Study on warped product of screen real lightlike submanifolds of a Golden semi-Riemannian manifold

G. Shanker, R. Kaur and A. Yadav

Abstract. The geometry of screen-real lightlike submanifolds of a golden semi-Riemannian manifold is investigated in this research. On screen semi-invariant lightlike submanifolds of a golden semi-Riemannian manifold, the intergrability conditions of distributions $S(TN)$ and $Rad(TN)$ are obtained. We also deduce the necessary and sufficient requirements for the aforesaid distributions to be completely geodesic foliations. We also look at the conditions of a warped product and provide an example.

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1 Introduction

Semi-Riemannian manifolds and their submanifolds are generally recognised to be more difficult to understand than Riemannian manifolds and their submanifolds. The induced metric on the submanifolds of semi-Riemannian manifolds is found to have two cases: non-degenerate and degenerate. There is no complexion to do calculus on these submanifolds when the submanifolds are non-degenerate. If a submanifold has a degenerate metric, the intersection of the tangent and normal bundles is not trivial. As a result, it's impossible to create numerous buildings that are unique and have the same character as ambient space structures.

Lightlike geometry is the study of degenerate submanifolds. Mathematicians and physicists are interested in it since it is widely used as a tool to explain the theory of relativity. In 1996, Duggal and Bejancu [7] accumulated all the researches of lightlike geometry. Many research articles on lightlike geometry since then have been published.

Because of its application in paintings, photographs, temples, and fractals, the root $x = \dfrac{1+\sqrt{5}}{2}$ of the equation $x^2 - x - 1 = 0$, sometimes known as the golden ratio, is particularly intriguing.The Golden ratio has been discovered in musical compositions,

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harmonic sound frequency ratios, and human body dimensions. In modern physical study, the Golden ratio is becoming more important, and it plays a vital role in atomic physics. Flower petal is an example of the Golden Ratio. A flower's number of petals is usually one of the following: 3,5,8,13,21,34, or 55. The flower, for example, has three petals, buttercups have five, chicory has 21, daisies have 34 or 55 petals, and so on. In [3, 5, 9, 10, 11, 12, 13, 14], we look at a variety of metallic and golden we look at a variety of metallic and golden Riemannian manifolds. Moreover, the geometry of numerous submanifolds of metallic and golden semi-Riemannian manifolds has been studied [1, 15, 16]. The presence of warped product irrotational screen-real lightlike submanifolds of metallic semi-Riemannian manifolds was investigated in [17].
Crasmeranue and Hretcanu [2] created the golden manifold by defining the tensor ϕ on a Riemannian manifold such that $\phi^2 = \phi + 1$. Later, Spinadel [4]-[6] introduced a generalisation of golden means called metallic means. If p and q are both positive integers, the positive solutions of the equation $y^2 - py - q = 0$ are known as metallic means and

$$\sigma_{p,q} = \frac{\sqrt{p^2 - 4q}}{2}$$

is known as (p, q) metallic number.
For $p = q = 1$, we have the Golden semi-Riemannian manifold.
This paper is divided into the following sections:
In the second section, we review the fundamentals of lightlike submanifolds and define golden semi-Riemannian manifolds. The intergrability criteria of the distributions $S(TN)$ and $RadTN$ are obtained in section 3. We also deduce the necessary and sufficient conditions for the aforementioned distributions to be entirely geodesic foliations in section 4. We also define the conditions of a warped product in section 5.

