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REVERSES OF THE GOLDEN–THOMPSON TYPE INEQUALITIES DUE TO ANDO-HIAI-PETZ

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This paper is dedicated to Professor Josip E. Pečarić

Submitted by M. Frank

ABSTRACT. In this paper, we show reverses of the Golden–Thompson type inequalities due to Ando, Hiai and Petz: Let H and K be Hermitian matrices such that $mI \le H, K \le MI$ for some scalars $m \le M$, and let $\alpha \in [0,1]$. Then for every unitarily invarint norm

$$|||e^{(1-\alpha)H+\alpha K}||| \ \le \ S(e^{p(M-m)})^{\frac{1}{p}} \ ||| \left(e^{pH} \ \sharp_\alpha \ e^{pK}\right)^{\frac{1}{p}} |||$$

holds for all $p > 0$ and the right-hand side converges to the left-hand side as $p \downarrow 0$, where $S(a)$ is the Specht ratio and the α-geometric mean $X \ \sharp_\alpha \ Y$ is defined as

$$X \ \sharp_\alpha \ Y = X^{\frac{1}{2}} \left(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}}\right)^{\alpha} X^{\frac{1}{2}} \qquad \text{for all} \ \ 0 \le \alpha \le 1$$

for positive definite X and Y.

1. INTRODUCTION.

Let $\mathbb{M}_n$ denote the space of n-by-n complex matrices and I stands for the identity matrix. For a pair X, Y of Hermitian matrices, the order relation $X \ge Y$ means as usual that $X - Y$ is positive semidefinite. In particular, $X > 0$ means

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that X is positive definite. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is said to be unitarily invariant if

$$|||UXV||| = |||X|||, \quad X \in \mathbb{M}_n$$

for all unitary U, V. Throughout the paper, the symbol $|||\cdot|||$ denotes the unitarily invariant norm.

Motivated by quantum statistical mechanics, Golden [5], Symanzik [12] and Thompson [13] independently proved that

$$\mathrm{Tr}\ e^{H+K} \leq \mathrm{Tr}\ e^{H}e^{K}$$

holds for Hermitian matrices H and K. This so-called Golden–Thompson trace inequality has been generalized in several ways [8, 2]. Hiai and Petz [6] gave a lower bound on $\mathrm{Tr}\ e^{H+K}$ in terms of the geometric mean of Hermitian matrices H and K, and it complements the Golden–Thompson upper bound: For each $\alpha \in [0,1]$

$$\mathrm{Tr}\left(e^{pH}\ \sharp_\alpha\ e^{pK}\right)^{\frac{1}{p}} \ \leq\ \mathrm{Tr}\ e^{(1-\alpha)H+\alpha K}$$

holds for all $p > 0$. Here $X \sharp_\alpha Y$ denotes the α-geometric mean of positive definite X and Y in the sense of Kubo–Ando [7] (in particular, $X \sharp Y = X \sharp_{\frac{1}{2}} Y$ is the geometric mean), i.e.,

$$X \sharp_\alpha Y = X^{\frac{1}{2}}\left(X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\right)^{\alpha}X^{\frac{1}{2}} \qquad \text{for all}\ \ 0 \leq \alpha \leq 1.$$

Afterwards, Ando and Hiai [1] showed that for every unitarily invariant norm $|||\cdot|||$

$$|||\left(e^{pH}\ \sharp_\alpha\ e^{pK}\right)^{\frac{1}{p}}||| \leq |||e^{(1-\alpha)H+\alpha K}||| \tag{1.1}$$

holds for all $p > 0$ and the left-hand side of (1.1) increases to the right-hand side as $p \downarrow 0$. In particular,

$$|||e^{2H}\ \sharp\ e^{2K}||| \leq |||e^{H+K}|||.$$

The purpose of this paper is to find a upper bound on $|||e^{(1-\alpha)H+\alpha K}|||$ in terms of scalar multiples of $|||\left(e^{pH}\ \sharp_\alpha\ e^{pK}\right)^{\frac{1}{p}}|||$ for every unitarily invariant norm, and it shows reverses of the Golden–Thompson type inequalities (1.1): Let H and K be Hermitian matrices such that $mI \leq H, K \leq MI$ for some scalars $m \leq M$, and let $\alpha \in [0,1]$. Then

$$|||e^{(1-\alpha)H+\alpha K}||| \ \leq\ S(e^{p(M-m)})^{\frac{1}{p}}\ |||\left(e^{pH}\ \sharp_\alpha\ e^{pK}\right)^{\frac{1}{p}}||| \tag{1.2}$$

holds for all $p > 0$ and the right-hand side of (1.2) converges to the left-hand side as $p \downarrow 0$, where $S(h)$ is the Specht ratio.

