Solitons and gradient solitons on perfect fluid spacetime in $f(\mathcal{R}, T)$-gravity

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Abstract. This research article attempts to examine the attribute of perfect fluid spacetime in $f(\mathcal{R}, T)$ gravity with a Killing velocity vector field ρ in terms of Ricci soliton, gradient Ricci soliton, Yamabe soliton, and gradient Yamabe soliton. Besides this, we evaluate a specific situation when the potential vector field ρ is of the form of gradient, we extract a modified Poisson equation, and modified Liouville equation from the Ricci soliton equation in $f(\mathcal{R}, T)$-gravity stuffing with perfect fluids. In addition, we explore some harmonic significance of Ricci soliton on perfect fluid spacetime in $f(\mathcal{R}, T)$ gravity with a harmonic potential function Ψ.

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1 Introduction

General Relativity (GR) is the best way to study the large-scale structure of the Universe theoretically. It is observed that GR without taking into account the dark energy can not describe the acceleration of the early and late Universe. GR does not explain precisely gravity and it is quite reasonable to modify in order to get theories that admit inflation and imitate the *Dark Energy* (DE). To explain the observed cosmological dynamics, the standard approach is given by the modification of the Einstein gravitational field equations, introduced by Einstein [28, 23]. The Einstein field equations provide the best fit to the observed data, with a further assumption of another hypothetical component of the Universe know as *Dark Matter*[19].

The Universe is filled with the mysterious component called DE which is considered to be the main reason for the accelerated expansion of the Universe and balances the matter-energy ratio. This scenario inspired several mathematician and physicists to developed more mature gravity theories that flourished by the Einstein-Hilbert action and by using modified gravity theories for example $f(\mathcal{R})$-gravity [26], Gauss-Bonnet, $f(G)$-gravity [21], and $f(T)$ theory [7] etc. These theory are different from

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the standard Einstein gravity theory. These modified gravity theories may also provide the effective approximation to quantum gravity [24].

Extending the Einstein-Hilbert Lagrangian density to a function $f(\mathcal{R})$, where $\mathcal{R}$ is the Ricci scalar, allows one to extend GR to the $f(\mathcal{R})$ gravity. The equations of motion of $f(\mathcal{R})$ gravity have higher degrees and solved the issue of massive neutron stars by higher order curvature, for references see [3, 4, 5]. However, the $f(\mathcal{R})$ gravity also has some limitations as regards consistency with solar system and also fails to justify the existence of some cosmic models like stable stellar configuration (for more details see [6, 18]), which raises questions about its validity. Further investigation led to a more generalized gravity models, whose Lagrangian is an arbitrary function of the trace of the energy-momentum tensor T and Ricci scalar $\mathcal{R}$, known as $f(\mathcal{R},T)$-gravity theory proposed by Harko et al. [15]. This theory was successfully applied for a description of the late time accelerated expansion of the Universe.

On other hand, a spacetime can be designed as a 4-dimensional time orientated Lorentzian manifold M which is a particular category of pseudo-Riemannian manifolds with Lorentzian metric g. The geometry of Lorentzian manifold begins with the study of nature of vectors on the manifold. Therefore, Lorentzian manifold M becomes most fitting alternative for the study of cosmological model. In typical cosmological models, the material content of the universe is known to behave like a perfect fluid spacetime [22].

The energy-momentum tensor T of a perfect fluid spacetime is in the following form ([22], [20])

$$T_{\alpha\beta} = pg_{\alpha\beta} + (\sigma + p)\eta_\alpha\eta_\beta \tag{1.1}$$

where σ, p indicates the energy density and isotropic pressure, respectively for the perfect fluid. In modern cosmology, it is assumed as a candidate for dark energy, the acceleration of the universe expansion.

On the other hand, physical matters symmetry is specially relating to the spacetimes geometry. More specifically, the metric of symmetry usually simplifies for the classification of solutions of Einstein's field equations. An important symmetry is soliton that connected to geometrical flow of spacetimes geometry. In fact Ricci flow and Yamabe flow are used to understand the idea of kinematics.

Hamilton introduced the notions of Ricci flow and Yamabe flow [16, 17]. These are intrinsic geometric flows on a (semi) Riemnnian manifold, whose fixed points are solitons. Ricci solitons and Yamabe solitons, which are generate self-similar solutions of Ricci flow and Yamabe flow are give by

$$\frac{\partial}{\partial t}g(t) = -2\mathcal{R}ic, \qquad \frac{\partial}{\partial t}g(t) = -\mathcal{R}g(t). \tag{1.2}$$

Moreover, a metric of M is said to be Ricci soliton if it satisfies [16]

$$\frac{1}{2}\mathcal{L}_V g + \mathcal{R}ic + \lambda g = 0, \tag{1.3}$$

and is said to be Yamabe soliton if it satisfies[17]

$$\frac{1}{2}\mathcal{L}_V g = (\mathcal{R} - \lambda)g \tag{1.4}$$

for some vector filed V and a real scalar λ. Here $\mathcal{L}_V$ indicates the Lie derivative operator along the soliton vector field V, $\mathcal{R}ic$ is Ricci curvature and $\mathcal{R}$ is the scalar curvature of M. The data (g, V, λ) known as Ricci soliton and Yamabe soliton follow by the (1.3) and (1.4), respectively. A Ricci soliton and Yamabe soliton on M is said to be shrinking, expanding or steady if λ is negative, positive or zero, respectively.

