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A FIXED POINT APPROACH TO THE STABILITY OF A GENERALIZED CAUCHY FUNCTIONAL EQUATION

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Abstract. We investigate the following generalized Cauchy functional equation

$$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$$

where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, and use a fixed point method to prove its generalized Hyers–Ulam–Rassias stability in Banach modules over a C^*-algebra.

1. Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [22] concerning the stability of group homomorphisms : *Let* $(G_1, *)$ *be a group and let* $(G_2, \diamond, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\epsilon > 0$, *does there exist* $\delta(\epsilon) > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d(h(x*y), h(x)\diamond h(y)) < \delta$$

for all $x, y \in G_1$, *then there is a homomorphism* $H : G_1 \to G_2$ *with*

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations where homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a homomorphism near it. D.H. Hyers [6] gave a first affirmative answer to the question of Ulam

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for Banach spaces. Let X and Y be Banach spaces: *Assume that* $f : X \to Y$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for some $\varepsilon \geq 0$ *and all* $x, y \in X$. *Then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$.
T. Aoki [2] and Th.M. Rassias [20] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1. (Th.M. Rassias). *Let* $f : E \to E'$ *be a mapping from a normed vector space* E *into a Banach space* E' *subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, *where* ϵ *and* p *are constants with* $\epsilon > 0$ *and* $p < 1$. *Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. *If* $p < 0$ *then inequality* (1.1) *holds for* $x, y \neq 0$ *and* (1.2) *for* $x \neq 0$. *Also, if for each* $x \in E$ *the mapping* $t \mapsto f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* L *is linear.*

The above inequality has provided a lot of influence in the development of what is now known as a *generalized Hyers–Ulam–Rassias stability* of functional equations. P. Găvruta [5] provided a further generalization of the Th.M. Rassias' theorem. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam–Rassias stability to a number of functional equations and mappings (see [14]–[19]). We also refer the readers to the books [4], [7], [9] and [21].

Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a *generalized metric* on E if d satisfies

(*i*) $d(x, y) = 0$ if and only if $x = y$;
(*ii*) $d(x, y) = d(y, x)$ for all $x, y \in E$;
(*iii*) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.2. [12] *Let* (E, d) *be a complete generalized metric space and let* $J : E \to E$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then for each given element* $x \in E$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all non-negative integers n *or there exists a non-negative integer* n_0 *such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{ y \in E : d(J^{n_0}x, y) < \infty \}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, let A be a unital C^*-algebra with unitary group $U(A)$, unit e and norm $|\cdot|$. Assume that X and Y are left Banach A-modules. An additive mapping $T : X \to Y$ is called *A-linear* if $T(ax) = aT(x)$ for all $a \in A$ and all $x \in X$.

In this paper, we investigate an A-linear mapping associated with the generalized Cauchy functional equation

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}\backslash\{0\}$, and using the fixed point method (see [1, 3, 10, 13]), we prove the generalized Hyers–Ulam–Rassias stability of A-linear mappings in Banach A-modules associated with the functional equation (1.3). The first systematic study of fixed point theorems in nonlinear analysis is due to G. Isac and Th.M. Rassias; cf. [8].

Throughout this paper, α and β are fixed non-zero real numbers. For convenience, we use the following abbreviation for a given $a \in A$ and a mapping $f : X \to Y$,

$$D_a f(x, y) := f(\alpha x + \beta a y) - \alpha f(x) - \beta a f(y)$$

for all $x, y \in X$.

2. Main Results

Lemma 2.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$D_a f(x, y) = 0 \tag{2.1}$$

for all $x, y \in X$ and all $a \in U(A)$. Then f is A-linear.

Proof. Letting $y = 0$ in (2.1), we get $f(\alpha x) = \alpha f(x)$ for all $x \in X$. Similarly, we have $f(\beta y) = \beta f(y)$ for all $y \in X$. Hence (2.1) implies that

$$f(\alpha x + \beta a y) = f(\alpha x) + a f(\beta y) \tag{2.2}$$

for all $x, y \in X$ and all $a \in U(A)$.

Replacing x and y by $\frac{x}{\alpha}$ and $\frac{y}{\beta}$, respectively, in (2.2), we get

$$f(x + ay) = f(x) + af(y) \tag{2.3}$$

for all $x, y \in X$ and all $a \in U(A)$. Letting $a = e \in U(A)$ in (2.3), we infer that f is additive and so $f(rx) = rf(x)$ for $x \in X$ and all rational numbers r. By letting $x = 0$ in (2.3), we get

$$f(ay) = af(y) \tag{2.4}$$

for all $a \in U(A)$ and all $y \in X$. It is clear that (2.4) holds for $a = 0$.

