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ON A FUNCTIONAL EQUATION CONTAINING FOUR WEIGHTED ARITHMETIC MEANS

ADRIENN VARGA[1]

Submitted by T. Reidel

ABSTRACT. In this paper we solve the functional equation

$$f\big(\alpha x+(1-\alpha)y\big)+f\big(\beta x+(1-\beta)y\big)=f\big(\gamma x+(1-\gamma)y\big)+f\big(\delta x+(1-\delta)y\big)$$

which holds for all $x, y \in I$, where $I \subset \mathbb{R}$ is a non-void open interval, $f\colon I \to \mathbb{R}$ is an unknown function and $\alpha, \beta, \gamma, \delta \in (0,1)$ are arbitrarily fixed.

1. INTRODUCTION AND PRELIMINARIES

Consider the functional equation

$$f\big(\alpha x+(1-\alpha)y\big)+f\big(\beta x+(1-\beta)y\big)=f\big(\gamma x+(1-\gamma)y\big)+f\big(\delta x+(1-\delta)y\big) \tag{1.1}$$

which holds for all $x, y \in I$, where $I \subset \mathbb{R}$ is a non-void open interval, $f\colon I \to \mathbb{R}$ is an unknown function and the parameters $\alpha, \beta, \gamma, \delta \in [0,1]$ are arbitrarily fixed. The particular case $\gamma = 1$, $\delta = 0$ has been investigated in Daróczy-Maksa-Páles [3], Daróczy-Lajkó-Lovas-Maksa-Páles [11], and also in Maksa [12] in connection with the equivalence of certain functional equations involving means. The purpose of this paper is to extend these results for arbitrary possible values of the weights α, β, γ, δ. The paper is organized as follows. First of all we study the special cases when at least two parameters are the same. The condition that $\alpha = \gamma$ and $\beta = \delta$ (or $\alpha = \delta$ and $\beta = \gamma$) do not hold at the same time is natural to avoid the trivialities. To investigate the general case with pairwise different parameters we use a relation to divide the space of the parameters into regions. By the help

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of these regions we can discuss the possible cases easier. As we shall see, the solutions of (1.1) have the general form

$$f(x) = A_2(x,x) + A_1(x) + A_0 \quad (x \in I),$$

where $A_k\colon \mathbb{R}^k \to \mathbb{R}$ are symmetric k-additive functions ($k = 0,1,2$) with the property

$$A_2(\alpha x, \beta x) = A_2(\gamma x, \delta x) \quad (x \in \mathbb{R}).$$

The existence of the solutions with non-zero biadditive part depends on the algebraic properties of the parameters. Here we introduce some basic notions we need in the following. Throughout the paper I denotes a non-void open interval.

Definition 1.1. For a fixed $p \in (0,1)$ the function $f\colon I \to \mathbb{R}$ is called *p-Wright affine* on I if

$$f\big(px + (1-p)y\big) + f\big((1-p)x + py\big) = f(x) + f(y)$$

holds for every $x, y \in I$. If $p = \frac{1}{2}$ then f is called *Jensen affine*.

It is well-known that every Jensen affine function on the interval I has the form

$$f(x) = A(x) + b \quad (x \in I),$$

where $A\colon \mathbb{R} \to \mathbb{R}$ is an additive function and $b \in \mathbb{R}$ is a constant, see Lajkó [8]. As a basic result for p-Wright affine functions in general we need the following theorem due to Lajkó [7] (for the terminology see Székelyhidi [10]).

Theorem 1.2. *The function f is p-Wright affine on the interval I if and only if there exist symmetric k-additive functions $A_k\colon \mathbb{R}^k \to \mathbb{R}$ ($k = 0,1,2$) with the property*

$$A_2(px, (1-p)x) = 0 \quad (x \in \mathbb{R})$$

such that

$$f(x) = A_2(x,x) + A_1(x) + A_0 \quad (x \in I).$$

We also need the localizability theorem due to Gilányi-Páles [4].

Theorem 1.3. *The function f is p-Wright affine on the interval I if and only if for any $\xi \in I$ there is an $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subset I$ and the restriction $f|_{(\xi-\varepsilon,\xi+\varepsilon)}$ is p-Wright affine function on the interval $(\xi - \varepsilon, \xi + \varepsilon)$.*

We will use the following simple remarks very frequently .

