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ON THE STABILITY OF MIXED TRIGONOMETRIC FUNCTIONAL EQUATIONS

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by J. Chmieliński

ABSTRACT. The aim of this paper is to study the superstability problem of the mixed trigonometric functional equations and the Hyers-Ulam-Rassias stability for a Jensen type functional equation.

1. INTRODUCTION

J. Baker, J. Lawrence and F. Zorzitto in [3] introduced the following : if f satisfies the inequality $|E_1(f) - E_2(f)| \le \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. The stability of this type is called the superstability.

The superstability of the cosine functional equation (also called the d'Alembert functional equation)

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{$\mathcal{C}$}$$

and the sine functional equation

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = f(x)f(y) \tag{$\mathcal{S}$}$$

were investigated by J. Baker [2] and P.W. Cholewa [4], respectively.

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The cosine functional equation $(\mathcal{C})$ is generalized by the following functional equations

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{$\mathcal{C}_{fg}$}$$

$$f(x+y)+f(x-y)=2g(x)f(y), \tag{$\mathcal{C}_{gf}$}$$

$$f(x+y)+f(x-y)=2g(x)g(y), \tag{$\mathcal{C}_{gg}$}$$

where the two unknown functions f, g are to be determined. The equation $(\mathcal{C}_{fg})$ is sometimes referred to as the Wilson equation. The above cosine type equations have been investigated by Badora, Ger, Kannappan, Kim, ([1], [2], [4], [8], [9], [10]) and others.

Motivated by some trigonometric identities (for example, $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$) we consider the following trigonometric functional equations:

$$f(x+y)-f(x-y)=2f(x)f(y), \tag{$\mathcal{T}$}$$

$$f(x+y)-f(x-y)=2f(x)g(y), \tag{$\mathcal{T}_{fg}$}$$

$$f(x+y)-f(x-y)=2g(x)f(y), \tag{$\mathcal{T}_{gf}$}$$

$$f(x+y)-f(x-y)=2g(x)g(y). \tag{$\mathcal{T}_{gg}$}$$

Let $g(x)\equiv 1$ in $(\mathcal{T}_{gf})$, then we also obtain the Jensen type functional equation

$$f(x+y)-f(x-y)=2f(y). \tag{$\mathcal{J}$}$$

In this paper, we will investigate the superstability problem for the mixed trigonometric functional equations $(\mathcal{T}_{fg})$, $(\mathcal{T}_{gf})$, $(\mathcal{T}_{gg})$ on the Abelian group, and the Hyers-Ulam-Rassias stability for the Jensen type functional equation $(\mathcal{J})$.

As a consequence, we obtain the results for the equation $(\mathcal{T})$ as a corollary, extending the obtained results to the Banach algebra.

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers. Whenever we deal with $(\mathcal{S})$, we need to assume additionally that $(G,+)$ is a uniquely 2-divisible group. We will write then "under 2-divisibility" for short. We may assume that f and g are nonzero functions and ε is a nonnegative real constant.

2. Stability of the equation $(\mathcal{T}_{gf})$

In this section, we will investigate the superstability of the trigonometric functional equations $(\mathcal{T}_{gf})$ related to the cosine functional equation $(\mathcal{C})$.

Theorem 2.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)-f(x-y)-2g(x)f(y)| \le \varepsilon \quad (x,y\in G). \tag{2.1}$$

Then either f is bounded or g satisfies $(\mathcal{C})$.

Proof. Let f be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that

$$0 \neq |f(y_n)| \to \infty \quad \text{as} \quad n\to\infty. \tag{2.2}$$

Substituting y_n for y in (2.1) we obtain

$$\left|\frac{f(x+y_n)-f(x-y_n)}{2f(y_n)}-g(x)\right|\leq\frac{\varepsilon}{2|f(y_n)|},$$

that implies

$$\lim_{n\to\infty}\frac{f(x+y_n)-f(x-y_n)}{2f(y_n)}=g(x)\qquad(x\in G).\tag{2.3}$$

