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NONCOMMUTATIVE INTEGRATION

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To the Memory of Two Distinguished Operator Algebraists: *William B. Arveson and Gert K. Pedersen.*

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Abstract. We will show that if $\mathcal{M}$ is a factor, then for any pair $\varphi, p \in \mathcal{M}_*^+$ of normal positive linear functionals on $\mathcal{M}$, the inequality:

$$\|\varphi\| \leq \|\psi\|$$

is equivalent to the fact that there exist a countable family $\{\varphi_i : i \in I\} \subset \mathcal{M}_*^+$ in $\mathcal{M}_*^+$ and a family $\{u_i : i \in I\} \subset \mathcal{M}$ of partial isometries in $\mathcal{M}$ such that

$$\varphi = \sum_{i\in I} \varphi_i, \quad \sum_{i\in I} u_i \varphi_i u_i^* \leq \psi, \quad \text{and} \quad u_i^* u_i = s(\varphi_i), i \in I,$$

where $s(\omega), \omega \in \mathcal{M}_*^+$, means the support projection of ω. Furthermore, if $\|\varphi\| = \|\psi\|$, then the equality replaces the inequality in the second statement. In the case that $\mathcal{M}$ is not of type III_1, the family of partial isometries can be replaced by a family of unitaries in $\mathcal{M}$. One cannot expect to have this result in the usual integration theory. To have a similar result, one needs to bring in some kind of non-commutativity. Let $\{X, \mu\}$ be a σ-finite semifinite measure space and G be an ergodic group of automorphisms of $L^\infty(X, \mu)$, then for a pair f and g of μ-integrable positive functions on X, the inequality:

$$\int_X f(x)\mathrm{d}\mu(x) \leq \int_X g(x)\mathrm{d}\mu(x)$$

is equivalent to the existence of a countable families $\{f_i : i \in I\} \subset L^1(X, \mu)$ of positive integrable functions and $\{\gamma_i : i \in I\}$ in G such that

$$f = \sum_{i\in I} f_i \quad \text{and} \quad \sum_{i\in I} \gamma_i(f_i) \leq g,$$

where the summation and inequality are all taken in the ordered Banach space $L^1(X, \mu)$ and the action of G on $L^1(X, \mu)$ is defined through the duality between $L^\infty(X, \mu)$ and $L^1(X, \mu)$, i.e.,

$$(\gamma(f))(x) = f\big(\gamma^{-1}x\big)\frac{\mathrm{d}\mu{\circ}\gamma^{-1}}{\mathrm{d}\mu}(x), \quad f \in L^1(X, \mu).$$

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