Now let $a \in A$ $(a \neq 0)$ and m an integer greater than $4|a|$. Then $|\frac{a}{m}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By Theorem 1 of [11], there exist three elements $u_1, u_2, u_3 \in U(A)$

such that $\frac{3}{m}a = u_1 + u_2 + u_3$. So $a = \frac{m}{3}(\frac{3}{m}a) = \frac{m}{3}(u_1 + u_2 + u_3)$. Hence by (2.4) we have

$$\begin{aligned} f(ax) &= \frac{m}{3} f(u_1x + u_2x + u_3x) = \frac{m}{3}[f(u_1x) + f(u_2x) + f(u_3x)] \\ &= \frac{m}{3}(u_1 + u_2 + u_3)f(x) = \frac{m}{3}.\frac{3}{m}af(x) = af(x) \end{aligned}$$

for all $x \in X$. So $f : X \to Y$ is A-linear, as desired. □

Now we prove the generalized Hyers–Ulam–Rassias stability of A-linear mappings in Banach A-modules.

Theorem 2.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\lim_{n\to\infty} 2^n \varphi\big(\frac{x}{2^n}, \frac{y}{2^n}\big) = 0, \tag{2.5}$$

$$\|D_a f(x, y)\| \le \varphi(x, y) \tag{2.6}$$

for all $x, y \in X$ and all $a \in U(A)$. If there exists a constant $L < 1$ such that the function

$$x \mapsto \psi(x) := \varphi\big(\frac{x}{2\alpha}, \frac{x}{2\beta}\big) + \varphi\big(\frac{x}{2\alpha}, 0\big) + \varphi\big(0, \frac{x}{2\beta}\big)$$

has the property

$$2\psi(x) \le L\psi(2x)$$

for all $x \in X$, then there exists a unique A-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{1-L}\psi(x) \tag{2.7}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.6), we get

$$\|f(\alpha x) - \alpha f(x)\| \le \varphi(x, 0) \tag{2.8}$$

for all $x \in X$. Similarly, letting $x = 0$ and $a = e \in U(A)$ in (2.6), we get

$$\|f(\beta y) - \beta f(y)\| \le \varphi(0, y) \tag{2.9}$$

for all $y \in X$. So it follows from (2.6), (2.8) and (2.9) that

$$\|f(\alpha x + \beta y) - f(\alpha x) - f(\beta y)\| \le \varphi(x, y) + \varphi(x, 0) + \varphi(0, y)$$

for all $x, y \in X$. Hence

$$\|f(x + y) - f(x) - f(y)\| \le \varphi\big(\frac{x}{\alpha}, \frac{y}{\beta}\big) + \varphi\big(\frac{x}{\alpha}, 0\big) + \varphi\big(0, \frac{y}{\beta}\big) \tag{2.10}$$

for all $x, y \in X$. Letting $y = x$ in (2.10), we get

$$\|f(2x) - 2f(x)\| \le \varphi\big(\frac{x}{\alpha}, \frac{x}{\beta}\big) + \varphi\big(\frac{x}{\alpha}, 0\big) + +\varphi\big(0, \frac{x}{\beta}\big)$$

for all $x, y \in X$. Hence

$$\Big\|f(x) - 2f\big(\frac{x}{2}\big)\Big\| \le \psi(x) \tag{2.11}$$

for all $x \in X$. Let $E := \{ g : X \to Y \mid g(0) = 0 \}$. We introduce a generalized metric on E as follows:

$$d(g,h) := \inf\{ C \in [0,\infty] : \|g(x) - h(x)\| \le C\psi(x) \text{ for all } x \in X \}.$$

It is easy to show that (E,d) is a generalized complete metric space [3].