Remark 1.4. Let $(x,y) \in I^2$ and $\alpha, \beta \in (0,1)$ are different real numbers. Consider the linear transformation having the matrix

$$P := \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}.$$

Then $\det P = \alpha - \beta \neq 0$. Since every regular linear transformation is an open mapping and

$$P\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} \qquad (x \in I),$$

every point of

$$diag\ I^2 := \{(\xi,\xi) \mid \xi \in I\}$$

is an interior point of the set $P(I^2)$ (the image of I^2 under P). Thus for any point $\xi \in I$ there is an $\varepsilon > 0$ such that

$$(\xi - \varepsilon, \xi + \varepsilon)^2 \subset P(I^2).$$

Remark 1.5. Every locally constant function on an open interval is constant. This means that f is constant on the interval I if and only if for any point $\xi \in I$ there is an $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subset I$ and the restriction $f|_{(\xi-\varepsilon,\xi+\varepsilon)}$ is constant on $(\xi - \varepsilon, \xi + \varepsilon)$. Indeed, if f is constant on $(\xi - \varepsilon, \xi + \varepsilon)$ then there exists the derivate of f at the point ξ and $f'(\xi) = 0$ for all $\xi \in I$. Therefore f is constant on I. The converse is trivial.

2. Special cases

The scheme below shows the special and trivial cases in terms of the parameters. Following the arrows we can find the classes of the solutions:

$$\begin{array}{ccccccc}
\alpha = \beta & \to & \gamma \neq \delta & \to & \alpha \neq \frac{\gamma+\delta}{2} & \to & \text{constant functions} \\
 & & \downarrow & & & & \\
 & & \alpha = \frac{\gamma+\delta}{2} & \to & & & \text{Jensen affine functions} \\
\alpha = \beta & \to & \gamma = \delta & \to & \alpha \neq \gamma & \to & \text{constant functions} \\
 & & \downarrow & & & & \\
 & & \alpha = \gamma & \to & & & \text{all functions}
\end{array}$$

A similar method can be used to illustrate the case $\gamma = \delta$, i.e. when the weights on the same side coincide. Another possible special cases are considered in the next scheme:

$$\begin{array}{ccccc}
\alpha = \gamma & \to & \beta \neq \delta & \to & \text{constant functions} \\
\downarrow & & & & \\
\beta = \delta & \to & & & \text{all functions}
\end{array}$$

A similar method can be used to illustrate the cases $\alpha = \delta$ or $\beta = \gamma$ or $\beta = \delta$. As we can see it is a natural condition to avoid the trivialities that $\alpha = \gamma$ and $\beta = \delta$ (or $\alpha = \delta$ and $\beta = \gamma$) do not hold at the same time. For simplicity we shall restrict our consideration to the following special cases:

(i) $\alpha = \beta$ and $\gamma = \delta$; then our equation

$$f\big(\alpha x + (1-\alpha)y\big) = f\big(\gamma x + (1-\gamma)y\big) \quad (x, y \in I),$$

(ii) $\alpha = \beta$ and $\gamma \neq \delta$; then our equation

$$2f\big(\alpha x + (1-\alpha)y\big) = f\big(\gamma x + (1-\gamma)y\big) + f\big(\delta x + (1-\delta)y\big) \quad (x, y \in I).$$

It is easy to see that in the further special cases listing in the schemes above we get a similar form of our equation as in (i) and (ii).

Theorem 2.1. *Let $\alpha, \gamma \in (0,1)$ be fixed such that $\alpha \neq \gamma$. The function $f\colon I \to \mathbb{R}$ satisfies the equation*

$$f\big(\alpha x + (1-\alpha)y\big) = f\big(\gamma x + (1-\gamma)y\big) \quad (x, y \in I)$$

if and only if f is constant on I.

Proof. Consider the transformation

$$u = \alpha x + (1-\alpha)y, \qquad v = \gamma x + (1-\gamma)y \quad \text{ if } \ (x,y) \in I^2$$

which takes any point $(x,y) \in I^2$ to the point $(u,v) \in P_1(I^2)$ (the image of I^2 under P_1), where

$$P_1 := \begin{pmatrix} \alpha & 1-\alpha \\ \gamma & 1-\gamma \end{pmatrix}.$$

It is easy to see that $f(u) = f(v)$ holds for all $(u,v) \in P_1(I^2)$. Since every point of $\operatorname{diag} I^2$ is an interior point of the set $P_1(I^2)$, for any $u \in I$ there exists an $\varepsilon > 0$ such that

$$\{u\} \times (u-\varepsilon, u+\varepsilon) \subset P_1(I^2).$$

This means that $f(u) = f(v)$ holds for all $v \in (u-\varepsilon, u+\varepsilon)$ from which it follows that f is constant on $(u-\varepsilon, u+\varepsilon)$. According to Remark 1.5 the statement follows easily. The converse is trivial. □