Ricci soliton and Yamabe soliton with $V = \mathcal{D}\psi$ gives the gradient Ricci soliton and gradient Yamabe soliton on semi-Riemannian manifold M, where $\mathcal{D}$ denotes the gradient operator and ψ is some smooth function on M. Thus, equation (1.3) and (1.4 reduces to the following form

$$\mathcal{H}ess\psi - \mathcal{R}ic = \lambda g, \tag{1.5}$$

$$\mathcal{H}ess\psi = (\mathcal{R} - \lambda)g, \tag{1.6}$$

where $Hess$ denotes the Hessian, $\mathcal{D}$ is the gradient operator of g and the smooth function ψ is called potential function of the gradient Ricci soliton and gradient Yamabe soliton, respectively.

In $f(\mathcal{R}, T)$-gravity theory scalar fields are supposed to play a fundamental role in physics and cosmology. However, obtaining more general gravitational models with scalar fields as a source may give a better insight in the general properties of the gravitational field. In [27] Singh and Singh have reconstructed flat scalar and exponential model of $f(\mathcal{R}, T)$ gravity in scalar field cosmology

In [9, 10] Capozziello et al. discussed the perpoties of Cosmological perfect fluid in $f(R)$ gravity. Chaubey [11] is also determine some important results about $f(\mathcal{R}, \mathcal{T})$-gravity theory, which is closely related with this study.

Recently, Siddiqi et al. [33] studied $f(\mathcal{R}, \mathcal{T})$-gravity model with perfect fluid in terms of Einstein solitons. Moreover, many authors also studied perfect fluid spacetime with various solitons for more details see [1, 2, 29, 31, 30, 32, 33, 34].

The above studies inspire the author, to study $f(\mathcal{R}, T)$-gravity with perfect fluids in terms of with Ricci soliton, gradient Ricci soliton, Yamabe soliton and gradient Yamabe soliton.

2 Perfect fluid spacetime stuffing in $f(\mathcal{R}, T)$-gravity

Perfect fluid spacetime (M^4, g) satisfying $f(\mathcal{R}, T)$-gravity, depends on the physical nature of the matter field and therefore we get a theoretical model [15], we choose

$$f(\mathcal{R}, T) = \mathcal{R} + 2f(T), \tag{2.1}$$

where $f(T)$ is an arbitrary function on the trace T of the energy-momentum tensor, and the term $2f(T)$ in the gravitational action modifies the gravitational interaction between matter and curvature.

We assume a modified Einstein-Hilbert action term

$$\mathcal{H} = \frac{1}{16\pi} \int [f(\mathcal{R}, T) + \mathcal{L}_m] \sqrt{(-g)} d^4x, \tag{2.2}$$

where $f(\mathcal{R},T)$ is an arbitrary function of Ricci scalar $\mathcal{R}$ and the trace T of the energy-momentum tensor, and $\mathcal{L}_\varphi$ is the matter Lagrangian of the scalar field. The stress energy tensor of the matter is given by

$$T_{\alpha\beta} = \frac{-2\delta(\sqrt{-g})\mathcal{L}_m}{\sqrt{-g}\delta^{\alpha\beta}}. \tag{2.3}$$

Let us consider that the matter Lagrangian of the scalar field depends only on the metric tensor $g_{\alpha\beta}$, and not on its derivatives.
The variation of action (2.2) with respect to the metric tensor g_{ab} yields the field equations of $f(\mathcal{R},T)$ gravity

$$f_{\mathcal{R}}(\mathcal{R},T)Ric_{\alpha\beta} - \frac{1}{2}f(\mathcal{R},T)g_{\alpha\beta} + (g_{\alpha\beta}\nabla_c\nabla^c - \nabla_\alpha\nabla_\beta)f_{\mathcal{R}}(\mathcal{R},T) \tag{2.4}$$

$$8\pi T_{\alpha\beta} - f_T(\mathcal{R},T)T_{\alpha\beta} - f_T(\mathcal{R},T^)\psi_{\alpha\beta},$$

where $f_{\mathcal{R}}$ and f_T denote the partial derivatives of $f(\mathcal{R},T)$ with respect to $\mathcal{R}$ and T, respectively. As per usual notation, ∇_a is the covariant derivative, $\Box \equiv \nabla_c\nabla^c$ is the d'Alembert operator and $\Omega_{\alpha\beta}$ is defined by

$$\Omega_{\alpha\beta} = -2T_{\alpha\beta} + g_{\alpha\beta}\mathcal{L}_m - 2g^{lk}\frac{\partial^2\mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{lk}}. \tag{2.5}$$

If we consider $f(\mathcal{R},T) = f(\mathcal{R})$, then (2.2) and (2.3) provide the field equations of $f(\mathcal{R})$-gravity.

