is an Einstein Sasakian structure on the manifold M.*

We note that the *geodesic flow* $\tilde{\xi}$ on the unit tangent sphere bundle T_1M of a riemannian manifold (M, g) is the Reeb vector field of the so called *g-natural contact metric structures* over T_1M, which depend on three real parameters a, b, c (see Subsection 3.3, for a presentation of these structures). The *standard contact metric structure* on T_1M, i.e. the one induced from the Sasaki metric, is defined by the parameters $a = 1/4, b = c = 0$. Then, an application of Corollary 1.4 gives the following

Theorem 1.6. *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ and $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$ a g-natural contact metric structure over the unit tangent sphere bundle T_1M. Then, $(\tilde{G}, \tilde{\xi}, \lambda)$ is an almost Ricci soliton if and only if $(T_1M, \tilde{G}, \tilde{\eta})$ is Einstein Sasakian with $\lambda = 2(n-1) > 0$. Besides, in this case the base manifold (M, g) has constant sectional curvature $\kappa > 0$, and the metric $\tilde{G}$ is of Kaluza-Klein type defined by the parameters*

$$a = (n-1)/2n, \quad b = 0, \quad c = (\kappa - 1)a\,.$$

In particular: $\tilde{G}$ is Kaluza-Klein (resp. induced from the Sasaki metric) if and only if M is a surface of constant Gaussian curvature $\kappa = 1 + 4c > 0$ (resp. $\kappa = 1$).

2 Preliminaries on contact metric manifolds

In this Section we collect some basic facts about contact Riemannian geometry and refer to the monograph [3] for more information. All manifolds are supposed to be connected and smooth. If (M, g) is a Riemannian manifold, in what follows we shall denote by ∇ the Levi-Civita connection, by *Ric* the Ricci tensor, by Q the corresponding Ricci operator and by r the scalar curvature.

A *contact manifold* is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Then, there exists a unique vector field ξ, called the *Reeb vector field*, such that $\eta(\xi) = 1$ and $(\mathrm{d}\eta)(\xi, \cdot) = 0$. A Riemannian metric g is said to be an associated metric if there exists a tensor φ, of type $(1,1)$, such that

$$\eta = g(\xi, \cdot), \quad \mathrm{d}\eta = g(\cdot, \varphi\cdot) \quad \varphi^2 = -I + \eta \otimes \xi.$$

Associated metrics are known to exist.We refer to (M, η, g), or $(M, \xi, \eta, \varphi, g)$, as a contact metric (or contact Riemannian) manifold. The tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, where $\mathcal{L}$ denotes the Lie derivative, plays a fundamental role in contact Riemannian geometry, it is symmetric and satisfies:

$$\text{(2.1)} \qquad a)\ h\varphi = -\varphi h,\ h\xi = 0,\ \mathrm{tr}h^2 = 2n - Ric(\xi, \xi); \qquad b)\ \nabla\xi = -\varphi - \varphi h.$$

In particular, from (2.1) follows that the Reeb vector field ξ is geodesic and divergence-free. Since ξ is geodesic, we have

$$\mathcal{L}_\xi \eta = 0 \quad \text{and} \quad (\mathcal{L}_\xi g)(\xi, \cdot) = 0.$$

Besides, if $X \in \ker \eta$, by using $(2.1)_b$ one gets

$$g([X, \varphi X], \xi) = g(\nabla_{\varphi X}\xi, X) - g(\nabla_X \xi, \varphi X) = 2g(X, X),$$

and this implies the following

Lemma 2.1. *If f is a smooth function on a contact metric manifold satisfying $X(f) = 0$ for any $X \in \ker \eta$, then f is a constant function. In particular, if f_1, f_2 are two smooth functions with the gradient $\nabla f_1 = f_2\, \xi$, then f_1 is a constant and $f_2 = 0$.*

A contact metric manifold (M, η, g) is said to be a *K-contact* manifold if the Reeb vector field ξ is a Killing vector field with respect to the associated metric g. If the almost complex structure J on $M \times \mathbb{R}$ defined by $J(X, fd/dt) = (\phi X - f\xi, \eta(X)d/dt)$ is integrable, i.e., the almost contact structure (η, ξ, φ) is normal, then M is said to be *Sasakian*. Any Sasakian manifold is K-contact and the converse also holds in dimension three.