Theorem 2.2. *Let $\alpha, \gamma, \delta \in (0,1)$ be pairwise different real numbers. Suppose that $f\colon I \to \mathbb{R}$ satisfies the equation*

$$2f\big(\alpha x + (1-\alpha)y\big) = f\big(\gamma x + (1-\gamma)y\big) + f\big(\delta x + (1-\delta)y\big) \quad (x, y \in I).$$

(i) *If $\alpha = \frac{\gamma+\delta}{2}$ then f is Jensen affine.*
(ii) *If $\alpha \neq \frac{\gamma+\delta}{2}$ then f is constant.*

Proof. The proof is similar to that of Theorem 2.1. Using the transformation

$$u = \gamma x + (1-\gamma)y, \qquad v = \delta x + (1-\delta)y \quad \text{ if } \ (x,y) \in I^2$$

we get that

$$2f\left(\frac{\alpha-\delta}{\gamma-\delta}u + \frac{\gamma-\alpha}{\gamma-\delta}v\right) = f(u) + f(v) \quad \text{if} \quad (u,v) \in P_2(I^2), \tag{2.1}$$

where

$$P_2 := \begin{pmatrix} \gamma & 1-\gamma \\ \delta & 1-\delta \end{pmatrix}.$$

With the notation $p := \frac{\alpha-\delta}{\gamma-\delta}$ equation (2.1) goes over into

$$2f(pu + (1-p)v) = f(u) + f(v) \quad \text{if} \quad (u,v) \in P_2(I^2).$$

If $p = \frac{1}{2}$ then we have that $\alpha = \frac{\gamma+\delta}{2}$. Then the equation above is

$$f\left(\frac{1}{2}u + \frac{1}{2}v\right) = \frac{f(u)+f(v)}{2} \quad \text{if} \quad (u,v) \in P_2(I^2).$$

Using Remark 1.4 and the localizability theorem we get that f is Jensen affine on I if $\alpha = \frac{\gamma+\delta}{2}$.

If $p \neq \frac{1}{2}$ then we repeat the argumentation as above. Since for all $\xi \in I$ there is an $\varepsilon > 0$ such that

$$(\xi - \varepsilon, \xi + \varepsilon)^2 \subset P_2(I^2),$$

for all $(u, v) \in {I_\xi}^2 := (\xi - \varepsilon, \xi + \varepsilon)^2$ the role of u and v is commutable. Therefore we have

$$f(pu + (1-p)v) = f(pv + (1-p)u) \quad \text{if} \quad (u, v) \in {I_\xi}^2.$$

Consider the transformation

$$t = pu + (1-p)v, \qquad s = pv + (1-p)u$$

we get that $f(t) = f(s)$ for any $(t, s) \in P_3({I_\xi}^2) \cap {I_\xi}^2$, where

$$P_3 := \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

Since every point of diag ${I_\xi}^2$ is an interior point of the set $P_3({I_\xi}^2) \cap {I_\xi}^2$, for any $t \in I_\xi$ there is an $\varepsilon > 0$ such that

$$\{t\} \times (t - \varepsilon, t + \varepsilon) \subset P_3({I_\xi}^2) \cap {I_\xi}^2.$$

This means that $f(t) = f(s)$ holds for all $s \in (t-\varepsilon, t+\varepsilon)$ from which it follows that f is constant on $(t - \varepsilon, t + \varepsilon)$. Therefore f is locally constant and, consequently, constant on I_ξ. Since $\xi \in I$ was arbitrary we can use Remark 1.5 again to prove that f is constant on I. □

Remark 2.3. Note that the converse statements of Theorem 2.2 are also valid.

3. The general case

Now we may restrict the consideration of the functional equation (1.1) to the case of pairwise different parameters $\alpha, \beta, \gamma, \delta$.