Using (2.1) we have

$$\begin{aligned}&|f(x+(y+y_n))-f(x-(y+y_n))-2g(x)f(y+y_n)\\&\qquad-f(x+(y-y_n))+f(x-(y-y_n))+2g(x)f(y-y_n)|\\&\leq|f(x+(y+y_n))-f(x-(y+y_n))-2g(x)f(y+y_n)|\\&\qquad+|f(x+(y-y_n))-f(x-(y-y_n))-2g(x)f(y-y_n)|\\&\leq 2\varepsilon,\end{aligned}$$

so that

$$\begin{aligned}&\left|\frac{f((x+y)+y_n)-f\left((x+y)-y_n\right)}{2f(y_n)}\right.\\&\left.+\frac{f((x-y)+y_n)-f((x-y)-y_n)}{2f(y_n)}-2g(x)\frac{f(y+y_n)-f(y-y_n)}{2f(y_n)}\right|\\&\leq\frac{\varepsilon}{|f(y_n)|}\end{aligned}$$

for all $x,y\in G$. By virtue of (2.2) and (2.3), we have

$$|g(x+y)+g(x-y)-2g(x)g(y)|\leq 0$$

for all $x,y\in G$. Therefore g satisfies $(\mathcal{C})$. □

Theorem 2.2. *Suppose that $f,g:G\to\mathbb{C}$ satisfy the inequality* (2.1).
If g fails to be bounded, then
(i) g satisfies $(\mathcal{C})$,
(ii) f and g are solutions of $(\mathcal{T}_{gf})$ and $(\mathcal{C}_{fg})$,
(iii) f satisfies $(\mathcal{S})$ under 2-divisibility.

Proof. (i) If f is bounded, choose $y_0\in G$ such that $f(y_0)\neq 0$, and then by (2.1) we obtain

$$\begin{aligned}|g(x)|-\Big|&\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)}\Big|\\&\leq\Big|\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)}-g(x)\Big|\\&\leq\frac{\varepsilon}{2|f(y_0)|},\end{aligned}$$

from which it follows that g is also bounded on G. Namely, since f is nonzero, the unboundedness of g implies the unboundedness of f. Hence g satisfies $(\mathcal{C})$ by Theorem 2.1.

(ii) For the unbounded g, we can choose a sequence $\{x_n\}$ in G such that $0 \neq |g(x_n)| \to \infty$ as $n \to \infty$.

A similar reasoning as in the proof applied in Theorem 2.1 with $x = x_n$ in (2.1) gives us

$$\lim_{n\to\infty} \frac{f(x_n+y) - f(x_n-y)}{2g(x_n)} = f(y) \qquad (y \in G). \tag{2.4}$$

Replacing x by $x_n + x$ and $x_n - x$ in (2.1), since g satisfies $(\mathcal{C})$ by (i), we obtain that $(\mathcal{T}_{gf})$ validates in an application of (2.4). The equation $(\mathcal{C}_{fg})$ also validates by replacing x by $x_n + y$ and $x_n - y$, and replacing y by x in (2.1), respectively.

(iii) Since g satisfies $(\mathcal{C})$, $g(0) = 1$ holds, the equation $(\mathcal{T}_{gf})$ implies that f is odd, hence $f(0) = 0$. Putting $x = y$ in $(\mathcal{T}_{gf})$, we get

$$f(2y) = 2g(y)f(y) \qquad (y \in G).$$

Keeping this in mind, by means of $(\mathcal{T}_{gf})$ and $(\mathcal{C}_{fg})$, we infer the equality

$$\begin{aligned} f(x+y)^2 - f(x-y)^2 &= [f(x+y)+f(x-y)][f(x+y)-f(x-y)] \\ &= 2[f(x+y)+f(x-y)]g(x)f(y) \\ &= \big[f(2x+y)+f(2x-y)\big]f(y) \\ &= \big[f(y+2x)-f(y-2x)\big]f(y) \\ &= 2g(y)f(2x)f(y) \\ &= f(2x)f(2y) \qquad (x, y \in G). \end{aligned}$$

This validates $(\mathcal{S})$ in the light of the unique 2-divisibility of G. □

Applying $g = f$ in Theorem 2.1 and Theorem 2.2, the following corollary may be stated.