2 Preliminaries

A semi-Riemannian manifold $(\tilde{\Omega}^{n+m}, \tilde{g})$ has a submanifold $(\Omega^n, \tilde{g})$. If the induced metric $\tilde{g}$ is degenerate [8], then with constant index q $(1 \leq q \leq n + m - 1,\ n, m \geq 1)$ is known as degenerate(lightlike) submanifold.
Due to the degenerate generate induced metric on $T\Omega$, for any $u \in \Omega$, there exists non zero intersection of $T_u\Omega$ (n-dimensional) and $T_u\Omega^\perp$ (m-dimensional), which is called $Rad(T\Omega)$. A lightlike submanifold is known as r-lightlike, if there exist a smooth distribution $Rad(T\Omega)$ of rank $r > 0$, such that every member u of Ω goes to r-dimension subspace $Rad(T_u\Omega)$ of $T_u\Omega$. Let $S(T\Omega)$ and $S(T\Omega^\perp)$ are non-degenerate complementary subbundles of $Rad(T\Omega)$ in $T\Omega$ and $T\Omega^\perp$ respectively. Let $ltr(T\Omega)$and $tr(T\Omega)$be complementary but not orthogonal vector bundles to $Rad(T\Omega)$ in $S(T\Omega^\perp)^\perp$ and $T\Omega$ in $T\tilde{\Omega}|_\Omega$ respectively.
Then, the orthogonal decomposition of $tr(T\Omega)$ and $T\tilde{\Omega}|_\Omega$ are given by(for detail see [8])

$$tr(T\Omega) = ltr(T\Omega) \perp S(T\Omega^\perp) \tag{2.1}$$

and

$$T\tilde{\Omega}|_\Omega = T\Omega \oplus tr(T\Omega) = [Rad(T\Omega) \oplus ltr(T\Omega)] \perp S(T\Omega) \perp S(T\Omega^\perp) \tag{2.2}$$

respectively.

Theorem 2.1. *[8] Let $(\tilde{\Omega}, \tilde{g})$ be a semi-Riemannian manifold $(\Omega, \tilde{g}, S(T\Omega), S(T\Omega^{\perp}))$ be its r-lightlike submanifold. Then there exists a vector bundle $ltr(T\Omega)$ and, for a coordinate neighbourhood u of Ω, there exist a basis of $\Gamma(ltr(T\Omega)|u)$ contains a smooth section $\{N_i\}$ of $S(T\Omega^{\perp})^{\perp}|_u$ such that*

$$\tilde{g}_{ij}(N_i, \xi_j) = \delta_{ij}, \tilde{g}_{ij}(N_i, N_j) = 0, \tag{2.3}$$

for any i, $j \in \{1, 2, ..., r\}$, where $\{\eta_i\}(1 \leq i \leq r,)$ is a lightlike basis of $\Gamma(Rad(T\Omega))$.

For any $T_1,\ T_2 \in \Gamma(T\Omega)$ and $\tilde{W} \in \Gamma(tr(T\Omega))$, the Gauss and Weingarten formula are

$$\tilde{\nabla}_{T_1} T_2 = \nabla_{T_1} T_2 + h(T_1, T_2), \tag{2.4}$$

$$\tilde{\nabla}_{T_1} \tilde{W} = -A_{\tilde{W}} T_1 + \nabla^{\perp}_{T_1} \tilde{W}, \tag{2.5}$$

where $\{\nabla_{T_1} T_2, A_{\tilde{W}} T_1\}$ and $\{h(T_1, T_2), \nabla^{\perp}_{Y} \tilde{W}\}$ belong to $\Gamma(T\Omega)$ and $\Gamma(tr(T\Omega))$ respectively, and ∇ is induced connection on Ω. . Further (2.4) and (2.5) reduce to

$$\tilde{\nabla}_{T_1} T_2 = \nabla_{T_1} T_2 + h^l(T_1, T_2) + h^s(T_1, T_2), \tag{2.6}$$

$$\tilde{\nabla}_{T_1} N = -A_N T_1 + \nabla^l_{T_1}(N) + D^s(T_1, N),\ N \in \Gamma(ltr(T\Omega)), \tag{2.7}$$

$$\tilde{\nabla}_{T_1} \tilde{W}_1 = -A_{\tilde{W}_1} T_1 + \nabla^s_{T_1}(\tilde{W}_1) + D^l(T_1, \tilde{W}_1),\ \tilde{W}_1 \in \Gamma(S(T\Omega^{\perp})). \tag{2.8}$$