Having fixed $(a, b) \in (0, 1)^2$ consider the relation on the set $(0, 1)^2$ by

$$(\tilde{a}, \tilde{b}) \lhd (a, b) \text{ if } \tilde{a} < a, \quad \tilde{b} < b.$$

The sets

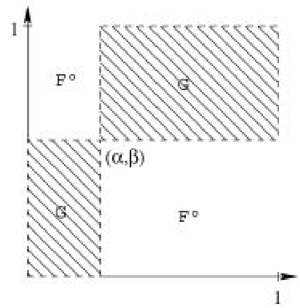
$$G := \{(\tilde{a}, \tilde{b}) \in (0, 1)^2 \mid (\tilde{a}, \tilde{b}) \lhd (a, b) \text{ or } (a, b) \lhd (\tilde{a}, \tilde{b})\}$$

and the interior of its complement F° with respect to $(0, 1)^2$ will be important for us. The point (a, b) is called the *appointed pair*. We distinguish two cases:

(I) The case $\alpha + \beta \neq \gamma + \delta$,
(II) The case $\alpha + \beta = \gamma + \delta$.

(I) Without loss of generality we may assume that $\alpha < \beta$. Using the relation with the appointed pair (α, β) introduced above, we have to investigate the cases $(\gamma, \delta) \in G$ and $(\gamma, \delta) \in F^\circ$, where the parameters are pairwise different.

(i) It is easy to see that $(\gamma, \delta\ ) \in F^\circ$ if and only if one of the following cases holds:



$$\min\{\gamma,\delta\} < \alpha < \beta < \max\{\gamma,\delta\} \quad \text{or} \quad \alpha < \min\{\gamma,\delta\} < \max\{\gamma,\delta\} < \beta$$

i.e. the parameters α, β are between γ, δ or the parameters γ, δ are between α, β. It is enough to investigate the case $\alpha < \gamma < \delta < \beta$. Using the transformation

$$u = \alpha x + (1-\alpha)y, \quad v = \beta x + (1-\beta)y \quad (x,y) \in I^2$$

we get the equation

$$f(u) + f(v) = f\left(\frac{\gamma-\beta}{\alpha-\beta}u + \frac{\alpha-\gamma}{\alpha-\beta}v\right) + f\left(\frac{\delta-\beta}{\alpha-\beta}u + \frac{\alpha-\delta}{\alpha-\beta}v\right) \tag{3.1}$$

where $(u,v) \in P_4(I^2)$ (the image of I^2 under P_4) and

$$P_4 := \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}.$$

The coefficients of u and v are between 0 and 1 as one can easily check. According to the conditions (I) and (i) equation (1.1) has the form

$$f\big(pu + (1-p)v\big) + f\big(qu + (1-q)v\big) = f(u) + f(v) \quad (u,v) \in P_4(I^2)$$

$$\text{where } p, q \in (0,1) \text{ and } p + q \neq 1.$$

Using Remark 1.4 for any $\xi \in I$ there is an $\varepsilon > 0$ such that

$$f\big(pu + (1-p)v\big) + f\big(qu + (1-q)v\big) = f(u) + f(v)$$

holds on the interval $J_\xi := (\xi - \varepsilon, \xi + \varepsilon) \subset I$. Let $\xi \in I$ be fixed. Results in Maksa [12], see also Theorem 1 in Daróczy [2], imply that f is constant on J_ξ for any $\xi \in I$. Using Remark 1.5 we get that f is a constant function on the entire interval I. We have just proved the following result.

Theorem 3.1. *Let $\alpha, \beta, \gamma, \delta \in (0,1)$ be pairwise different parameters such that $\alpha + \beta \neq \gamma + \delta$. Suppose that $f\colon I \to \mathbb{R}$ satisfies the equation*

$$f\big(\alpha x + (1-\alpha)y\big) + f\big(\beta x + (1-\beta)y\big) = f\big(\gamma x + (1-\gamma)y\big) + f\big(\delta x + (1-\delta)y\big)$$

for all $x, y \in I$. If the parameters α, β are between γ, δ or the parameters γ, δ are between α, β then f is constant.

(ii) Now we investigate the case $(\gamma, \delta) \in G$.

Note that in the case of pairwise different parameters $\alpha, \beta, \gamma, \delta$ the property $(\gamma, \delta) \in G$ means that at most one of the parameters γ and δ is between the parameters α and β. Without loss of generality we may assume that

$$\alpha < \beta < \gamma < \delta \quad \text{or} \quad \alpha < \gamma < \beta < \delta.$$

At first we prove the following lemma.