Corollary 2.3. *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \leq \varepsilon \qquad (x, y \in G). \tag{2.5}$$

Then f is bounded.

Proof. Suppose that f is unbounded. Applying $g = f$ in Theorem 2.1 and Theorem 2.2, we obtain that f satisfies $(\mathcal{C})$ and $(\mathcal{T})$, simultaneously. This would mean that f must be a zero mapping, hence bounded – a contradiction. □

3. Stability of the equation $(\mathcal{T}_{fg})$

In this section, we will investigate the stability of the trigonometric functional equations $(\mathcal{T}_{fg})$ related to the sine functional equation $(\mathcal{S})$.

Theorem 3.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \leq \varepsilon \qquad (x, y \in G). \tag{3.1}$$

If f fails to be bounded, then

(i) g satisfies $(\mathcal{S})$ under 2-divisibility,

(ii) if, additionally, f satisfies $(\mathcal{C})$, then f and g are solutions of $g(x+y) - g(x-y) = 2f(x)g(y)$.

Proof. (i) For the unbounded f, we can choose a sequence $\{x_n\}$ in G such that $0 \neq |f(x_n)| \to \infty$ as $n \to \infty$.

A similar reasoning as in the proof applied in Theorem 2.1 with $x = x_n$ in (3.1) gives us

$$\lim_{n\to\infty} \frac{f(x_n + y) - f(x_n - y)}{2f(x_n)} = g(y) \qquad (y \in G). \tag{3.2}$$

Substitute $x_n + x$ and $x_n - x$ for x in (3.1), dividing by $2f(x_n)$, then it gives us the existence of a limit function

$$h(x) := \lim_{n\to\infty} \frac{f(x_n + x) + f(x_n - x)}{2f(x_n)}, \tag{3.3}$$

where the function $h : G \to \mathbb{C}$ satisfies the equation

$$g(x + y) - g(x - y) = 2h(x)g(y) \qquad (x, y \in G). \tag{3.4}$$

From the definition of h, we obtain the equality $h(0) = 1$, which jointly with (3.4) implies that g is an odd function. Hence $g(0) = 0$.

A similar procedure to that applied in (iii) of Theorem 2.2 in equation (3.4) allows us to show that g satisfies $(\mathcal{S})$.

(ii) In case f satisfies $(\mathcal{C})$, (3.3) means $h = f$. Hence, from equation (3.4), f and g are solutions of $g(x + y) - g(x - y) = 2f(x)g(y)$. □

Theorem 3.2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality* (3.1).

If g fails to be bounded, then

(i) g satisfies $(\mathcal{S})$ under 2-divisibility,

(ii) f and g are solutions of $(\mathcal{T}_{fg})$,

(iii) f satisfies $(\mathcal{S})$ under 2-divisibility and one of the cases $f(0) = 0$, $f(x) = f(-x)$ for all $x \in G$,

(iv) if g satisfies $(\mathcal{C})$ or $(\mathcal{T})$, then f and g are solutions of $(\mathcal{C}_{fg})$.

Proof. (i) Similar to (i) of Theorem 2.2, we know that g is also bounded whenever f is bounded. Hence, by contraposition, g satisfies $(\mathcal{S})$ by (i) of Theorem 3.1.

(ii) A slight change to Theorem 2.1 gives us

$$f(x) = \lim_{n\to\infty} \frac{f(x + y_n) - f(x - y_n)}{2g(y_n)} \quad (x \in G). \tag{3.5}$$

Replacing x by $x + y_n$ and $x - y_n$ in (3.1), which gives, with an application of (3.5), the required result$(\mathcal{T}_{fg})$.

(iii) Using the same method of proof as in Theorem 2.2, i.e., replacing y by $y + y_n$ and $-y + y_n$ in (3.1), taking the limit as $n \to \infty$ with the use of (3.5), we conclude that, for every $x \in G$, there exists

$$h_2(y) := \lim_{n\to\infty} \frac{g(y_n + y) + g(y_n - y)}{2g(y_n)},$$

where the function $h_2 : G \to \mathbb{C}$ satisfies the equation

$$f(x + y) + f(x - y) = 2f(x)h_2(y) \qquad (x, y \in G). \tag{3.6}$$

In the case $f(0) = 0$ in (3.6), we see that f is an odd function.