Let consider the matter is a perfect fluid spacetime M with isotropic pressure p, energy density σ and velocity vector η^α. Also, we know that there is no unique value of Lagrangian, therefore we assume that $\mathcal{L}_m = -p$ and using (1.1) we turn up

$$T_{\alpha\beta} = -pg_{\alpha\beta} + (\sigma + p)\eta_\alpha\eta_\beta, \tag{2.6}$$

where

$$\eta^\alpha\nabla_\beta\eta_\alpha = 0, \quad \eta_\alpha.\eta^\alpha = 1. \tag{2.7}$$

Infer (2.6), we can easily obtain the variation of stress energy in the following form

$$\Omega_{\alpha\beta} = -pg_{\alpha\beta} - 2T_{\alpha\beta}. \tag{2.8}$$

After adopting (2.1) and (2.4) we get the form

$$\mathcal{R}ic_{\alpha\beta} = \frac{1}{2}g_{\alpha\beta} - 2f^{'}(T)T_{\alpha\beta} - 2f^{'}(T)\Omega_{\alpha\beta} + f(T)g_{\alpha\beta} + 8\pi T_{\alpha\beta}. \tag{2.9}$$

In view of (2.6), (2.7) and (2.8), (2.9) becomes

$$\mathcal{R}ic_{\alpha\beta} = \left\{\frac{1}{2}\mathcal{R} + f(T) - 8p\pi\right\}g_{\alpha\beta} + \left\{(\sigma + p)(8\pi + 2f^{'}(T))\right\}\eta_\alpha\eta_\beta. \tag{2.10}$$

Contracting (2.10), we get

$$\mathcal{R} = -4[f(T) - 8p\pi] + (\sigma + p)(8\pi + 2f^{'}(T)). \tag{2.11}$$

Thus for the perfect fluid spacetime in $f(\mathcal{R},T)$-gravity the Ricci tensor is of the form

$$\mathcal{R}ic_{\alpha\beta} = ag_{\alpha\beta} + b\eta_\alpha\eta_\beta, \tag{2.12}$$

where

$$a = -\frac{1}{2}\mathcal{R} + f(T) - 8p\pi \quad and \quad b = (\sigma + p)(4\pi + f^{'}(T)). \tag{2.13}$$

Now with the help of (2.11) we get $a = -[f(T) - 8p\pi]$ and $b = (\sigma + p)(4\pi + f^{'}(T))$. Thus we have the following conclusion:

Theorem 2.1. *The Ricci tensor for the perfect fluid spacetime in $f(\mathcal{R},T)$-gravity is of the form*

$$\mathcal{R}ic_{\alpha\beta} = \left\{\frac{1}{2}\mathcal{R} + f(T) - 8p\pi\right\} g_{\alpha\beta} + \left\{(\sigma + p)(8\pi + 2f^{'}(T))\right\} \eta_\alpha\eta_\beta.$$

Corollary 2.2. *The scalar curvature tensor for the perfect fluid spacetime in $f(\mathcal{R},T)$-gravity is given by $\mathcal{R} = -4[f(T) - 8p\pi] + (\sigma + p)(8\pi + 2f^{'}(T))$.*

Now, equation (2.10) can be written in the index free notation equation as

$$\mathcal{R}ic(U,V) = ag(U,V) + b\eta(U)\eta(V) \tag{2.14}$$

and

$$QU = aU + b\eta(U)\rho, \quad \forall U \in \chi(M), \tag{2.15}$$

where $a = -[f(T) - 8p\pi]$ and $b = (\sigma + p)(4\pi + f^{'}(T))$. Now, in light of (2.11), we have

$$p + \sigma = \frac{\mathcal{R} + 4[f(T) - 8p\pi]}{[8\pi + 2f'(T)]}. \tag{2.16}$$

Thus, we can observe that the scalar curvature $\mathcal{R}$ is non vanishing in $PFST$ in $f(\mathcal{R},T)$-gravity, it follows from (2.16) that $(p + \sigma) \neq 0$. Therefore, we turn up the following result.

Theorem 2.3. *The $PFST$ in $f(\mathcal{R},T)$-gravity with non-zero scalar curvature $\mathcal{R}$ cannot concede the dark matter fluid.*

Since, if $f(T) = 0$ then $f(\mathcal{R},T)$ gravity recover $f(R)$-gravity. Thus we have the following corollary.

Corollary 2.4. *The $PFST$ in $f(\mathcal{R})$-gravity with non-zero scalar curvature $\mathcal{R}$ cannot concede the dark matter fluid and the equation of state is $5p + \sigma = \frac{\mathcal{R}}{8\pi}$.*

3 Ricci soliton on perfect fluid spacetime in $f(\mathcal{R},T)$-gravity

Consider the equation (1.3)

$$\mathcal{R}ic(U,V) = -\lambda g(U,V) - \frac{1}{2}(\mathcal{L}_\rho g)(U,V). \tag{3.1}$$

Use explicit form the Lie derivative in (1.3), we get

$$\mathcal{R}ic(U,V) = -\lambda g(U,V) - \frac{1}{2}[g(\nabla_U \rho, V) + g(U, \nabla_V \rho)], \tag{3.2}$$

for any $U, V \in \chi(M)$.
Contracting (3.2) we get

$$\mathcal{R} = -4\lambda - div\rho. \tag{3.3}$$

In view of (2.11) and (3.3), we turn up

$$4a + b = -4\lambda - div\rho. \tag{3.4}$$

Using (2.14)and (3.1) together, and putting $U = V = \rho$, we obtain

$$a + b = -\lambda. \tag{3.5}$$

and

$$a = -\lambda - \frac{div\rho}{3}. \tag{3.6}$$

Let us assume that ρ is Killing, which implies $\lambda = -a$ and $b = 0$. Thus we have the following result.