2. Preliminaries.

In order to prove our results, we need some preliminaries. As a converse of the arithmetic-geometric mean inequality, Specht [11] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m, M]$ with $0 < m \leq M$,

$$\frac{x_1 + \cdots + x_n}{n} \leq S(h)\sqrt[n]{x_1 \cdots x_n}, \tag{2.1}$$

where $h = \frac{M}{m}$ (≥ 1) is a generalized condition number in the sense of Turing [15] and the Specht ratio is defined for $h > 0$ as

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \ (h \neq 1) \quad \text{and} \quad S(1) = 1. \tag{2.2}$$

Pečarić [10] showed the noncommutative operator version of (2.1): For positive definite A and B such that $0 < mI \leq A, B \leq MI$ for some scalars $0 < m \leq M$

$$(1-\alpha)A + \alpha B \leq S(h)\ A \sharp_\alpha B \qquad \text{for all } \alpha \in [0,1], \tag{2.3}$$

also see [14].

We collect basic properties of the Specht ratio ([4, Lemma2.47], [16]):

Lemma 2.1. *Let $h > 0$ be given. Then the Specht ratio has the following properties:*

(1) $S(h^{-1}) = S(h)$.

(2) *A function $S(h)$ is strictly decreasing for $0 < h < 1$ and strictly increasing for $h > 1$.*

(3) $\lim_{p\to 0} S(h^p)^{\frac{1}{p}} = 1$.

For positive definite A such that $mI \leq A \leq MI$ for some scalars $0 < m \leq M$, the following inequality is called the Kantorovich inequality:

$$(Ax, x)(A^{-1}x, x) \leq \frac{(M+m)^2}{4Mm} \qquad \text{for every unit vector } x. \tag{2.4}$$

We call the constant $\frac{(M+m)^2}{4Mm}$ the Kantorovich constant. Furuta [3] showed the following extension of (2.4) as a reverse of Hölder-McCarthy inequality:

Theorem A. Let A be a positive definite matrix such that $mI \leq A \leq MI$ for some scalars $0 < m < M$ and x a unit vector. Put $h = \frac{M}{m}$. Then

$$(Ax, x)^p \leq (A^p x, x) \leq K(h,p)(Ax, x)^p \qquad \text{for all } p \notin [0,1]. \tag{i}$$

$$K(h,p)(Ax, x)^p \leq (A^p x, x) \leq (Ax, x)^p \qquad \text{for all } p \in [0,1], \tag{ii}$$

where a generalized Kantorovich constant $K(h,p)$ is defined for $h > 0$ as

$$K(h,p) = \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p \quad \text{for any real number } p \in \mathbb{R}. \tag{2.5}$$

In fact, if we put $p = -1$, then $K(\frac{M}{m}, -1) = \frac{(M+m)^2}{4Mm}$.

Remark 2.2. By using the Mond–Pečarić method, Mond and Pečarić [9] showed more general form of Theorem A in 1993: Let A be a Hermitian matrix such that $mI \leq A \leq MI$. If f is a strictly convex twice differentiable function on $[m, M]$ such that $f(t) > 0$ for all $t \in [m, M]$, then for all unit vectors x, the inequality

$$(f(A)x, x) \leq \lambda\ f((Ax, x))$$

holds for some $\lambda > 1$. In fact, if we put $f(t) = t^p$, then we have Theorem A.