Using the same method of proof as that applied in (iii) of Theorem 2.2 in equation (3.6) shows us that f satisfies $(\mathcal{S})$.

Next, for the case $f(x) = f(-x)$, it is enough to show that $f(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (3.1), from the above assumption and a given condition, we obtain the inequality

$$|g(y)| \le \frac{\varepsilon}{2|f(0)|} \qquad (y \in G).$$

This inequality means that g is globally bounded – a contradiction. Thus, the claim $f(0) = 0$ holds, so the proof of theorem is completed.

(iv) For the case g satisfies $(\mathcal{C})$, we know that the limit function h_2 is g. So (3.6) becomes $(\mathcal{C}_{fg})$

Finally, consider the case g satisfies $(\mathcal{T})$. Replacing y by $y + y_n$ and $y - y_n$ in (3.1), and taking the limit as $n \to \infty$ with the use of (3.5), we conclude that f and g are solutions of $(\mathcal{C}_{fg})$. □

4. Stability of the equation $(\mathcal{T}_{gg})$

In this section, we will investigate the stability of the trigonometric functional equation $(\mathcal{T}_{gg})$.

Theorem 4.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2g(x)g(y)| \le \varepsilon \qquad (x, y \in G). \tag{4.1}$$

Then either g is bounded or g satisfies $(\mathcal{S})$ under 2-divisibility.

Proof. (i) Let g be an unbounded solution of the inequality (4.1). Then, there exists a sequence $\{x_n\}$ in G such that $0 \ne |g(x_n)| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ in (4.1), dividing both sides by $|2g(x_n)|$ and passing to the limit as $n \to \infty$ we obtain

$$g(y) = \lim_{n\to\infty} \frac{f(x_n + y) - f(x_n - y)}{2g(x_n)} \quad (y \in G). \tag{4.2}$$

Using the same method of proof as that applied after (2.4) of Theorem 2.2 in (4.1) shows, with an application of (4.2), the existence of a limit function such that

$$h_3(x) := \lim_{n\to\infty} \frac{g(x_n + x) + g(x_n - x)}{2g(x_n)},$$

where the function $h_3 : G \to \mathbb{C}$ satisfies the equation

$$g(x+y) - g(x-y) = 2h_3(x)g(y) \qquad (x, y \in G). \tag{4.3}$$

From the definition of h_3, we get $h_3(0) = 1$, which with (4.3) implies that g is odd. The oddness of g forces it to vanish at 0. Putting $x = y$ in (4.3), we have $g(2y) = 2h_3(y)g(y)$ for all $y \in G$.

A slight modification in (iii) of Theorem 2.2 runs, by means of (4.3), that g satisfies $(\mathcal{S})$. □

5. Extension to the Banach algebra

All results in Sections 2–4 can be extended to the Banach algebra. For simplicity, we only will represent the case of Theorem 2.2, and the extension to the other theorems will be omitted.

Given mappings $f, g : G \to \mathbb{C}$, we will denote their differences $D : G \times G \to \mathbb{C}$ as

$$\begin{aligned} D\mathcal{C}_g(x,y) &:= g(x+y) + g(x-y) - 2g(x)g(y), \\ D\mathcal{S}(x,y) &:= f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 - f(x)f(y), \\ D\mathcal{T}_{gf}(x,y) &:= f(x+y) - f(x-y) - 2g(x)f(y), \\ D\mathcal{C}_{fg}(x,y) &:= f(x+y) + f(x-y) - 2f(x)g(y) \end{aligned}$$

for $x, y \in G$.

Theorem 5.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - 2g(x)f(y)\| \le \varepsilon \qquad (x, y \in G). \tag{5.1}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, if the superposition $x^* \circ g$ fails to be bounded, then
(i) g satisfies $(\mathcal{C})$,
(ii) f and g are solutions of $(\mathcal{T}_{gf})$ and $(\mathcal{C}_{fg})$,
(iii) f satisfies $(\mathcal{S})$ under 2-divisibility.