Theorem 3.1. *If a perfect fluid spacetime M in $f(\mathcal{R},T)$ gravity admits a Ricci soliton (g, λ, ρ) with Killing velocity vector field ρ, then M is steady if $p = \frac{f(T)}{8\pi}$, expanding if $p > \frac{f(T)}{8\pi}$, and shrinking if $p < \frac{f(T)}{8\pi}$.*

In particular, if $f(T) = 0$ then $f(\mathcal{R},T)$ gravity recover $f(R)$-gravity. Thus we have the following corollary.

Corollary 3.2. *If a perfect fluid spacetime M in $f(\mathcal{R})$ gravity admits a Ricci soliton (g, λ, ρ) with Killing velocity vector field ρ, then its represents a dust era with expanding Ricci soliton.*

4 Ricci soliton on perfect fluid spacetime in $f(\mathcal{R},T)$-gravity with velocity vector field $\rho = grad\Psi$

In this segment, we study a particular case when the velocity vector field ρ of Ricci soliton is of gradient type, $\rho =: grad\Psi$ in perfect fluid spacetime M in $f(\mathcal{R},T)$-gravity.

Let $\rho = grad\Psi$, where Ψ is a smooth function on M. Now, from equation (3.4), we can conclude the following results as

Theorem 4.1. *Let M be a perfect fluid sapectime in $f(\mathcal{R},T)$-gravity admits Ricci soliton and the velocity vector field ρ of the Ricci soliton is of gradient type, then the modified Poisson equation of the $f(\mathcal{R},T)$-gravity satisfying by Ψ is*

$$\nabla^2\Psi = 3[\lambda + 8p\pi - f(T)]. \tag{4.1}$$

Corollary 4.2. *Let M be a perfect fluid sapectime in $f(\mathcal{R})$-gravity admits Ricci soliton and the velocity vector field ρ of the Ricci soliton is of gradient type, then the modified Poisson equation of the $f(\mathcal{R})$-gravity satisfying by Ψ is*

$$\nabla^2\Psi = 3[\lambda + 8p\pi]. \tag{4.2}$$

The above equations (4.1) and (4.2) could be taken as general relativistic analog of Poisson's equation in stationary spacetime.

Remark 4.1. Also, for $\Psi \in C^\infty(M)$ and the vector field ρ a straight forward calculation gives

$$div(\Psi\rho) = \rho(d\Psi) + \Psi div\rho. \tag{4.3}$$

The function $\Psi \in C^\infty(M)$ is a last multiplier of vector field ρ with respect to g if $div(\Psi\rho) = 0$. The corresponding equation

$$\rho(d\ ln\rho) = -div(\rho) \tag{4.4}$$

is called the **Liouville equation** of the vector field ζ with respect to g (for more details see [25]).

Now, infer the above remark and equation (3.4), we obtain the following result:

Theorem 4.3. *Let M be a perfect fluid sapectime in $f(\mathcal{R}, T)$-gravity admits Ricci soliton and the velocity vector field ρ of the Ricci soliton is of gradient type, then the modified Liouville equation of $f(\mathcal{R}, T)$-gravity satisfying by Ψ and ρ is,*

$$\rho(d\ ln\Psi) = -3[\lambda + 8p\pi - f(T)]. \tag{4.5}$$

Again, using the fact that, if $f(T) = 0$ then $f(\mathcal{R}, T)$ gravity recover $f(R)$-gravity. Thus we have the following corollary.

Corollary 4.4. *Let M be a perfect fluid sapectime in $f(\mathcal{R})$-gravity admits Ricci soliton and the velocity vector field ρ of the Ricci soliton is of gradient type, then the modified Liouville equation of $f(\mathcal{R})$-gravity satisfying by Ψ and ρ is,*

$$\rho(d\ ln\Psi) = -3[\lambda + 8p\pi]. \tag{4.6}$$

5 Harmonic aspect of Ricci soliton on perfect fluid sapectime in $f(\mathcal{R}, T)$-gravity

This section is based on the situation that a function $f : \mathbf{M} \longrightarrow \mathbb{R}$ is said to be harmonic if $\nabla^2 f = 0$, where ∇^2 is the Laplacian operator on $\mathbf{M}$ [35], we turn up the following results:

Theorem 5.1. *Let M be a perfect fluid sapectime in $f(\mathcal{R}, T)$-gravity admits Ricci soliton and the velocity vector field ρ of the Ricci soliton is of gradient type and if ψ is a harmonic function on M , then M in $f(\mathcal{R}, T)$-gravity admits Ricci soliton is expanding, steady and shrinking according as*