By the use of metric connection $\tilde{\nabla}$ and (2.4), (2.5), (2.6), (2.7), (2.8), we obtain the following conditions:

$$\tilde{g}(h^s(T_1, T_2), \tilde{W}^{'}) + \tilde{g}(T_1, D^l(T_2, \tilde{W}^{'})) = \tilde{g}(A_{\tilde{W}'} T_1, T_2), \tag{2.9}$$

$$\tilde{g}(h^l(T_1, T_2), \eta) + \tilde{g}(T_1, h^l(T_2, \eta)) = -\tilde{g}(T_1, \nabla_{T_2} \eta), \tag{2.10}$$

for any $\eta \in \Gamma(Rad(T\Omega))$ $T_1, T_2 \in \Gamma(T\Omega)$, and $\tilde{W}^{'} \in \Gamma(S(T\Omega^{\perp}))$.
Since the induced connection is not necessarily the Levi-Civita connection, for any $T_1', T_2', T_3' \in \Gamma(T\Omega)$ and $T_1, T^{'} \in \Gamma(tr(T\Omega))$ we have the following formulas

$$(\nabla_{T_1'} \tilde{g})(T_2', T_3') = \tilde{g}(h^l(T_1', T_2'), T_3') + \tilde{g}(h^l(T_1', T_3'), T_2'). \tag{2.11}$$

Let S denote projection map on $S(T\Omega)$ from $T\Omega$. Then, we have the following equations, for any $T_1, T_2 \in \Gamma(T\Omega)$ and $\eta \in \Gamma(Rad(T\Omega))$:

$$\nabla_{T_1} S T_2 = \nabla^*_{T_1} S T_2 + h^*(T_1, S T_2). \tag{2.12}$$

$$\nabla_{T_2} \xi = A^*_{\eta} T_2 + \nabla^{*t}_{T_2}(\eta), \tag{2.13}$$

where $\{h^*(T_1, P' T_2), \nabla^{*t}_{T_2}(\eta)\}$ and $\{\nabla^*_{T_1} S T_2, A^*_{\eta} T_2\}$ belong to $\Gamma(Rad(T\Omega))$ and $\Gamma(S(T\Omega)))$ respectively.

3 Golden semi-Riemannian manifold

Definition 3.1. If (Ω', g') is a semi-Riemannian manifold, then Ω' is called golden semi-Riemannian manifold if $(1,1)$ tensor field P' exists on Ω' and Ω' is a semi-Riemannian manifold, s.t

$$P'^2 = P' + I, \tag{3.1}$$

where I denotes the Ω' identity map.

$$\tilde{g}(P'\tilde{W}, T_1) = (\tilde{g}, \tilde{W}, P'T_1). \tag{3.2}$$

The semi-Riemannian metric is known as P'-compatible, while the golden semi-Riemannian manifold is known as $(\Omega', \tilde{g}, P')$. Also we get

$$\nabla'_{\tilde{W}} P'T_1 = P'\nabla'_{\tilde{W}} T_1. \tag{3.3}$$

In addition, (3.3) is equal to (3.4) if P' is a golden structure.

$$\tilde{g}(P'\tilde{W}, P'T_1) = \tilde{g}(P'\tilde{W}, T_1) + \tilde{g}(\tilde{W}, T_1), \tag{3.4}$$

for any $\tilde{W}, T_1 \in \Gamma(T\Omega')$.

4 Screen-real lightlike submanifolds

Definition 4.1. A lightlike submanifold of a golden semi-Riemannian manifold $(\Omega, \tilde{g}, S(T\Omega))$ is $(\tilde{\Omega}, \tilde{g}, P')$. If meets the following requirements, it is known as a screen-real lightlike submanifold:

$$P'(Rad(T\Omega) = Rad(T\Omega) \ \ \& \ \ P'(S(T\Omega)) \subseteq S(T\Omega^{\perp}).$$

Clearly,

$$P'(ltr(T\Omega) = ltr(T\Omega) \ \ \& \ \ P'(\mu) = \mu.$$

We get the breakdown of distributions from the above conditions.

$$T\Omega|_{\Omega} = [Rad(T\Omega) \oplus ltr(T\Omega)] \oplus_{orth.} S(T\Omega) \oplus_{orth.} P'(S(T\Omega)) \oplus_{orth.} \mu. \tag{4.1}$$

For any $T_1 \in \Gamma(T\Omega)$, by use of (4.1), we get

$$T_1 = R'T_1 + S'T_1,$$

where R' and S' are $Rad(T\Omega)$ and $S(T\Omega)$ projection maps, respectively. We get, applying P' to the above equation and using (4.1).