Proof. (i) Fix arbitrarily a linear multiplicative functional $x^* \in E^*$, we have $\|x^*\| = 1$ as known. In (5.1), we have

$$\begin{aligned} \varepsilon &\ge \|f(x+y) - f(x-y) - 2g(x)f(y)\| \\ &= \sup_{\|y^*\|=1} \big|y^*\big(f(x+y) - f(x-y) - 2g(x)f(y)\big)\big| \\ &\ge \big|x^*\big(f(x+y)\big) - x^*\big(f(x-y)\big) - 2x^*\big(g(x)\big)x^*\big(f(y)\big)\big| \qquad (x, y \in G). \end{aligned}$$

In the above inequality, we know that the superpositions $x^* \circ f$ and $x^* \circ g$ yield a solution of inequality (2.1) in Theorem 2.1. Assume that the superposition $x^* \circ g$ is unbounded, then Theorem 2.2 forces the following results:
(i) the function $x^* \circ g$ solves $(\mathcal{C})$,
(ii) the functions $x^* \circ f$ and $x^* \circ g$ are solutions of $(\mathcal{T}_{gf})$ and $(\mathcal{C}_{fg})$,
(iii) the function $x^* \circ f$ solves $(\mathcal{S})$.

These statements mean, keeping the linear multiplicativity of x^* in mind, that the each difference $D\mathcal{C}_g(x,y), D\mathcal{S}(x,y), D\mathcal{T}_{gf}(x,y), D\mathcal{C}_{fg}(x,y)$ for all $x, y \in G$ falls into the kernel of x^*. Since x^* is arbitrary, we deduce that

$$\begin{aligned} &D\mathcal{C}_g(x,y), D\mathcal{S}(x,y), D\mathcal{T}_{gf}(x,y), D\mathcal{C}_{fg}(x,y) \\ &\qquad\qquad\qquad\qquad \in \bigcap\{\ker x^* : x^* \in E^*\} \end{aligned}$$

for all $x, y \in G$.

Since the Banach algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$D\mathcal{C}_g(x,y) = D\mathcal{S}(x,y) = D\mathcal{T}_{gf}(x,y) = D\mathcal{C}_{fg}(x,y) = 0 \quad (x,y \in G),$$

as claimed. □

Corollary 5.2. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f : G \to E$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - 2f(x)f(y)\| \leq \varepsilon \qquad (x,y \in G).$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, the superposition $x^* \circ f$ is bounded.

6. Solution and Stability of the functional equation $(\mathcal{J})$

In 1940, the stability problem raised by S. M. Ulam [13] was solved by D. H. Hyers [5]. The generalized results of Hyers have been introduced in Hyers, Isac and Rassias [6]. In this section, let E_1 be a real normed space and E_2 a Banach space. We prove the Hyers-Ulam-Rassias stability for the functional equation $(\mathcal{J})$. As a consequence, we obtain the Hyers-Ulam stability.

Theorem 6.1. *Suppose that $f : E_1 \to E_2$ satisfies the inequality*

$$||f(x+y) - f(x-y) - 2f(y)|| \leq \varepsilon(||x||^p + ||y||^p) \tag{6.1}$$

for all $x, y \in E_1$ and for some fixed $p \in \mathbb{R}$ such that $0 \leq p < 1$.

Then there exists an unique additive mapping $A : E_1 \to E_2$ as a solution of $(\mathcal{J})$, which satisfies

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{6.2}$$

and

$$||f(x) + f(0) - A(x)|| \leq \frac{2\varepsilon||x||^p}{2-2^p} \qquad (x \in E_1). \tag{6.3}$$