1. $f(T) > 8p\pi$,
2. $f(T) = 8p\pi$ *and*
3. $f(T) < 8p\pi$ *respectively.*

Proof. Form equation (4.1) we can easily obtain the desired result. □

Theorem 5.2. *Let M be a perfect fluid sapectime in $f(\mathcal{R},T)$-gravity admits Ricci soliton and the velocity vector field ρ of the Ricci soliton is of gradient type and ψ is a harmonic function on M in $f(\mathcal{R},T)$, then the isotropic pressure is $p = \frac{f(T)-\lambda}{8\pi}$.*

Since $f(T) = 0$ in $f(\mathcal{R})$ gravity theory. Thus we have

Corollary 5.3. *Let M be a perfect fluid sapectime in $f(\mathcal{R})$-gravity admits Ricci soliton and the velocity vector field ρ of the Ricci soliton is of gradient type and if ψ is a harmonic function on M , then M in $f(\mathcal{R})$-gravity admits a shrinking Ricci soliton.*

Corollary 5.4. *Let M be a perfect fluid sapectime in $f(\mathcal{R})$-gravity admits Ricci soliton and the velocity vector field ρ of the Ricci soliton is of gradient type and ψ is a harmonic function on M in $f(\mathcal{R})$ gravity , then the isotropic pressure is $p = \frac{-\lambda}{8\pi}$.*

6 Gradient Ricci soliton on perfect fluid spacetime in $f(\mathcal{R},T)$-gravity

Let us consider that vector field V of the Ricci soliton in n-dimensional perfect fluid spacetime M of $f(\mathcal{R},T)$-gravity.
From (1.5), we can write

$$\nabla_U \mathcal{D}\psi + QU + \lambda U = 0 \tag{6.1}$$

for all $U \in \chi(M)$. The equation (6.1) along with the relation

$$R(U,V)\mathcal{D}\psi = \nabla_U \nabla_V \mathcal{D}\psi - \nabla_V \nabla_U \mathcal{D}\psi - \nabla_{[U,V]}\mathcal{D}\psi \tag{6.2}$$

give

$$R(U,V)\mathcal{D}\psi = (\nabla_U Q)V - (\nabla_V Q)U. \tag{6.3}$$

The covariant derivative of (6.1) along vector field V gives

$$\nabla_V \nabla_U \mathcal{D}\psi = -((\nabla_V Q)(U) - Q(\nabla_V)U - \lambda \nabla_V U \tag{6.4}$$

Interchanging U and V in (6.4) and then using the foregoing equation together with (1.3) and (6.4) in (6.2), we get
(6.5)

$$R(U,V)\mathcal{D}\psi = (\nabla_U Q)V - (\nabla_V Q)U + \mu[(\nabla_V \eta)(U)\rho + \eta(U)\nabla_V \rho - (\nabla_U \eta)(V)\rho - \eta(V)\nabla_U \rho]$$

Now, differentiating equation (2.15) covariantly along vector field U, we turn up

$$(\nabla_U Q)(V) = U(a)V + U(b)\eta(V)\rho + b(\nabla_U \eta)(V)\rho + b\eta(V)\nabla_U \rho. \tag{6.6}$$

In view of (6.5) and (6.6), we lead

$$R(U,V)\mathcal{D}\psi = U(a)V - V(a)U + [U(b)\eta(V) - V(b)\eta(U) + b(\nabla_U\eta)(V) \\ -b(\nabla_V\eta)(U)]\rho + b[\eta(V)\nabla_U\rho - \eta(U)\nabla_V\rho]. \tag{6.7}$$

Taking a set of orthonormal frame field and contracting (6.7) along the vector field U, we have

(6.8)

$$\mathcal{R}ic(V,\mathcal{D}\psi) = (1-n)V(a) + V(b) + \rho(b)\eta(V) + b[(\nabla_\rho\eta)(V) - (\nabla_V\eta)(\rho) + \eta(V)div\rho].$$

Again, from (2.12) we have

$$\mathcal{R}ic(V,\mathcal{D}\psi) = aV(\psi) + b\eta(V)\rho(\psi). \tag{6.9}$$

Setting $V=\rho$ in (6.8) and (6.9) and then equating the values of $\mathcal{R}ic(\rho,\mathcal{D}\psi)$, we get

$$(a-b)\rho(\psi) = (1-n)\rho(a) - bdiv\rho. \tag{6.10}$$

Let the velocity vector field ρ of the perfect fluid spacetime is Killing, that is $\mathcal{L}_\rho g = 0$ and scalar a remains invariant under the velocity vector field ρ that is $\rho(a) = 0$. Then we get $div\rho = 0$. Thus, from equations (2.13) and (6.10) we get

$$(a-b)\rho(\psi) = 0, \tag{6.11}$$

which shows that either $a = b$ or $\rho(\psi) = 0$ on a perfect fluid spacetime in $f(\mathcal{R},T)$ gravity with the gradient Ricci soliton. Now, we classify our study into two cases as:
Case I. We consider that $a = b$ and $\rho(\psi) \neq 0$ and therefore from (2.13), we conclude that

$$p = \frac{1}{(4\pi - f'(T))}\Big\{4\pi + f^{'}(T) - f(T)\Big\}\sigma. \tag{6.12}$$

this gives the equation of state in a perfect fluid spacetime in $f(\mathcal{R},T)$ gravity. Also, $\lambda = b - a = 0$ and hence the gradient Ricci soliton is steady.
Case II. Now, consider that $\rho(\psi) = 0$ and $a \neq b$. The covariant derivative of $g(\rho,\mathcal{D}\psi) = 0$ along the vector field U gives