In [18] we introduced the H-contact manifolds. A contact metric manifold (M, η, g) is said to be an *H-contact manifold* if its Reeb vector field ξ is a harmonic vector field, that is, ξ satisfies the critical point condition for the energy functional $\mathcal{E}(U) = (1/2)\int_M \|dU\|^2 dv$ defined on the space of all unit vector fields (we refer to the monograph [7] for more information about harmonic vector fields on a Riemannian manifold). Moreover, in [18] we proved that a contact metric manifold (M, η, g) is H-contact if and only if ξ is an eigenvector of the Ricci operator Q. It should be noted that the class of H-contact manifolds is very large. In particular, Sasakian manifolds, K-contact manifolds, (k, μ)-spaces, (strongly) locally φ-symmetric spaces are all examples of H-contact manifolds.

Now, let us quickly recall that a strictly pseudoconvex almost-CR structure is equivalent to the notion of contact metric structure (we refer to [21] for more informstion). Let M be a $(2n+1)$-dimensional manifold. A *strictly pseudo-convex almost CR structure* on M is a pair (θ, J) where θ is an 1-form, J is an almost complex structure on $\mathcal{H} = \ker \theta$: $J^2 = -I$, and the *Levi form* $L_\theta(X, Y) := -(\mathrm{d}\theta)(X, JY),\ X, Y \in \mathcal{H}$, is positive definite. In particular, θ is a *contact form* and denote by T its Reeb vector field. We extend J to an endomorphism φ of the tangent bundle by requesting that $\varphi_{|\mathcal{H}} = J$ and $\varphi(T) = 0$. Then, $\varphi^2 = -I + \theta \otimes T$ and the *Webster metric* g_θ, defined by

$$g_\theta(X, Y) = (d\theta)(X, JY), \;\; g_\theta(X, T) = 0, \;\; g_\theta(T, T) = 1,$$

for any $X, Y \in \mathcal{H}$, is a Riemannian metric on M. In this case the synthetic object $(\eta = -\theta, \xi = -T, \varphi, g = g_\theta)$ is a contact metric structure on M. Vice versa, a contact metric structure (η, ξ, φ, g) defines a strictly pseudo-convex almost CR structure on M given by $(\theta = -\eta, J = \varphi_{|\mathcal{H}})$. A strictly pseudo-convex almost CR structure is called *strictly pseudo-convex CR structure* if the almost CR structure is *integrable*, that is, the following condition is satisfied

$$J\big([JX, Y] + [X, JY]\big) = [JX, JY] - [X, Y], \quad X, Y \in \mathcal{H}. \tag{2.2}$$

Next, let (M, η) be an oriented compact contact manifold. Denote by $\mathcal{M}(\eta)$ the set of all Riemannian metrics associated to the contact form η. Tanno [23] considered the *Dirichlet energy*

$$E(g) = \frac{1}{2}\int_M \|\tau\|^2 dv, \quad \tau = \mathcal{L}_\xi g,$$

defined for any $g \in \mathcal{M}(\eta)$. Then, he found the critical point condition ([23], Theorem 5.1)

$$\nabla_\xi \tau = 2\tau\varphi, \text{ equivalently } \nabla_\xi h = -2\varphi h. \tag{2.3}$$

The Dirichlet energy $E(g)$ was first studied by Chern and Hamilton [6] for compact contact three-manifolds (there was an error in their calculation of the critical point condition, as was pointed out by Tanno). This functional is known in literature also with the name of *Chern-Hamilton energy functional*. We note that K-contact metrics and Sasakian metrics are trivial critical metrics, besides we note that the critical point condition (2.3) has a tensorial character, so it holds also in the non compact case.