We state some properties of $K(h,p)$ (see [4, Theorem 2.54, 2.56], [16]):

Lemma 2.3. *Let $h > 0$ be given. Then a generalized Kantorovich constant $K(h,p)$ has the following properties:*

$$K(h,p) = K(h^{-1},p) \qquad \text{for all } p \in \mathbb{R}. \tag{i}$$

$$K(h,p) = K(h,1-p) \qquad \text{for all } p \in \mathbb{R}. \tag{ii}$$

$$K(h,0) = K(h,1) = 1 \quad \text{and} \quad K(1,p) = 1 \quad \text{for all } p \in \mathbb{R}. \tag{iii}$$

$$K(h^r, \frac{p}{r})^{\frac{1}{p}} = K(h^p, \frac{r}{p})^{-\frac{1}{r}} \qquad \text{for } pr \neq 0. \tag{iv}$$

$$\lim_{p \to 0} K(h^p, \frac{r}{p}) = S(h^r) \qquad \text{for all } r \in \mathbb{R}. \tag{v}$$

3. Specht ratio version.

Let A and B be positive definite matrices. Ando and Hiai [1] showed the following inequality by using the log-majorization: For each $\alpha \in [0,1]$

$$\|\!| (A^p \natural_\alpha B^p)^{\frac{1}{p}} \|\!| \ \leq \ \|\!| (A^q \natural_\alpha B^q)^{\frac{1}{q}} \|\!| \qquad \text{for all } 0 < q < p \tag{3.1}$$

for every unitarily invariant norm. In particular,

$$\|\!| A^r \natural_\alpha B^r \|\!| \ \leq \ \|\!| (A \natural_\alpha B)^r \|\!| \qquad \text{for all } r \geq 1.$$

First of all, we investigate order relations between $(A^q \natural_\alpha B^q)^{\frac{1}{q}}$ and $(A^p \natural_\alpha B^p)^{\frac{1}{p}}$ in terms of the Specht ratio. In fact a stronger result holds. We show that a reverse of (3.1) can be extended to all eigenvalues. Given two positive definite matrices X and Y, recall that the eigenvalues of Y dominate the corresponding eigenvalues of X iff there exists a unitary matrix U such that $X \leq UYU^*$. For a Hermitian matrix H, let $\lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_n(H)$ be the eigenvalues of H arranged in decreasing order.

Lemma 3.1. *Let A and B be positive definite matrices such that $0 < mI \leq A, B \leq MI$ for some scalars $0 < m < M$, and let $\alpha \in [0,1]$. Put $h = \frac{M}{m}$. Then for each $0 < q \leq p$, there exist unitary matrices U and V such that*

$$S(h^p)^{-\frac{1}{p}}\ V (A^p \natural_\alpha B^p)^{\frac{1}{p}} V^* \leq (A^q \natural_\alpha B^q)^{\frac{1}{q}} \ \leq S(h^p)^{\frac{1}{p}}\ U (A^p \natural_\alpha B^p)^{\frac{1}{p}} U^*, \tag{3.2}$$

where $S(h)$ is defined as (2.2).

Proof. By the arithmetic-geometric mean inequality and its reverse (2.3), we have

$$A \natural_\alpha B \ \leq \ (1-\alpha)A + \alpha B \ \leq \ S(h)\ A \natural_\alpha B.$$

Since $0 < \frac{q}{p} < 1$, it follows from the operator concavity of $t^{\frac{q}{p}}$ that

$$A^{\frac{q}{p}} \natural_\alpha B^{\frac{q}{p}} \leq (1-\alpha)A^{\frac{q}{p}} + \alpha B^{\frac{q}{p}} \leq ((1-\alpha)A + \alpha B)^{\frac{q}{p}} \leq S(h)^{\frac{q}{p}} (A \natural_\alpha B)^{\frac{q}{p}}.$$

Replacing A and B by A^p and B^p respectively, we have

$$A^q \natural_\alpha B^q \ \leq \ S(h^p)^{\frac{q}{p}}\ (A^p \natural_\alpha B^p)^{\frac{q}{p}}. \tag{3.3}$$

In the case of $q \geq 1$, the Löwner-Heinz inequality asserts

$$(A^q \natural_\alpha B^q)^{\frac{1}{q}} \leq S(h^p)^{\frac{1}{p}} (A^p \natural_\alpha B^p)^{\frac{1}{p}}.$$