Proof. Putting $y = x$ in (6.1) we have

$$||f(2x) - f(0) - 2f(x)|| \leq 2\varepsilon\|x\|^p \tag{6.4}$$

for all $x \in E_1$. Let $F(x) := f(x) + f(0)$ for all $x \in E_1$. Then $F(0) = 2f(0)$ and

$$||F(2x) - 2F(x)|| \leq 2\varepsilon\|x\|^p \tag{6.5}$$

for all $x \in E_1$. Replacing x by $2^n x$ in (6.5) and dividing its result by 2^{n+1} we get

$$\left\|\frac{F(2^n x)}{2^n} - \frac{F(2^{n+1}x)}{2^{n+1}}\right\| \leq \frac{2\varepsilon}{2^{n+1}} \cdot \|2^n x\|^p \tag{6.6}$$

for all $x \in E_1$ and all nonnegative integers n. Using (6.6) and the triangle inequality we have

$$\left\|\frac{F(2^m x)}{2^m} - \frac{F(2^n x)}{2^n}\right\| \leq \varepsilon \cdot \sum_{k=m}^{n-1} \frac{1}{2^k}\|2^k x\|^p \tag{6.7}$$

for all $x \in E_1$ and all nonnegative integers m and n with $m < n$. This shows that $\left\{\frac{F(2^n x)}{2^n}\right\}$ is a Cauchy sequence for all $x \in E_1$ because the right side of (6.7) converges to zero, since $0 \leq p < 1$, when $m \to \infty$. Consequently, since E_2 be a Banach space, we can obtain a mapping $A : E_1 \to E_2$ of (6.2) defined by

$$A(x) := \lim_{n\to\infty} \frac{F(2^n x)}{2^n} \tag{6.8}$$

for all $x \in E_1$. Putting $m = 0$ in (6.7) and taking the limit as $n \to \infty$, we obtain (6.3). Also, we have

$$\begin{aligned}
&||A(x+y) - A(x-y) - 2A(y)|| \\
&\qquad \leq \lim_{n\to\infty} \frac{1}{2^n} \left[||f(2^n x + 2^n y) - f(2^n x - 2^n y) - 2f(2^n y)|| \right] \\
&\qquad \leq \lim_{n\to\infty} \varepsilon \cdot 2^{n(p-1)} (\|x\|^p + \|y\|^p) = 0
\end{aligned}$$

for all $x, y \in E_1$, which means that A satisfies the equation $(\mathcal{J})$. It also follows that A is additive mapping with $A(0) = 0$.

Now, let $A' : E_1 \to E_2$ be another additive mapping satisfying (6.3). Then we have

$$\begin{aligned}
&||A(x) - A'(x)|| \\
&= 2^{-n} ||A(2^n x) - A'(2^n x)|| \\
&\leq 2^{-n} (||A(2^n x) - f(2^n x) - f(0)|| + ||A'(2^n x) - f(2^n x) - f(0)||) \\
&\leq \frac{2 \cdot 2\varepsilon}{2 - 2^p} \cdot \sum_{k=n}^{\infty} 2^{k(p-1)} \cdot \|x\|^p
\end{aligned}$$

for all $x \in E_1$ and all positive integers n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $A(x) = A'(x)$ for all $x \in E_1$. This proves the uniqueness of A. $\square$

Corollary 6.2. *Suppose that $f : E_1 \to E_2$ satisfies the inequality*

$$||f(x+y) - f(x-y) - 2f(y)|| \leq \begin{cases} (i) & \varepsilon ||x||^p \\ \\ (ii) & \varepsilon ||y||^p \end{cases}$$

for all $x, y \in E_1$ and for some fixed $p \in \mathbb{R}$ such that $0 \leq p < 1$.

Then there exists an unique additive mapping $A : E_1 \to E_2$ as a solution of $(\mathcal{J})$, which satisfies (6.2) and

$$||f(x) + f(0) - A(x)|| \leq \frac{\varepsilon ||x||^p}{2 - 2^p} \qquad (x \in E_1).$$

Corollary 6.3. *Suppose that $f : E_1 \to E_2$ satisfies the inequality*

$$||f(x+y) - f(x-y) - 2f(y)|| \leq \varepsilon.$$

Then there exists an unique additive mapping $A : E_1 \to E_2$ as a solution of $(\mathcal{J})$, which satisfies (6.2) and

$$||f(x) + f(0) - A(x)|| \leq \varepsilon \qquad (x \in E_1).$$

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