$$g(\nabla_U\rho,\mathcal{D}\psi) = -[\lambda + (a-b)]\eta(U), \tag{6.13}$$

where (2.14) and and (6.1) are used. Since the velocity vector field ρ is Killing in a perfect fluid spacetime in $f(\mathcal{R},T)$ gravity, that is $g(\nabla_U\rho,V) + g(U,\nabla_Y\rho,\rho) = 0$. Putting $V=\rho$ in this equation, we get that $g(U,\nabla_\rho\rho) = 0$ because $g(\nabla_U\rho,\rho) = 0$. Thus we conclude that $\nabla_\rho\rho = 0$. Changing U with ρ in equation (6.13) and using the last equation, we infer that

$$\lambda = b - a. \tag{6.14}$$

$$\lambda = [f^{'}(T) + f(T) + 4\pi] - \frac{8p\pi}{(p+\sigma)}. \tag{6.15}$$

This reflects that the gradient Ricci soliton in a perfect fluid spacetime of $f(\mathcal{R},T)$ gravity is expanding or shrinking if $[f^{'}(T)+f(T)+4\pi] > \frac{8p\pi}{(p+\sigma)}$,
$[f^{'}(T)+f(T)+4\pi] < \frac{8p\pi}{(p+\sigma)}$, respectively.
Next, the equations (6.8) and (6.9) together with the hypothesis take the form

$$\mathcal{R}ic(V,\mathcal{D}\psi)=aV(\psi) \tag{6.16}$$

and

$$\mathcal{R}ic(V,\mathcal{D}\psi)=(1-n)V(a)+V(b). \tag{6.17}$$

In view of (6.14)-(6.17), we conclude

$$aV(\psi)+(n-2)V(a)=0 \Leftrightarrow \mathcal{D}\psi+(n-2)\mathcal{D}a=0. \tag{6.18}$$

Considering a set of orthonormal frame and contracting equation (6.2) along vector field U and using the fact that *trace* $\left\{V \longrightarrow \frac{1}{2}(\nabla_V Q)U\right\}=\frac{1}{2}\nabla_U\mathcal{R}$, we lead

$$\mathcal{R}ic(V,\mathcal{D}\psi)=-\frac{1}{2}V(\mathcal{R})=aV(\psi), \tag{6.19}$$

where (6.16) has been used. Again from (2.15) and (6.14), we infer that

$$V(\mathcal{R})=(n-1)V(a). \tag{6.20}$$

In consequence of equations (6.18)-(6.20), we conclude that

$$(n-3)V(a)=0. \tag{6.21}$$

Since the dimension of the perfect fluid spacetime ≥ 4. therefore equation (6.21) shows that $a=$ *constant*. Consequently, $b=$ *constant* and the scalar curvature of the perfect fluid spacetime in $f(\mathcal{R},T)$ gravity is constant. Now, using (6.21) in (6.18) we have

$$aV(\psi)=0, \tag{6.22}$$

which implies that either $a=0$ or $V(\psi)=0 \Downarrow \psi=$ *constant*. If $a=0$ and ψ is a non-zero constant function on a perfect fluid spacetime of $f(\mathcal{R},T)$ gravity, then from (6.7) we have

$$\mathcal{R}ic=-\mathcal{R}\eta\otimes\eta, \tag{6.23}$$

where $\mathcal{R}=-b=$ *constant* $\neq 0$. From (6.23), we observe that the perfect fluid spacetime in $f(\mathcal{R},T)$ gravity is *Ricci simple* [12]. Next, we consider that $a\neq 0$ and $\mathcal{D}\psi=0$ and therefore $\psi=$ *constant*. Thus the gradient Ricci soliton on a perfect fluid spacetime in $f(\mathcal{R},T)$ gravity is trivial. Thus by concluding the above facts, we can write our results as:

Theorem 6.1. *Let the perfect fluid spacetime M in $f(\mathcal{R},T)$ gravity admits a gradient Ricci soliton and its velocity vector field ρ is Killing. Then either*
(i) the equation of state of the perfect fluid in $f(\mathcal{R},T)$ gravity is governed by $p=\frac{1}{(4\pi-f^{'}(T))}\left\{4\pi+f^{'}(T)-f(T)\right\}\sigma$ *and the soliton is steady,or*
(ii) the perfect fluid spacetime in $f(\mathcal{R},T)$ gravity is Ricci simple or the gradient Ricci soliton is trivial.

Let M be a perfect fluid spacetime in $f(\mathcal{R},T)$ gravity with $a \neq b$. If M admits a gradient Ricci soliton and ρ is Killing, then from Theorem (6.1), we can state the following corollary

Corollary 6.2. *Let a perfect fluid spacetime M in $f(\mathcal{R},T)$ gravity admits a gradient Ricci soliton with $a \neq b$. If the velocity vector field of M is Killing, then M possesses a constant scalar curvature.*

Corollary 6.3. *Let a perfect fluid spacetime M in $f(\mathcal{R})$ gravity admits a gradient Ricci soliton and its velocity vector field ρ is Killing. Then either*
(i) the equation of state of the perfect fluid in $f(\mathcal{R})$ gravity is governed by $\frac{p}{\sigma} = \frac{4\pi+f^{'}(T)}{(4\pi-f^{'}(T))}$ and the soliton is steady,or
(ii) the perfect fluid spacetime in $f(\mathcal{R})$ gravity is Ricci simple or the gradient Ricci soliton is trivial.