3 Proofs of the results

3.1 Proof of Theorem 1.1

Proof. Let $(M, \eta, g, \xi, \varphi)$ be a complete H-contact $(2n+1)$-manifold satisfying the conditions of Theorem 1.1. Consider the new contact metric structure defined by the so called D-homothetic deformation ([22]):

$$\tilde{g} = g_t = tg + t(t-1)\eta \otimes \eta, \;\; \tilde{\eta} = t\eta, \;\; \tilde{\xi} = (1/t)\xi, \;\; \tilde{\varphi} = \varphi, \quad t\text{=const.} > 0.$$

Recall that the Ricci tensors of the metrics $g, \tilde{g}$ are related by (see [11], p. 368)

$$\tilde{Ric} = Ric - 2(t-1)g + 2(t-1)(nt+n+1)\eta \otimes \eta + \frac{t-1}{t} g\big((\nabla_\xi h)\varphi + 2h, \cdot\big). \tag{3.1}$$

Since g satisfies the critical point condition (2.3). Formula (3.1) becomes

$$\tilde{Ric} = Ric - 2(t-1)g + 2(t-1)(nt+n+1)\eta \otimes \eta. \tag{3.2}$$

Now, we show that there exits $t > 0$ for which the Bakry-Emery Ricci tensor

$$\tilde{Ric}_{\tilde{\xi}} := \tilde{Ric} + \mathcal{L}_{\tilde{\xi}}\tilde{g}$$

of the Riemannian manifold $(M, \tilde{g})$ satisfies:

$$\tilde{Ric}_{\tilde{\xi}} \geq c\tilde{g}, \quad \text{for some constant } c > 0. \tag{3.3}$$

Let X be an arbitrary smooth vector field, $X = X_1 + X_2$ where X_1 is vertical, i.e. $X_1 = f\xi$, and X_2 is horizontal, i.e., $X_2 \in \ker\eta$. Since $\mathcal{L}_\xi \eta = 0$, one gets $\mathcal{L}_{\tilde{\xi}}\tilde{g} = \mathcal{L}_\xi g$, and thus

$$(\mathcal{L}_{\tilde{\xi}}\tilde{g})(X_1, \cdot) = f(\mathcal{L}_\xi g)(\xi, \cdot) = 0. \tag{3.4}$$

Since (M, η, g) is H-contact, from (3.1) follows that also $(M, \tilde{\eta}, \tilde{g})$ is H-contact, i.e., $\tilde{\xi}$ is an eigenvector of $\tilde{Q}$, and thus by using (3.4) we get

$$\tilde{Ric}_{\tilde{\xi}}(X_1, X_2) = 0. \tag{3.5}$$

Moreover, by using (3.2), (3.4) and $(2.1)_a$, we have

$$\begin{aligned}
\tilde{Ric}_{\tilde{\xi}}(X_1, X_1) &= Ric(X_1, X_1) - 2(t-1)g(X_1, X_1) + 2(t-1)(nt + n + 1)\eta(X_1)\eta(X_1) \\
&= f^2\big(Ric(\xi, \xi) + 2n(t^2 - 1)\big) \\
&= f^2\big(2n - \mathrm{tr}h^2 + 2n(t^2 - 1)\big) \\
&= f^2\big(2nt^2 - \mathrm{tr}h^2\big).
\end{aligned}$$

Since the tensors $\mathcal{L}_\xi g$ and h are related by $\mathcal{L}_\xi g = 2g(h\varphi\cdot, \cdot)$, we have $\|\mathcal{L}_\xi g\|^2 = 4\mathrm{tr}h^2$ and thus the condition $\mu \geq \frac{\|\tau\|}{\sqrt{2n}}$ gives

$$-\mathrm{tr}h^2 \geq -(n/2)\mu^2.$$

Then, from above equation we obtain

$$\tilde{Ric}_{\tilde{\xi}}(X_1, X_1) \geq f^2\big(2nt^2 - (n/2)\mu^2\big) = \frac{n}{2t^2}\big(4t^2 - \mu^2\big)\tilde{g}(X_1, X_1).$$