Lemma 3.2. *Let $\alpha, \beta, \gamma, \delta \in (0,1)$ be pairwise different real numbers, $(\gamma, \delta) \in G$ such that $\alpha < \beta < \gamma < \delta$ or $\alpha < \gamma < \beta < \delta$, and $p := \frac{\gamma-\delta}{\alpha-\delta}$, $q := \frac{\beta-\delta}{\alpha-\delta}$. Suppose that $f\colon I \to \mathbb{R}$ satisfies functional equation* (1.1)*. Then $p,\ q \in (0,1)$ and for all $\xi \in I$ there exists $\varepsilon > 0$ such that*

$$f(u) - f(v) = f\big(pu + (1-p)v\big) - f\big(qu + (1-q)v\big), \tag{3.2}$$

holds for all $u,\ v \in J_\xi := (\xi - \varepsilon, \xi + \varepsilon) \subset I$.

Proof. Let the least and the highest parameters be on the same side of equation (1.1). Then

$$f\big(\alpha x + (1-\alpha)y\big) - f\big(\delta x + (1-\delta)y\big) = f\big(\gamma x + (1-\gamma)y\big) - f\big(\beta x + (1-\beta)y\big)$$

holds for all $x, y \in I$. Using the transformation $u = \alpha x + (1-\alpha)y$ and $v = \delta x + (1-\delta)y$ we have

$$f(u) - f(v) = f\left(\frac{\gamma-\delta}{\alpha-\delta}u + \frac{\alpha-\gamma}{\alpha-\delta}v\right) - f\left(\frac{\beta-\delta}{\alpha-\delta}u + \frac{\alpha-\beta}{\alpha-\delta}v\right)$$

for all $(u,v) \in P_5(I^2)$ (the image of I^2 under P_5), where

$$P_5 := \begin{pmatrix} \alpha & 1-\alpha \\ \delta & 1-\delta \end{pmatrix}.$$

Let $\xi \in I$ be arbitrarily fixed. According to Remark 1.4 there exists $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon)^2 \subset P_5(I^2)$ and the equation can be written in the form

$$f(u) - f(v) = f\big(pu + (1-p)v\big) - f\big(qu + (1-q)v\big)$$

for all $u, v \in J_\xi$, where $p, q \in (0,1)$ because of $(\gamma, \delta) \in G$ such that $\alpha < \beta < \gamma < \delta$ or $\alpha < \gamma < \beta < \delta$. □

Lemma 3.3. *Let $\xi \in I$ be arbitrarily fixed and assume that f satisfies the functional equation* (3.2) *for all $u, v \in J_\xi$. Then there exists $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that $\tilde{f}$ satisfies* (3.2) *for all $u, v \in \mathbb{R}$ and $\tilde{f}|_{J_\xi} = f$.*

Proof. The lemma is a simple consequence of Theorem 5 in Páles [5] in the following setting

$$F = X = \mathbb{R}, \quad K = I, \quad \varphi_0 = 0, \quad \varphi_i\colon \mathbb{R} \to \mathbb{R} \quad i = 1,2,3$$

$$\varphi_1(x) = \varphi_2(x) = x, \quad \varphi_3(x) = -x,$$

$$a_1 = 0, \quad b_1 = 1, \quad a_2 = p, \quad b_2 = 1-p \quad a_3 = q, \quad b_3 = 1-q.$$

□

Lemma 3.4. *Let $\varphi_i, \psi_i\colon \mathbb{R} \to \mathbb{R}$ be homomorphisms of $\mathbb{R}$ onto itself such that*

$$\mathrm{Rg}\ (\psi_j \circ \psi_i^{-1} - \varphi_j \circ \varphi_i^{-1}) = \mathbb{R} \quad for \quad i \neq j \quad (i,j = 1,2,3). \tag{3.3}$$

If the functions $f_i\colon \mathbb{R} \to \mathbb{R}$ $(i = 0,1,2,3)$ satisfy the functional equation

$$f_0(x) + \sum_{i=1}^{3} f_i\big(\varphi_i(x) + \psi_i(y)\big) = 0 \qquad (x, y \in \mathbb{R})$$

then there exist $A_k^i\colon \mathbb{R}^k \to \mathbb{R}$ $(k = 0,1,2;\ i = 0,1,2,3)$ *k-additive symmetric functions such that*

$$f_i(x) = A_2^i(x,x) + A_1^i(x) + A_0^i \quad (i = 0,1,2,3) \quad (x \in \mathbb{R}).$$