$$P'T_1 = R'T_1 + S''T_1, \tag{4.2}$$

where $P'R'T_1 = R'T_1$, $P'S'T_1 = S''T_1$ and R', S'' are projection maps on $Rad(T\Omega)$ and $S(T\Omega^{\perp})$ respectively.
For any $w' \in tr(T\Omega)$,

$$P'(w') = Bw' + C_1w' + C_2w' + C_3w', \tag{4.3}$$

where the projection maps B, C_1, C_2 and C_3 are defined on $S(T\Omega)$, $ltr(T\Omega)$, $P'S(T\Omega)$ and μ respectively.
For any $w_1' \in \Gamma(ltr(T\Omega))$, $w_2' \in \Gamma(P'S(T\Omega))$ and $w_3' \in \Gamma(\mu)$, (4.3) reduces, respectively

$$P'(w_1') = C_1 w_1', \tag{4.4}$$

$$P'(w_2') = B w_2' + C_2 w_2', \tag{4.5}$$

$$P'(w_3') = C_3 w_3'. \tag{4.6}$$

Example 4.2. Assume $(\tilde{\Omega} = \mathbb{R}^5_2, \tilde{g}, P')$ is a six-dimensional semi-Euclidean space, and $\tilde{g}$ is a semi-Euclidean metric with the signature $(+ - + + +)$. $P'(z_1, z_2, z_3, z_4, z_5) = (\sigma z_1, \sigma z_2, (1-\sigma)z_3, \sigma z_4, \sigma z_5)$, where $(z_1, z_2, z_3, z_4, z_5)$ is the $\mathbb{R}^5_2$ standard coordinate system. The fact that P' is a golden structure may then be simply proven.
Let us define a Ω submanifold of $\tilde{\Omega}$ that has the property that

$$z_1 = Sinh\sigma y_1,$$

$$z_2 = Cosh\sigma y_1,$$

$$z_3 = \sigma y_4,$$

$$z_4 = y_4,$$

$$z_5 = y_1,$$

Then, for the above submanifold, we may find the tangent vectors
$Q_1 = Sinh\sigma \frac{\partial}{\partial y_1} + Cosh\sigma \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_5}$,
$Q_2 = \sigma \frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_4}$,
such that $T\Omega$= Span$\{Q_1, Q_2\}$. Definitely, Ω is a lightlike submanifold with
Rad$(T\Omega)$ = Span $\{Q_1 = \eta\}$,
S$(T\Omega)$ = Span $\{Q_2\}$,
ltr$(T\Omega)$ = Span $\{\mathcal{N} = \frac{1}{2}(-Sinh\sigma \frac{\partial}{\partial y_1} - Cosh\sigma \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_5})\}$,
S$(T\Omega^{\perp})$ = Span $\{\mathcal{W} = -\frac{\partial}{\partial y_3} + \sigma \frac{\partial}{\partial y_4}\}$
where $P'\eta = \sigma\eta$, $P'(\mathcal{N}) = \sigma\mathcal{N}$, $P'(Q_2) = \mathcal{W}$. Because P' satisfies,

$$P'(Rad(T\Omega)) = Rad(T\Omega),$$

and

$$P'(S(T\Omega)) = S(T\Omega^{\perp}).$$

Ω is a screen real lightlike submanifold.

Theorem 4.1. *Let $(\Omega, \tilde{g}, S(T\Omega))$ be a golden semi-Riemannian manifold with a screen-real lightlike submanifold, then following equations are hold*

$$R\nabla_{T_1} T_2 + B(h^s(T_1, T_2)) = \nabla_{T_1} RT_2 - A_{S'T_2} T_1,$$

$$C_1 h^l(T_1, T_2) = h^l(T_1, RT_2) + D^l(T_1, S^{''} T_2)$$

and

$$h^s(T_1, RT_2) + \nabla^s_{T_1} S' T_2 = S'' \nabla_{T_1} T_2 + C_2 h^s(T_1, T_2) + C_3 h^s(T_1, T_2).$$