Therefore, if we put $t = t_1 > (\mu/2) \geq 0$ and $c_1 = \frac{n}{2t^2}\big(4t^2 - \mu^2\big) > 0$, we get

$$\tilde{Ric}_{\tilde{\xi}}(X_1, X_1) \geq c_1\tilde{g}(X_1, X_1), \quad \text{where } \tilde{g} = g_t \text{ with } t = t_1 > (\mu/2) \geq 0. \tag{3.6}$$

Now, we consider $\tilde{Ric}_{\tilde{\xi}}(X_2, X_2)$. By using (3.2), (3.4) and (1.2) we get

$$\begin{aligned}
\tilde{Ric}_{\tilde{\xi}}(X_2, X_2) &= Ric(X_2, X_2) + (\mathcal{L}_\xi g)(X_2, X_2) - 2(t-1)g(X_2, X_2) \\
&\geq \lambda\, g(X_2, X_2) - 2(t-1)g(X_2, X_2) \\
&= \frac{1}{t}(2 - 2t + \lambda)\tilde{g}(X_2, X_2).
\end{aligned}$$

Consequently, if we put $0 < t = t_2 < \frac{2+\lambda}{2}$, where $2 + \lambda > \mu \geq 0$, and $c_2 = \frac{1}{t}(2 - 2t + \lambda)$, then

$$\tilde{Ric}_{\tilde{\xi}}(X_2, X_2) \geq c_2\, \tilde{g}(X_2, X_2), \quad \text{where } \tilde{g} = g_t \text{ with } t = t_2 < \frac{2+\lambda}{2}. \tag{3.7}$$

Now, taking $t_1 = t_2 = t$ and $\tilde{g} = g_t$, where t is a positive real number satisfying $0 \leq \mu < 2t < 2 + \lambda$, then from (3.6) and (3.7), we have

$$\tilde{R}ic_{\tilde{\xi}}(X_2, X_2) \geq c_1\, \tilde{g}(X_2, X_2), \quad \text{where } c_1 = \frac{n}{2t^2}(4t^2 - \mu^2),$$
$$\tilde{R}ic_{\tilde{\xi}}(X_2, X_2) \geq c_2\, \tilde{g}(X_2, X_2), \quad \text{where } c_2 = \frac{1}{t}(2 - 2t - \lambda),$$
$$\tilde{R}ic_{\tilde{\xi}}(X_1, X_2) = 0.$$

Therefore, if we put $c = min\{c_1, c_2\}$, then $\tilde{R}ic_{\tilde{\xi}}(X, X) \geq c\, \tilde{g}(X, X)$, that is, (3.3). On the other hand, for any $t > 0$, $(\tilde{\eta}, \tilde{g} = g_t)$ is again a complete contact metric structure on M ([22], Lemma 11.1). Then, $(M, \tilde{g})$ is a complete Riemannian manifold, $\tilde{\xi}$ is a unit vector field, and the condition (3.3) is satisfied. Hence, by Theorem A, we obtain that M is compact and the fundamental group $\pi_1(M)$ is finite. Besides, the homology group $H_1(M, \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$, for which the first Betti number $b_1(M) = 0$. □

Proof. of Corollary 1.2
If the contact metric structure is K-contact, that is, $\mathcal{L}_\xi g = 0$, then for $\|\tau\| = 0$ and $\mu = 0$ the condition (1.2) becomes $Ric \geq \lambda\, g >$ with $\lambda > -2$. Thus, Corollary 1.2 follows from Theorem 1.1. □

3.2 Proof of Theorem 1.3, Corollary 1.4 and Corollary 1.5

Before to start the proof of Theorem 1.3, we recall that a vector field X_0 on a Riemannian manifold (M, g) is called an *infinitesimal harmonic transformation* if the one-parameter group of local transformations of (M, g) generated by X_0 are local harmonic diffeomorphisms. Moreover, X_0 is infinitesimal harmonic transformation if and only if holds the condition (see [16], p.574)

$$\Delta X_0 = 2QX_0,$$

where Δ is the Hodge-Laplacian.