In the case of $0 < q \leq 1$, by the minimax principle, there exists a subspace F of codimension $k-1$ such that

$$\lambda_k((A^q \natural_\alpha B^q)^{\frac{1}{q}}) = \max_{x \in F, \|x\|=1} (x, (A^q \natural_\alpha B^q)^{\frac{1}{q}} x) = \max_{x \in F, \|x\|=1} (x, (A^q \natural_\alpha B^q)x)^{\frac{1}{q}}.$$

Therefore, by (3.3) we have

$$\begin{aligned} &\lambda_k((A^q \natural_\alpha B^q)^{\frac{1}{q}}) \\ &\leq \max_{x \in F, \|x\|=1} \left(S(h^p)^{\frac{q}{p}} (x, (A^p \natural_\alpha B^p)^{\frac{q}{p}} x) \right)^{\frac{1}{q}} \\ &\leq \max_{x \in F, \|x\|=1} S(h^p)^{\frac{1}{p}} (x, (A^p \natural_\alpha B^p)^{\frac{1}{p}} x) \quad \text{by } 0 < q < 1 \text{ and Theorem A(ii)} \\ &\leq S(h^p)^{\frac{1}{p}} \lambda_k((A^p \natural_\alpha B^p)^{\frac{1}{p}}) \end{aligned}$$

and hence we obtain the right-hand side of (3.2).

To prove the left-hand side inequality, we replace A and B by their inverses and we use

$$A^{-1} \natural_\alpha B^{-1} = (A \natural_\alpha B)^{-1}.$$

Then we have

$$\left(A^{-q} \natural_\alpha B^{-q}\right)^{\frac{1}{q}} \leq S(h^{-p})^{\frac{1}{p}} V \left(A^{-p} \natural_\alpha B^{-p}\right)^{\frac{1}{p}} V^*$$

for some unitary V. By raising both sides to the inverse and (1) of Lemma 2.1 we obtain the desired one. □

As a corollary of Lemma 3.1, we have a reverse of (3.1):

Corollary 3.2. *Let A and B be positive definite matrices such that $0 < mI \leq A, B \leq MI$ for some scalars $0 < m \leq M$, and let $\alpha \in [0,1]$. Put $h = \frac{M}{m}$. Then*

$$|||\,(A^q \natural_\alpha B^q)^{\frac{1}{q}}\,||| \leq S(h^p)^{\frac{1}{p}} \,|||\,(A^p \natural_\alpha B^p)^{\frac{1}{p}}\,||| \quad \text{for all } 0 < q \leq p. \tag{3.4}$$

In particular,

$$|||A^p \natural_\alpha B^p||| \leq S(h)^p |||(A \natural_\alpha B)^p||| \quad \text{for all } 0 < p \leq 1 \tag{3.5}$$

and

$$|||(A \natural_\alpha B)^p||| \leq S(h^p) |||A^p \natural_\alpha B^p||| \quad \text{for all } p > 1. \tag{3.6}$$

Proof. By Lemma 3.1, we have (3.4). If we put $q = 1$ in (3.4), then we have

$$|||A \natural_\alpha B||| \leq S(h^p)^{\frac{1}{p}} \,|||\,(A^p \natural_\alpha B^p)^{\frac{1}{p}}\,|||$$

for all $p \geq 1$. Moreover, replacing A and B by $A^{\frac{1}{p}}$ and $B^{\frac{1}{p}}$ we have

$$|||A^{\frac{1}{p}} \natural_\alpha B^{\frac{1}{p}}||| \leq S(h^p)^{\frac{1}{p}} \,|||\,(A \natural_\alpha B)^{\frac{1}{p}}\,|||$$

and hence we have (3.5). Similarly we have (3.6). □

We show reverses of the Golden–Thompson type inequalities due to Ando,Hiai and Petz, which is our main result.