Proof. Using (4.2)-(4.6) in $(\tilde{\nabla}_{T_1}P')T_2 = 0$, for any $T_1, T_2 \in \Gamma(T\Omega)$ we obtain

$$\nabla_{T_1}RT_2 + h^l(T_1, RT_2) + h^s(T_1, RT_2) - A_{S'T_2}T_1 + \nabla^s_{T-1}S''T_2 + D^l(T_1, S^{''}T_2) = R\nabla_{T_1}T_2$$
$$+S''\nabla_{T_1}T_2 + C_1h^l(T_1, T_2) + B(h^s(T_1, T_2)) + C_2h^s(T_1, T_2) + C_3h^s(T_1, T_2). \quad (4.7)$$

By equating tangential, $ltr(T\Omega)$ and $S(T\Omega^{\perp})$ components in above equation, we get required results. □

Theorem 4.2. *Let $(\Omega, \tilde{g}, S(T\Omega))$ be a screen-real lightlike submanifold of a golden semi-Riemannian manifold, then*

(i) The distribution $Rad(T\Omega)$ is integrable if and only if

$$h^s(P'\eta_2, \eta_1) = h^s(P'\eta_1, \eta_2).$$

(ii) $S(T\Omega)$ is integrable if and only if

$$R(-A_{P'T_2}T_1 + A_{P'T_1}T_2) = (h^*(T_1, T_2) - h^*(T_2, T_1)).$$

Proof. (i) $Rad(T\Omega)$ is integrable if and only if, for any $\eta_1, \eta_2 \in \Gamma(Rad(T\Omega))$ and $T_1 \in \Gamma(S(T\Omega))$,

$$\tilde{g}([\eta_1, \eta_2], T_1) = 0.$$

Expanding $\tilde{g}([\eta_1, \eta_2], T_1)$, using (3.4), we get

$$\tilde{g}([\eta_1, \eta_2], T_1) = \tilde{g}(P'(\tilde{\nabla}_{\eta_1}\eta_2 - \tilde{\nabla}_{\eta_2}\eta_1), P'T_1) - \tilde{g}(\tilde{\nabla}_{\eta_1}\eta_2 - \tilde{\nabla}_{\eta_2}\eta_1, P'T_1). \quad (4.8)$$

Using (2.8), (2.6) and (2.12) in (4.8), we get

$$\tilde{g}([\eta_1, \eta_2], T_1) = \tilde{g}(h^s(P'\eta_2, \eta_1) - h^s(P'\eta_1, \eta_2), P'T_1). \quad (4.9)$$

From (4.11), we obtain $\tilde{g}([\eta_1, \eta_2]T_1) = 0$ if and only if

$$h^s(P'\eta_2, \eta_1) = h^s(P'\eta_1, \eta_2)$$

(ii) $S(T\Omega)$ is integrable if and only if, for any $T_1, T_2 \in \Gamma(S(T\Omega))$ and $H \in \Gamma(ltr(T\Omega))$,

$$\tilde{g}([T_1, T_2], H) = 0.$$

Expanding $\tilde{g}([T_1, T_2], H)$, using (3.4), we have

$$\tilde{g}([T_1, T_2], H) = \tilde{g}(P'(\tilde{\nabla}_{T_1}T_2 - \tilde{\nabla}_{T_2}T_1), P'H) - \tilde{g}(\tilde{\nabla}_{T_1}T - [2] - \tilde{\nabla}_{T_2}T_1, P'H). \quad (4.10)$$

By the use of (2.8), (2.6) and (2.12) in (4.10), we have

$$\tilde{g}([T_1, T_2], H) = \tilde{g}(-A_{P'T_2}T_1 + A_{P'T_1}T_2, P'H) - \tilde{g}(h^*(T_1, T_2) - h^*(T_2, T_1), P'H). \quad (4.11)$$

From (4.11), we obtain $\tilde{g}([T_1, T_2], H) = 0$ if and only if

$$\tilde{g}(R(-A_{P'T_2}T_1 + A_{P'T_1}T_2), P'H) = \tilde{g}(h^*(T_1, T_2) - h^*(T_2, T_1), P'H),$$

i.e.,

$$R(-A_{P'T_2}T_1 + A_{P'T_1}T_2) = (h^*(T_1, T_2) - h^*(T_2, T_1)).$$

□

Theorem 4.3. *Let $(\Omega, \tilde{g}, S(T\Omega))$ be a screen-real lightlike submanifold of a golden semi-Riemannian manifold, then*