Proof. The lemma is an easy consequence of Theorem 3.9 in Székelyhidi [9]. □

Theorem 3.5. *Let* $\alpha, \beta, \gamma, \delta \in (0,1)$ *be pairwise different real numbers and* $(\gamma, \delta) \in G$ *such that* $\alpha < \beta < \gamma < \delta$ *or* $\alpha < \gamma < \beta < \delta$*. The function* $f\colon I \to \mathbb{R}$ *satisfies equation* (1.1) *if and only if* f *is constant.*

Proof. We prove only the nontrivial part. Let $f\colon I \to \mathbb{R}$ be a solution of equation (1.1) and $\xi \in I$ be arbitrarily fixed. According to Lemma 3.2 and Lemma 3.3 there exists $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{f}(u) - \tilde{f}(v) = \tilde{f}\big(pu + (1-p)v\big) - \tilde{f}\big(qu + (1-q)v\big),$$

for all $u, v \in \mathbb{R}$, where $p := \frac{\gamma-\delta}{\alpha-\delta} \in (0,1)$ and $q := \frac{\beta-\delta}{\alpha-\delta} \in (0,1)$; moreover, $\tilde{f}|_{J_\xi} = f$ where $J_\xi := (\xi - \varepsilon, \xi + \varepsilon) \subset I$ for some $\varepsilon > 0$. Using the substitutions

$$u = x + y \quad \text{and} \quad v = y \qquad (x, y \in \mathbb{R})$$

it follows that

$$\tilde{f}(y) + \tilde{f}(y + px) - \tilde{f}(y + qx) - \tilde{f}(x + y) = 0 \quad (x, y \in \mathbb{R}). \tag{3.4}$$

If we show, that $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is constant then we have that f is locally constant because ξ was arbitrarily fixed. Applying Remark 1.5 we are ready with the proof. To prove this, apply Lemma 3.4 for equation (3.4) in the following setting

$$f_0 = \tilde{f}, \quad f_1 = \tilde{f}, \quad f_2 = -\tilde{f}, \quad f_3 = -\tilde{f}$$

$$\varphi_1(x) = px, \quad \varphi_2(x) = qx, \quad \varphi_3(x) = x,$$

$$\psi_1(x) = x, \quad \psi_2(x) = x, \quad \psi_3(x) = x \quad (x \in \mathbb{R}).$$

It is easy to check that conditions (3.3) hold because $p, q \in (0,1)$ and $\alpha, \beta, \gamma, \delta$ are pairwise different. Thus we get that there exist symmetric k-additive functions $A_k\colon \mathbb{R}^k \to \mathbb{R}$ $(k = 0,1,2)$ such that

$$\tilde{f}(x) = A_2(x,x) + A_1(x) + A_0 \quad (x \in \mathbb{R}).$$

Substituting this form of $\tilde{f}$ into (3.4) we get that

$$\begin{aligned} &A_2(px,px) - A_2(qx,qx) - A_2(x,x) + 2A_2(px,y) + A_1(px) - \\ &\quad -2A_2(qx,y) - A_1(qx) - 2A_2(x,y) - A_1(x) = 0 \quad (x, y \in \mathbb{R}). \end{aligned} \tag{3.5}$$

Since x is an arbitrary real number we can replace x by $-x$. Because of the rational homogeneity it follows that

$$\begin{aligned} &A_2(px,px) - A_2(qx,qx) - A_2(x,x) - 2A_2(px,y) - A_1(px) + \\ &\quad +2A_2(qx,y) + A_1(qx) + 2A_2(x,y) + A_1(x) = 0 \quad (x, y \in \mathbb{R}). \end{aligned} \tag{3.6}$$

According to (3.5) and (3.6) we get that

$$2A_2(px,y) + A_1(px) - 2A_2(qx,y) - A_1(qx) - 2A_2(x,y) - A_1(x) = 0$$

for all $x, y \in \mathbb{R}$, or equvivalently

$$2A_2\big((p-q-1)x, y\big) + A_1\big((p-q-1)x\big) = 0 \quad (x, y \in \mathbb{R}).$$

If $y = 0$ then we obtain that

$$A_1\big((p-q-1)x\big) = 0 \ \ (x \in \mathbb{R}) \ \text{ and thus } \ A_2\big((p-q-1)x, y\big) = 0$$

for all $x, y \in \mathbb{R}$. The condition $\alpha+\beta \neq \gamma+\delta$ with pairwise different real numbers implies that $p - q \neq 1$. Thus we get that $A_1(x) = 0$ and $A_2(x, x) = 0$ $(x \in \mathbb{R})$, that is $\tilde{f}$ is constant. □

(II) The case $\alpha+\beta = \gamma+\delta$. The following lemma shows that the investigation of the parameters is simplier than in the case of (I).