Proof. (*of Theorem 1.3*)
Recall the following Yano's formula (see, for example, [19] Prop. 9.6)

$$Ric(X_0, X_0) = \|\nabla X_0\|^2 - \frac{1}{2}\|\mathcal{L}_{X_0} g\|^2 + (divX_0)^2 + div(\nabla_{X_0} X_0) - div\big((divX_0)X_0\big). \tag{3.8}$$

Since X_0 is a unit, geodesic, divergence-free vector field, Yano's formula (3.8) becomes

$$Ric(X_0, X_0) = \|\nabla X_0\|^2 - \frac{1}{2}\|\mathcal{L}_{X_0} g\|^2. \tag{3.9}$$

Besides,

$$\big(\mathcal{L}_{X_0} g\big)(X_0, X) = (1/2)X\big(g(X_0, X_0)\big) + g(\nabla_{X_0} X_0, X) = 0. \tag{3.10}$$

From definition of almost Ricci soliton, by using (3.10), we get that X_0 is an eigenvector of the Ricci operator, i.e.,

$$QX_0 = \lambda X_0. \tag{3.11}$$

Now, we show that the properties $1), 2), 3)$ are equivalent.
$1) \Rightarrow 2)$.

Suppose that X_0 is a Killing vector field. Then X_0 satisfies the equation (see, for example, [19] p.266)

$$\bar{\Delta} X_0 = QX_0,$$

where $\bar{\Delta}$ is the rough Laplacian. Then, by Weitzenböck's formula, we have

$$\Delta X_0 = \bar{\Delta} X_0 + QX_0 = 2QX_0,$$

that is, X_0 is an infinitesimal harmonic transformation.
$2) \Rightarrow 3)$.

Suppose X_0 infinitesimal harmonic transformation, that is, $\Delta X_0 = 2QX_0$. Since $QX_0 = \lambda X_0$, we have that X_0 is an eigenvector of the Hodge-Laplacian with eigenvalue $\mu = 2\lambda$.
$3) \Rightarrow 1)$.

By using the definition of the the rough Laplacian $\bar{\Delta}$ and the definition of the Laplace-Beltrami Δ operator acting on the functions, one gets (cf. also [7] Lemma 2.15)

$$g(\bar{\Delta} X_0, X_0) = \frac{1}{2}\Delta \left\| X_0 \right\|^2 + \left\| \nabla X_0 \right\|^2. \tag{3.12}$$

Then, since X_0 is an eigenvector of the Hodge-Laplacian, that is $\Delta X_0 = \mu X_0$, with eigenvalue $\mu \leq 2\lambda$, by Weitzenböck's formula and (3.12), we get

$$g(\bar{\Delta} X_0, X_0) = \frac{1}{2}\Delta \left\| X_0 \right\|^2 + \left\| \nabla X_0 \right\|^2 = \left\| \nabla X_0 \right\|^2$$
$$g(\bar{\Delta} X_0, X_0) = g(\Delta X_0 - QX_0, X_0) = \mu - \lambda.$$

Then, these formulas and (3.9) give

$$\lambda = Ric(X_0, X_0) = \mu - \lambda - \tfrac{1}{2}\left\| \mathcal{L}_{X_0} g \right\|^2,$$

that is,

$$\left\| \mathcal{L}_{X_0} g \right\|^2 = 2(\mu - 2\lambda) \leq 0.$$

Therefore, X_0 is Killing and (M, g) is an Einstein Riemannian manifolds. □

Remark 3.1. Let (M, g, X_0, λ) be a Ricci soliton where the potential vector field X_0 is a unit vector field. Then, in this case, by using Proposition 3.1 and Theorem 3.2 of [20], we have that X_0 is Killing if and only if X_0 is geodesic and divergence free.

Remark 3.2. Let (M, g, X_0, λ) be an almost Ricci soliton. We observe that if the potential vector field X_0 is geodesic and eigenvector of the Hodge-Laplacian ($\Delta X_0 = \mu X_0$), then

$$2(\mu - 2\lambda)X_0 + (m-2)\nabla\lambda = 0, \quad dimM = m. \tag{3.13}$$

In fact, for an almost Ricci soliton, since X_0 is geodesic, $QX_0 = \lambda X_0$; besides holds the following ([2], Lemma 2):

$$\bar{\Delta}X_0 = QX_0 + \tfrac{m-2}{2}\nabla\lambda = \lambda X_0 + \tfrac{m-2}{2}\nabla\lambda.$$

Then, this equation, $\Delta X_0 = \mu X_0$ and Weitzenböck's formula, imply (3.13).