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = 2g\big(\frac{x}{2}\big), \quad \text{for all } g \in E \text{ and } x \in X.$$

Let $g, h \in E$ and let $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le C$. From the definition of d, we have

$$\|g(x) - h(x)\| \le C\psi(x)$$

for all $x \in X$. By the assumption and last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = 2\Big\|g\big(\frac{x}{2}\big) - h\big(\frac{x}{2}\big)\Big\| \le 2C\psi\big(\frac{x}{2}\big) \le CL\psi(x)$$

for all $x \in X$. So

$$d(\Lambda g, \Lambda h) \le L d(g,h)$$

for any $g, h \in E$. It follows from (2.11) that $d(\Lambda f, f) \le 1$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point T of Λ, i.e.,

$$T : X \to Y, \quad T(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} 2^n f\big(\frac{x}{2^n}\big)$$

and $T(2x) = 2T(x)$ for all $x \in X$. Also T is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f,g) < \infty \}$ and

$$d(T,f) \le \frac{1}{1-L} d(\Lambda f, f) \le \frac{1}{1-L},$$

i.e., inequality (2.7) holds true for all $x \in X$. It follows from the definition of T, (2.5) and (2.6) that

$$\begin{aligned} \|D_a T(x,y)\| &= \lim_{n\to\infty} 2^n \Big\|D_a f\big(\frac{x}{2^n}, \frac{y}{2^n}\big)\Big\| \\ &\le \lim_{n\to\infty} 2^n \varphi\big(\frac{x}{2^n}, \frac{y}{2^n}\big) = 0 \end{aligned}$$

for all $x, y \in X$ and all $a \in U(A)$. By Lemma 2.1, the mapping $T : X \to Y$ is A-linear. Finally it remains to prove the uniqueness of T. Let $P : X \to Y$ be another A-linear mapping satisfying (2.7). Since $d(f,P) \le \frac{1}{1-L}$ and P is additive, $P \in E^*$ and $(\Lambda P)(x) = 2P(x/2) = P(x)$ for all $x \in X$, i.e., P is a fixed point of Λ. Since T is the unique fixed point of Λ in E^*, $P = T$. □

Corollary 2.3. *Let $r > 1$ and θ be non-negative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\|D_a f(x,y)\| \le \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$ and all $a \in U(A)$. Then there exists a unique A-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2(|\alpha|^r + |\beta|^r)\theta}{(2^r - 2)|\alpha\beta|^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y) := \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. $\square$

Theorem 2.4. *Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\Phi : X^2 \to [0, \infty)$ *such that*

$$\lim_{n\to\infty} \frac{1}{2^n} \Phi(2^n x, 2^n y) = 0,$$

$$\|D_a f(x, y)\| \le \Phi(x, y)$$

for all $x, y \in X$ *and all* $a \in U(A)$. *If there exists a constant* $L < 1$ *such that the function*

$$x \mapsto \Psi(x) := \Phi\big(\frac{x}{\alpha}, \frac{x}{\beta}\big) + \Phi\big(\frac{x}{\alpha}, 0\big) + \Phi\big(0, \frac{x}{\beta}\big)$$

has the property

$$\Psi(2x) \le 2L\Psi(x)$$

for all $x \in X$, *then there exists a unique* A*-linear mapping* $T : X \to Y$ *such that*

$$\|f(x) - T(x)\| \le \frac{1}{2 - 2L} \Psi(x) \tag{2.12}$$

for all $x \in X$.

Proof. Using the same method as in the proof of Theorem 2.2, we have

$$\Big\|\frac{1}{2} f(2x) - f(x)\Big\| \le \frac{1}{2}\Psi(x) \tag{2.13}$$

for all $x \in X$. We introduce the same definitions for E and d (replacing Ψ by ψ) as in the proof of Theorem 2.2 such that (E, d) becomes a generalized complete metric space. Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda g)(x) = \frac{1}{2} g(2x), \quad \text{for all } g \in E \text{ and } x \in X.$$

One can show that $d(\Lambda g, \Lambda h) \le L d(g, h)$ for any $g, h \in E$. It follows from (2.13) that $d(\Lambda f, f) \le \frac{1}{2}$. Due to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point T of Λ, i.e.,

$$T : X \to Y, \quad T(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

and $T(2x) = 2T(x)$ for all $x \in X$. Also

$$d(T, f) \le \frac{1}{1 - L} d(\Lambda f, f) \le \frac{1}{2 - 2L},$$

i.e., inequality (2.12) holds true for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2 and we omit the details. $\square$

Corollary 2.5. *Let $0 < r < 1$ and θ, δ be non-negative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\|D_a f(x,y)\| \leq \delta + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$ and all $a \in U(A)$. Then there exists a unique A-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{3\delta}{2-2^r} + \frac{2(|\alpha|^r + |\beta|^r)\theta}{(2-2^r)|\alpha\beta|^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\Phi(x,y) := \delta + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get the desired result. □

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