Lemma 3.6. *If $\alpha, \beta, \gamma, \delta$ are pairwise different real numbers, $\alpha + \beta = \gamma + \delta$ and (α, β) is the appointed pair then $(\gamma, \delta) \in F^\circ$.*

Proof. It is sufficient to show that the statement holds for the case of $\alpha < \beta$ and $\gamma < \delta$. In this case the lemma says that if $\alpha + \beta = \gamma + \delta$ then

$$\alpha < \gamma < \delta < \beta \quad \text{or} \quad \gamma < \alpha < \beta < \delta.$$

In contrast with our assertion suppose that

$$\alpha < \beta < \gamma < \delta \quad \text{or} \quad \gamma < \delta < \alpha < \beta \quad \text{or}$$

$$\alpha < \gamma < \beta < \delta \quad \text{or} \quad \gamma < \alpha < \delta < \beta.$$

If $\alpha < \beta < \gamma < \delta$ then

$$\alpha + \beta < \beta + \beta < \gamma + \gamma < \gamma + \delta$$

which is a contradiction. In the case of $\gamma < \delta < \alpha < \beta$ the method of the argumentation is the same. If $\alpha < \gamma < \beta < \delta$ then adding the inequalities $\alpha < \gamma$ and $\beta < \delta$ we get a contradiction. In the case of $\gamma < \alpha < \delta < \beta$ the method of the argumentation is the same. □

Without loss of generality we may assume that $\alpha < \gamma < \delta < \beta$ because of Lemma 3.6. After proving the following lemma our equation can be reduced to the functional equation of p-Wright affine functions.

Lemma 3.7. *If the function $f : I \to \mathbb{R}$ satisfies the functional equation* (1.1) *and $\alpha < \gamma < \delta < \beta$ such that $\alpha + \beta = \gamma + \delta$ then f is p-Wright affine on the interval I, where*

$$p := \frac{\gamma - \beta}{\alpha - \beta}.$$

Proof. The transformation

$$u = \alpha x + (1-\alpha)y, \qquad v = \beta x + (1-\beta)y \quad \text{if } \ (x, y) \in I^2$$

leads us to equation (3.1) again. Using the notations

$$p := \frac{\gamma - \beta}{\alpha - \beta} \ \text{ and } \ q := \frac{\delta - \beta}{\alpha - \beta}$$

it follows that $p+q=1$ because $\alpha+\beta=\gamma+\delta$. It is also clear that if $\alpha<\gamma<\delta<\beta$ then $p,q\in(0,1)$. Therefore equation (3.1) can be written in the form

$$f\big(pu+(1-p)v\big)+f\big((1-p)u+pv\big)=f(u)+f(v)\quad (u,v)\in P_4(I^2) \tag{3.7}$$

where $p:=\frac{\gamma-\beta}{\alpha-\beta}$. So we get that f is p-Wright affine but it is only on the set $P_4(I^2)$ at this moment. According to Remark 1.4 for any $\xi\in I$ there is an $\varepsilon>0$ such that equation (3.7) holds on $J_\xi:=(\xi-\varepsilon,\xi+\varepsilon)\subset I$. Then we can apply Theorem 1.3. Therefore we have that f is p-Wright affine on the interval I, where $p:=\frac{\gamma-\beta}{\alpha-\beta}$. □

Finally we can formulate the main result of (II) as follows.