Proof. (of Corollary 1.4)
Recall that the Reeb vector field ξ of a contact metric $(2n+1)$-manifold is a unit, geodesic, divergence-free vector field. Besides $\Delta\xi = \mu\xi$, $\mu = 4n$ ([18], Corollary 3.7). Thus, if (g, ξ, λ) is an almost Ricci soliton, by (3.13) with $X_0 = \xi$ and Lemma 2.1, we have $\mu = 2\lambda$. Then, by Theorem 1.3, ξ is Killing, and thus M is an Einstein K-contact manifold. □

Proof. (of Corollary 1.5)
By using the notations of Section 2, $(\eta = -\theta, \xi = -T, \varphi, g_\theta)$ is a contact metric structure. Then, if (g_θ, ξ, λ) is an almost Ricci soliton, by Corollary 1.4 we get that g_θ is Einstein and ξ is Killing. Besides, the contact metric structure satisfies the integrability condition (2.2). Therefore, Theorem 11 of [21] gives that the structure (η, ξ, φ) is normal, and so $(\eta = -\theta, \xi = -T, \varphi, g_\theta)$ is Sasakian. □

3.3 Natural contact metric structures - proof of Theorem 1.6

g-natural contact metric structures
An interesting geometric situation in which a distinguished vector field, namely the *geodesic flow* $\tilde{\xi}$ (also called the *geodesic spray*), appears in a natural way, is given by the unit tangent sphere bundle T_1M of a Riemannian manifold (M, g). This vector field is the Reeb vector field of the natural contact metric structures on unit tangent sphere bundles of which we now give a quick presentation (for more details we refer to the papers [20], [1] and references therein).

Let (M, g) be an n-dimensional Riemannian manifold. Riemannian g-natural metrics (also called *Riemannian natural metrics*) form a wide family of Riemannian metrics on TM. These metrics depend on several smooth functions from $[0, +\infty)$ to $\mathbb{R}$ and as their name suggests, they arise from a very "natural" construction starting from the Riemannian metric g over M. The *Sasaki metric*, the *Cheeger-Gromoll metric*, the *Kaluza-Klein metrics*, and the so called *metrics of Kaluza-Klein type*, are special cases of Riemannian g-natural metrics.

Let G be a Riemannian g-natural metric on TM. The *unit tangent sphere bundle* over a Riemannian manifold (M, g), is the hypersurface

$$T_1M = \{(x, u) \in TM | g_x(u, u) = 1\}.$$

The tangent space of T_1M, at a point $(x, u) \in T_1M$, is given by

$$(T_1M)_{(x,u)} = \{X^h + Y^v : X \in M_x, Y \in \{u\}^\perp \subset M_x\},$$

where X^h and X^v are the *horizontal lift* and the *vertical lift* of X. By X^{t_G} we denote the *tangential lift* with respect to G, of a vector $X \in M_x$ to $(x,u) \in T_1M$; of course if $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$. The *geodesic flow on* T_1M, that we denote by $\tilde{\xi}$, is the restrictions of the geodesic flow on TM to its hypersurface T_1M, thus

$$\tilde{\xi}_{(x,u)} = u^h_{(x,u)}.$$

We call *g-natural metrics on* T_1M the restrictions $\tilde{G}$ of g-natural metrics G of TM to its hypersurface T_1M. These metrics possess a simpler form. Precisely, $\tilde{G}$ is completely determined by the identities

$$\tag{3.1} \begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)\, g_x(X,Y) + d\, g_x(X,u) g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^{t_G}) &= b\, g_x(X,Y), \\ \tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= a\, g_x(X,Y) - \frac{\phi}{a+c+d}\, g_x(X,u) g_x(Y,u), \end{cases}$$

for all $(x,u) \in T_1M$ and $X,\ Y \in M_x$, where a,b,c,d are constants and satisfy the following inequalities:

$$a > 0, \quad a(a+c) - b^2 > 0 \quad \text{and} \quad \phi := a(a+c+d) - b^2 > 0.$$

The standard *Sasaki metric* $\tilde{G}_S$ is defined by $a = 1$ and $b = c = d = 0$, a *Kaluza-Klein metric* is defined by $b = d = 0$, and *Kaluza-Klein type metric*, that is, horizontal and tangential lifts are mutually orthogonal with respect to $\tilde{G}$, is defined by $b = 0$.