As a consequences of Theorem (6.1) and equation (6.10) we have following observation.

Theorem 6.4. *If a perfect fluid spacetime M in $f(\mathcal{R},T)$ gravity admits a gradient Ricci soliton and its velocity vector field ρ is Killing. Then evolution of the universe is given in the following table through equation of state of the perfect fluid in $f(\mathcal{R},T)$ gravity*

Equation of state (EoS) $\frac{p}{\sigma} = \omega$	**Restrictions of** $f^{'}(T)$ **and** $f(T)$	**Evolution of the universe**
$\omega = 1$	$f^{'} = \frac{f(T)}{2}$	*Ultra relativistic era*
$\omega > -1$	$f(T) < 8\pi$	*Quintessence era*
$\omega < -1$	$f(T) > 8\pi$	*Phantom era*
$\omega = 0$	$f^{'}(T) = f(T) - 4\pi$	*dust era*

7 Yamabe soliton on a perfect fluid spacetime in $f(\mathcal{R},T)$-gravity

Let the Lorentzian metric of the perfect fluid spacetime of perfect fluid spacetime M in $f(\mathcal{R},T)$-gravity be a Yamabe soliton. Then we have

$$\mathcal{L}_W g = 2(\mathcal{R} - \lambda)g, \tag{7.1}$$

which is equivalent to

$$g(\nabla_U W, V) + g(U, \nabla_V W) = 2(\mathcal{R} - \lambda)g(E, F).$$

Taking an orthonormal frame field on M and contracting the above equation over U and V, we infer

$$div\, W = (\mathcal{R} - \lambda)n. \tag{7.2}$$

Using equation (7.2) in equation (7.1), we find

$$\mathcal{L}_W g = \frac{2\, div\, W}{n} g.$$

This shows that the vector field W is Killing if and only if $div\, W = 0$.
Let us assume that the potential vector field $V = \rho$. Then equation (7.1) can be written as

$$(\mathcal{L}_\rho g)(U,V) = g(\nabla_U \rho, V) + g(U, \nabla_V \rho) = 2(\mathcal{R}-\lambda)g(U,V). \tag{7.3}$$

Now replacing V by ρ and using equations (2.10) and the fact $g(\nabla_U \rho, \rho) = 0$ and $(\nabla_U \eta)(\rho) = 0$ in the above equation, we infer

$$\nabla_\rho \rho = 2(\mathcal{R}-\lambda)\rho. \tag{7.4}$$

Again setting $U = V = \rho$ in equation (7.3) we obtain

$$\lambda = \mathcal{R}. \tag{7.5}$$

Thus, we have

Theorem 7.1. *Let (g, λ, ρ) be a Yamabe soliton on a perfect fluid spacetime M in $f(\mathcal{R},T)$ gravity with Killing velocity vector field ρ, then Yamabe soliton is expanding, steady and shrinking according as*

1. $(\sigma + p)(8\pi + 2f^{'}(T)) > 4[f(T) - 8p\pi]$,
2. $(\sigma + p)(8\pi + 2f^{'}(T)) = 4[f(T) - 8p\pi]$ *and*
3. $(\sigma + p)(8\pi + 2f^{'}(T)) < 4[f(T) - 8p\pi]$ *respectively.*

Corollary 7.2. *Let (g, λ, ρ) be a Yamabe soliton on a perfect fluid spacetime M in $f(\mathcal{R})$ gravity with Killing velocity vector field ρ, then Yamabe soliton is expanding, steady and shrinking according as*

1. $(\sigma + p)(8\pi + 2f^{'}(T)) > 32p\pi$,
2. $(\sigma + p)(8\pi + 2f^{'}(T)) = 32p\pi$ *and*
3. $(\sigma + p)(8\pi + 2f^{'}(T)) < 32p\pi$ *respectively.*

8 Gradient Yamabe soliton on perfect fluid spacetimes in $f(\mathcal{R},T)$ gravity

From equation (1.6), we have

$$\nabla_V \mathcal{D}\psi = (\mathcal{R}-\lambda)V. \tag{8.1}$$

Differentiating (8.1) covariantly along the vector field V, we have

$$\nabla_U \nabla_V \mathcal{D}\psi = U(\mathcal{R})V + (\mathcal{R}-\lambda)\nabla_U V. \tag{8.2}$$