(i) The radical distribution $(T\Omega)$ defines a totally geodesic foliation if and only if

$$h^s(\eta_1, P'\eta_2) = h^s(\eta_1, \eta_2).$$

(ii) The screen distribution $S(T\Omega)$ defines a totally geodesic foliation if and only if

$$R(A_{P'T_2}T_1) = -h^*(T_2, T_1).$$

Proof. (i) $Rad(T\Omega)$ defines totally geodesic foliations if and only if, for any $\eta_1, \eta_2 \in \Gamma(Rad(T\Omega))$ and $T_1 \in \Gamma(S(T\Omega))$, $\tilde{g}(\nabla_{\eta_1}\eta_2, T_1) = 0$.
Using (3.4), we get

$$\tilde{g}(\nabla_{\eta_1}\eta_2, T_1) = \tilde{g}(\tilde{\nabla}_{\eta_1}\eta_2, T_1) = \tilde{g}(P'\tilde{\nabla}_{\eta_1}\eta_2, P'T_1) - \tilde{g}(\tilde{\nabla}_{\eta_1}\eta_2, P'T_1). \tag{4.12}$$

Using (2.8), (2.6) and (2.12) in (4.12), we obtain

$$\tilde{g}(\tilde{\nabla}_{\eta_1}\eta_2, T_1) = \tilde{g}(h^s(\eta_1, P'\eta_2), P'T_1) - \tilde{g}(h^s(\eta_1, \eta_2), P'T_1). \tag{4.13}$$

From (4.13), $\tilde{g}(\tilde{\nabla}_{\eta_1}\eta_2, T_1) = 0$ if and only if

$$h^s(\eta_1, P'\eta_2) = h^s(\eta_1, \eta_2).$$

(ii) $S(T\Omega)$ defines totally geodesic foliations if and only if, for any $T_1, T_2 \in S(T\Omega)$ and $H \in ltr(T\Omega)$, $\tilde{g}(\nabla_{T_1}T_2, H) = 0$.
Using (3.4), we get

$$\tilde{g}(\nabla_{T_1}T_2, H) = \tilde{g}(\tilde{\nabla}_{T_1}T_2, H) = \tilde{g}(P'\tilde{\nabla}_{T_1}T_2, P'H) - \tilde{g}(\tilde{\nabla}_{T_1}T_2, P'H). \tag{4.14}$$

Using (2.8), (2.6) and (2.12) in (4.14), we obtain

$$\tilde{g}(\tilde{\nabla}_{T_1}T_2, H) = \tilde{g}(-A_{P'T_2}T_1, P'H) - \tilde{g}(-A^*_{T_2}T_1 + h^*(T_2, T_1), P'H). \tag{4.15}$$

From (4.15), $\tilde{g}(\nabla_{T_1}T_2, H) = 0$ if and only if

$$R(A_{P'T_2}T_1) = -h^*(T_2, T_1).$$

□

5 Warped Product lightlike submanifolds

Definition 5.1. [8] Let $\Omega = \Omega_1 \times_\gamma \Omega_2$, be a product manifold, where (Ω_1, g_1) is m dimension totally lightlike submanifold and (Ω_2, g_2) is n-dimension of a semi-Riemannian manifold $\tilde{\Omega}$, is known as warped product lightlike submanifold with induced degenerate metric $\breve{g}$ defined as

$$\breve{g}(T_1, T_2) = \tilde{g}_1(\sigma_{1*}T_1, \sigma_{1*}T_2) + (\gamma \; o \; \sigma_1)^2 \tilde{g}_2(\sigma_{2*}T_1, \sigma_{2*}T_2), \tag{5.1}$$

where $T_1, T_2 \in \Gamma(T\Omega)$, $\gamma \in C^\infty(\Omega_1, \mathbb{R})$, σ_1 and σ_2 are projection maps from $\Omega_1 \times \Omega_2$ to Ω_1 and to Ω_2 respectively, and $*$ denotes tangent map.