In [1] Abbassi et al. have shown that there is a family of contact metric structures $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$ over T_1M, called *g-natural contact metric structures*. More precisely the set $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$ is described as follows:

$$\tilde{G} \text{ is defined by (3.1)}, \quad \tilde{\xi}_{(x,u)} = \varrho u^h, \quad \tilde{\eta}(X^h) = (1/\varrho) g(X,u), \quad \tilde{\eta}(X^{t_G}) = b\, \varrho\, g(X,u),$$

where ϱ being a positive constant satisfying

$$\tag{3.2} (1/\varrho^2) = 4(a(a+c) - b^2) = (a+c+d),$$

and $\tilde{\varphi}$ is completely determined by the relation $d\tilde{\eta} = \tilde{G}(\cdot, \tilde{\varphi})$. By using formula (3.2) we get the parameter d as a function of a,b,c, thus the g-natural contact metric structures depend on three real parameters. For $a = \frac{1}{4}, b = c = 0$, and consequently $d = 0$ and $\varrho = 2$, we get the *standard contact metric structure* on T_1M induced from the Sasaki metric $\tilde{G}_S$ (see [3]).

Proof. (**of Theorem 1.6**)
Suppose that $(\tilde{G}, \tilde{\xi}, \lambda)$ is an almost Ricci soliton. Then, by Corollary 1.4, $(T_1M, \tilde{G}, \tilde{\eta})$ is K-contact and Einstein. Since $(T_1M, \tilde{G}, \tilde{\eta})$ is K-contact, by Theorem 2 of [1], we obtain that (M,g) is of constant sectional curvature $\kappa = (a+c)/a > 0$, $\tilde{G}$ is of Kaluza-Klein type (i.e., $b = 0$) and $(T_1M, \tilde{G}, \tilde{\eta})$ is Sasakian. On the other hand, for a Riemannian manifold of constant sectional curvature $\kappa = (a+c)/a > 0$, by using (3.2), the natural contact metric structure over T_1M are exactly the ones determined by Riemannian natural metrics defined by the parameters

$$a > 0, \quad b = 0, \quad c = (\kappa - 1)a, \quad \text{and thus} \quad d = (a+c)(4a-1) - b^2 = \kappa a(4a-1)\,.$$

Then, in this case, the Ricci tensor is given by (4.9) of [20], that is,

$$\tilde{Ric} = \alpha\, \tilde{G} + \beta\, \tilde{\eta} \otimes \tilde{\eta}, \quad \text{where} \quad \alpha = \frac{(-2a+n-1)}{a} \quad \text{and} \quad \beta = \frac{(2an-n+1)}{a}\,.$$

Since $\tilde{G}$ is Einstein, must be $\beta = 0$, that is, $a = (n-1)/2n$, thus $d = \kappa(n-1)(n-2)/2n^2$ and $\lambda = \alpha = 2(n-1)$. In particular, $\tilde{G}$ is Kaluza-Klein, that is, $b = d = 0$ (equivalently, $a = 1/4$, $b = 0$ and $c = (\kappa-1)/4$) if and only if $n = 2$ and $\kappa = 1+4c > 0$. Finally, $\tilde{G}$ is induced from the Sasaki metric $\tilde{G}_S$, that is, $a = 1/4, b = c = 0$, if and only if $n = 2$ and $\kappa = 1$. □

Remark 3.3. Theorem 1.6 extends Theorem 4.3 of [20] where $(\tilde{G}, \tilde{\xi}, \lambda)$ was considered as a Ricci soliton.

Acknowledgements. The author is member of INDAM-GNSAGA (Italy).

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