Interchanging U and V in the above equation and then using the foregoing equation, (8.1) and (8.2) in $R(U,V)\mathcal{D}\psi = \nabla_U\nabla_V\mathcal{D}\psi - \nabla_V\nabla_U\mathcal{D}\psi - \nabla_{[U,V]}\mathcal{D}\psi$, we infer

$$R(U,V)\mathcal{D}\psi = U(\mathcal{R})V - V(\mathcal{R})U.$$

Considering an orthonormal frame field and contracting the above equation over U, we find

$$\mathcal{R}ic(V,\mathcal{D}\psi) = -(n-1)V(\mathcal{R}).$$

From equation (2.14) we have

$$\mathcal{R}ic(V,\mathcal{D}\psi) = aV(\rho) + b\rho(\psi)\eta(V).$$

The last two equations give

$$aV(\psi) + b\rho(\psi)\eta(V) = -(n-1)V(\mathcal{R}). \tag{8.3}$$

Setting $V = \rho$ in the above equation, we get

$$(a-b)\rho(\psi) = -(n-1)\rho(\mathcal{R}). \tag{8.4}$$

Let us assume that the velocity vector field ρ of the perfect fluid spacetime is Killing and scalars a and b remains invariant under the velocity vector field ρ. These facts together with equations (2.11) and (2.13) reveal that $\rho(a) = 0 = \rho(b) \implies \rho(\mathcal{R}) = 0$. Using this fact in equation (8.4), we find

$$(a-b)\rho(\psi) = 0,$$

which entails that either $a = b$ or $\rho(\psi) = 0$. If $a = b$, then from equation (2.13), we infer that

$$p = \frac{1}{(4\pi - f'(T))}\left\{4\pi + f^{'}(T) - f(T)\right\}\sigma, \tag{8.5}$$

which gives the equation of state.

Next, we suppose that $a \neq b$ and $\rho(\psi) = 0 \implies g(\rho,\mathcal{D}\psi) = 0$. The covariant derivative of this equation gives

$$g(\nabla_U\rho,\mathcal{D}\psi) + (\mathcal{R}-\lambda)\eta(U) = 0,$$

where equation (8.1) is used. Since ρ (by hypothesis) is Killing and therefore $\nabla_\rho\rho = 0$. Setting $U = \rho$ in the above equation and making use of $\nabla_\rho\rho = 0$, we find

$$\lambda = \mathcal{R}. \tag{8.6}$$

This shows that the scalar curvature of the manifold is constant. By considering the hypothesis $\rho(\psi) = 0$ and equation (8.6), we can infer from equation (8.3) that either $a = 0$ or $\mathcal{D}\psi = 0$. The equation $\mathcal{D}\psi = 0$ implies that f is constant and thus the gradient Yamabe soliton is trivial. Since we are interested in non-trivial gradient Yamabe soliton, therefore we consider $a = 0$ and thus from equation (2.12), we lead to

$$\mathcal{R}ic = -\mathcal{R}\eta\otimes\eta,$$

where $b = -\mathcal{R}$. Thus, we articulate our result as:

Theorem 8.1. *Let a perfect fluid spacetime M in $f(\mathcal{R},T)$-gravity admits gradient Yamabe soliton with a velocity vector field of M is Killing, then either the equation of state is given by (8.5) or the perfect fluid spacetime in $f(\mathcal{R})$ gravity is Ricci simple.*

Corollary 8.2. *Let a perfect fluid spacetime M in $f(\mathcal{R})$-gravity admits gradient Yamabe soliton with a velocity vector field of M is Killing, then the equation of state is of the form*

$$\frac{p}{\sigma} = \frac{4\pi + f^{'}(T)}{4\pi - f^{'}(T)}, \tag{8.7}$$

or the perfect fluid spacetime in $f(\mathcal{R})$ gravity is Ricci simple.

Let $a = b$ and $\rho(\psi) \neq 0$. Then from equation (2.11) and Theorem 8.1 we conclude that

$$\mathcal{R}ic = a(g + \eta \otimes \eta).$$

The Lie derivative of the above equation along ρ gives

$$(\mathcal{L}_\rho \mathcal{R}ic)(U,V) = a\{(\mathcal{L}_\rho \eta)(U)\eta(V) + \eta(U)(\mathcal{L}_\rho \eta)(V)\}, \tag{8.8}$$

since ρ is Killing. The Lie derivative of $\eta(U) = g(U,\rho)$ along the vector field ρ together with the assumption that ρ is Kiiling infer that

$$\mathcal{L}_\rho \eta = 0.$$

In view of last equation, equation (8.8) becomes $\mathcal{L}_\rho \mathcal{R}ic = 0$. This shows that the velocity vector field ρ of the perfect fluid spacetimes of $f(\mathcal{R},T)$ gravity is Ricci inheritance. Hence, we can state the following:

Corollary 8.3. *Let a perfect fluid spacetimes M in $f(\mathcal{R},T)$ gravity M admit a gradient Yamabe soliton with the Killing velocity vector field ρ of M and $\rho(\psi) \neq 0$, then ρ is Ricci inheritance.*

Now, we assume that $a \neq b$ on a perfect fluid spacetimes M in $f(\mathcal{R},T)$ gravity admitting a gradient Yamabe soliton. If the velocity vector field of M is Killing, then equation (8.6) is satisfied. Thus, we conclude that the nature of the flow vary according to the scalar curvature of M. Thus we write the following.

Corollary 8.4. *Let a perfect fluid spacetime M in $f(\mathcal{R},T)$ gravity of dimension n. If the metric of M is a gradient Yamabe soliton, velocity vector field of M is Killing and $a \neq b$, then M possesses the constant scalar curvature.*

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