Theorem 3.8. *Let $\alpha,\beta,\gamma,\delta\in(0,1)$ be pairwise different real numbers and $\alpha+\beta=\gamma+\delta$. The function $f\colon I\to\mathbb{R}$ satisfies the functional equation*

$$f\big(\alpha x+(1-\alpha)y\big)+f\big(\beta x+(1-\beta)y\big)=f\big(\gamma x+(1-\gamma)y\big)+f\big(\delta x+(1-\delta)y\big)$$

for all $x,y\in I$ if and only if there exist symmetric k-additive functions $A_k\colon\mathbb{R}^k\to\mathbb{R}$ ($k=0,1,2$) with the property

$$A_2(\alpha x,\beta x)=A_2(\gamma x,\delta x)\quad (x\in\mathbb{R})$$

such that

$$f(x)=A_2(x,x)+A_1(x)+A_0\quad (x\in I).$$

Proof. Taking into consideration Theorem 1.2 we have to prove the equivalence of the conditions

$$\text{(a)}\ \ A_2(px,(1-p)x)=0\ \ \text{and}\ \ \text{(b)}\ \ A_2(\alpha x,\beta x)=A_2(\gamma x,\delta x)\quad (x\in\mathbb{R})$$

where $p:=\frac{\gamma-\beta}{\alpha-\beta}$. To see that (a) implies (b) replace x by $(\alpha-\beta)x$ and use the symmetry and the biadditivity of A_2. Recall that $\alpha-\gamma=\delta-\beta$. To see that (b) implies (a) replace x by $\frac{x}{\alpha-\beta}$ and use the symmetry and the biadditivity of A_2. Conversely, the condition

$$A_2(\alpha x,\beta x)=A_2(\gamma x,\delta x)\quad (x\in\mathbb{R})$$

implies the identity

$$A_2(\gamma y,\delta x)+A_2(\gamma x,\delta y)=A_2(\alpha y,\beta x)+A_2(\alpha x,\beta y)$$

by replacing x by $x+y$. After a straightforward calculation we have that f is the solution of equation (1.1). □

4. Examples

To construct examples for the existence of solutions with non-zero biadditive part we use the following Lemma. This is the direct consequence of Lemma 1 and 2 in [11] based on Daróczy's known theorem [1], see also Kuczma [6].

Lemma 4.1. *There exists a not identically zero symmetric biadditive function $A_2\colon\mathbb{R}^2\to\mathbb{R}$ with the property*

$$A_2(\lambda x,x)=0\quad (x\in\mathbb{R})$$

if and only if λ is transcendental or if λ is algebraic and $-\lambda$ is an algebraic conjugate of λ.

Using the equivalence of (a) and (b) from the proof of Theorem 3.8 we can easily calculate that

$$A_2\left(\frac{1-\frac{\beta}{\gamma}}{1-\frac{\alpha}{\gamma}}x, x\right) = 0 \quad (x \in \mathbb{R})$$

if $\alpha < \gamma < \delta < \beta$. It easy to prove that if exactly one of the numbers $\frac{\beta}{\gamma}$ and $\frac{\alpha}{\gamma}$ is transcendental then $\frac{1-\frac{\beta}{\gamma}}{1-\frac{\alpha}{\gamma}}$ is transcendental. Combining these facts with the condition $\alpha < \gamma < \delta < \beta$ one can easily check that, in the case

$$\alpha = \frac{1}{2ce}, \quad \beta = \frac{1}{c}, \quad \gamma = \frac{1}{ce}, \quad \delta = \frac{2e-1}{2ce},$$

where $c > 1$ is a real constant and e is the Euler number, there exists a solution of (1.1) with non-zero biadditive part.

5. Summary

Omitting the trivial cases $\alpha = \gamma$ and $\beta = \delta$ or $\alpha = \delta$ and $\delta = \gamma$ all solutions of functional equation (1.1) have the general form

$$f(x) = A_2(x, x) + A_1(x) + A_0,$$

where $A_k\colon \mathbb{R}^k \to \mathbb{R}$ are symmetric k-additive functions and k=0,1,2.

Theorems 2.1, 2.2 (ii), 3.1 and 3.5 imply that

(I) in the case $\alpha + \beta \neq \gamma + \delta$ the solutions of (1.1) are only the constant functions.

Theorems 2.2 (i), 3.8 and Lemma 4.1 imply that

(II) in the case $\alpha + \beta = \gamma + \delta$
  (1) the solutions are Jensen affine if $\alpha = \beta$ or $\gamma = \delta$,
  (2) there exsist solutions with nonzero biadditive part such that

$$A_2(\alpha x, \beta x) = A_2(\gamma x, \delta x)$$

  if and only if $\lambda := \frac{1-\frac{\beta}{\gamma}}{1-\frac{\alpha}{\gamma}}$ transcendental or if λ algebraic and -λ is an algebraic conjugate of λ.

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[1] Institute of Mathematics, University of Debrecen, P.O. Box 12, Debrecen, Hungary.

E-mail address: vargaa@math.klte.hu