Theorem 5.1. *Let $\Omega = \Omega_1 \times_\gamma \Omega_2$ be a warped product lightlike submanifold then, for any $\eta \in \Gamma(Rad(T\Omega))$, $T_1 \in \Gamma(S(T\tilde{M}))$, $\nabla_\eta T_1 \in \Gamma(S(T\Omega))$.*

Definition 5.2. [8] An m-lightlike submanifold is called irrotational lightlike submanifold if and only if

$$h^l(\eta, T_1) = 0, \ \ h^s(\eta, T_1) = 0, \tag{5.2}$$

for any $T_1 \in S(T\Omega)$ and $\eta \in Rad(T\Omega)$.

Theorem 5.2. *The induced connection is metric if $(\Omega, g, S(T\Omega))$ is an irrotational screen-real m-lightlike submanifold of a golden semi-Riemannian manifold.*

Proof. Let ∇ be an induced connection from the ambient connection $\tilde{\nabla}$. Then ∇ is said to be a metric connection if and only if $h^l(T_1, T_2) = 0$, for any $T_1, T_2 \in \Gamma(T\Omega)$. From (5.2), $h^l(T_1, \eta) = 0$. Now, it is enough to show that $h(T_1, T_2) = 0$, if $T_1, T_2 \in S(T\Omega)$.
Using (3.4), we get

$$\tilde{g}(\tilde{\nabla}_{T_1} T_2, \eta) = \tilde{g}(\tilde{P'}\tilde{\nabla}_{T_1} T_2, \tilde{P'}\eta) - \tilde{g}(\tilde{\nabla}_{T_1} T_2, \tilde{P'}\eta). \tag{5.3}$$

Since $\tilde{\nabla}$ is a metric connection, equation (5.3) reduces to

$$\tilde{g}(h^l(T_1, T_2), \eta) = -\tilde{g}(\tilde{P'}T_2, \tilde{\nabla}_{t_1}\tilde{P'}\eta) + \tilde{g}(\tilde{P'}T_2, \tilde{\nabla}_{T_1}\eta). \tag{5.4}$$

Using (2.6) in (5.4), we get

$$\tilde{g}(h^l(T_1, T_2), \eta) = -\tilde{g}(\tilde{P'}T_2, h^s(T_1, \tilde{P'}\eta)) + \tilde{g}(\tilde{P'}T_2, h^s(T_1, \eta)). \tag{5.5}$$

Since $h^s(T_1, \tilde{P'}\eta) = 0$ and $h^s(T_1, \eta) = 0$, (5.4) becomes

$$\tilde{g}(h^l(T_1, T_2), \eta) = 0.$$

This implies $h^l(T_1, T_2) = 0$. □

Theorem 5.3. *There is no such idea as an irrotational screen-real m-lightlike submanifold that can be represented in the form of warped product lightlike submanifolds.*

Proof. Suppose there exist a class of irrotaional screen-real r- lightlike submanifolds such that any Ω in this class can be written as warped product lightlike submanifolds i.e., $\Omega = \Omega_1 \times_\gamma \Omega_2$.
Use of Theorem (4.1), we obtain

$$\tilde{g}(\tilde{\nabla}_\eta T_1, T_2) = \frac{\eta(\gamma)}{\gamma}\tilde{g}_2(T_1, T_2). \tag{5.6}$$

Since Ω is irrotational, using Theorem (5.2), we have
$h^l(T_1, T_2) = 0$, for any $T_1, T_2 \in \Gamma(T\Omega)$.
Using (2.4), we obtain

$$\tilde{g}(\eta, h^l(T_1, T_2)) = -\tilde{g}(\tilde{\nabla}_\eta T_1, T_2) = 0. \tag{5.7}$$

Using (5.6) and (5.7), we get

$$\frac{\eta(\gamma)}{\gamma}\tilde{g}_2(T_1, T_2) = 0.$$

This implies that either γ is constant function or $\tilde{g}_2$ is degenerate metric, which is a contradiction. □

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Authors' address:

Gauree Shanker, Ramandeep Kaur and Ankit Yadav
Department of Mathematics and Statistics,
Central University of Punjab, Bathinda, Punjab-151 401, India
E-mail: grshnkr2007@gmail.com , ramanaulakh1966@gmail.com , ankityadav93156@gmail.com