Theorem 3.3. *Let H and K be Hermitian matrices such that $mI \leq H, K \leq MI$ for some scalars $m \leq M$, and let $\alpha \in [0,1]$. Then for each $p > 0$ there exists unitary matrices U and V such that*

$$\begin{aligned} S(e^{p(M-m)})^{-\frac{1}{p}}\ V\left(e^{pH}\ \natural_\alpha\ e^{pK}\right)^{\frac{1}{p}} V^* \\ \leq\ e^{(1-\alpha)H+\alpha K}\ \leq\ S(e^{p(M-m)})^{\frac{1}{p}}\ U\left(e^{pH}\ \natural_\alpha\ e^{pK}\right)^{\frac{1}{p}} U^*. \end{aligned} \tag{3.7}$$

Proof. Replacing A and B by e^H and e^K in Lemma 3.1 respectively, it follows that for each $0 < q \leq p$ there exist unitary matrix $U_{p,q}$ such that

$$(e^{qH}\ \natural_\alpha\ e^{qK})^{\frac{1}{q}}\ \leq\ S(e^{p(M-m)})^{\frac{1}{p}} U_{p,q}(e^{pH}\ \natural_\alpha\ e^{pK})^{\frac{1}{p}} U_{p,q}^*.$$

By [6, Lemma 3.3], we have

$$e^{(1-\alpha)H+\alpha K} = \lim_{q\to 0}(e^{qH}\ \natural_\alpha\ e^{qK})^{\frac{1}{q}}$$

and hence it follows that for each $p > 0$ there exist unitary matrix U such that

$$e^{(1-\alpha)H+\alpha K}\ \leq\ S(e^{p(M-m)})^{\frac{1}{p}}\ U\left(e^{pH}\ \natural_\alpha\ e^{pK}\right)^{\frac{1}{p}} U^*.$$

We also have the left-hand side inequality of (3.7) by a similar method as the proof of Lemma 3.1. □

In particular, we have the following results by (3) of Lemma 2.1.

Theorem 3.4. *Let H and K be Hermitian matrices such that $mI \leq H, K \leq MI$ for some scalars $m \leq M$, and let $\alpha \in [0,1]$. Then*

$$|||e^{(1-\alpha)H+\alpha K}|||\ \leq\ S(e^{p(M-m)})^{\frac{1}{p}}\ |||\left(e^{pH}\ \natural_\alpha\ e^{pK}\right)^{\frac{1}{p}}||| \tag{3.8}$$

holds for all $p > 0$ and the right-hand side of (3.8) *converges to the left-hand side as $p \downarrow 0$. In particular,*

$$|||e^{H+K}|||\ \leq\ S(e^{2(M-m)})\ |||e^{2H}\ \natural\ e^{2K}|||. \tag{3.9}$$

4. Kantorovich constant version.

In this section, we want to show another estimate of the Golden–Thompson type inequalities due to Ando, Hiai and Petz. As a matter of fact, the upper bound $S(e^{p(M-m)})^{\frac{1}{p}}$ in (3.8) of Theorem 3.4 is constant for all $\alpha \in [0,1]$. We show another order relations between $(A^q\ \natural_\alpha\ B^q)^{\frac{1}{q}}$ and $(A^p\ \natural_\alpha\ B^p)^{\frac{1}{p}}$ in terms of the generalized Kantorovich constant.

Lemma 4.1. *Let A and B be positive definite matrices such that $0 < mI \leq A, B \leq MI$ for some scalars $0 < m < M$, and let $\alpha \in [0,1]$. Put $h = \frac{M}{m}$. Let $0 < q \leq 1$. Then for each $0 < q \leq p \leq 1$, there exist unitary matrices U_1 and U_2 such that*

$$K(h,p)^{\frac{\alpha}{p}} K(h^{2p},\alpha)^{\frac{1}{p}} U_1 (A^p \natural_\alpha B^p)^{\frac{1}{p}} U_1^* \leq (A^q \natural_\alpha B^q)^{\frac{1}{q}} \leq K(h,p)^{-\frac{\alpha}{p}} K(h^{2p},\alpha)^{-\frac{1}{p}} U_2 (A^p \natural_\alpha B^p)^{\frac{1}{p}} U_2^* \tag{4.1}$$

and for each $p \geq 1$, there exist unitary matrices V_1 and V_2 such that

$$K(h^{2p},\alpha)^{\frac{1}{p}} V_1 (A^p \natural_\alpha B^p)^{\frac{1}{p}} V_1^* \leq (A^q \natural_\alpha B^q)^{\frac{1}{q}} \leq K(h^{2p},\alpha)^{-\frac{1}{p}} V_2 (A^p \natural_\alpha B^p)^{\frac{1}{p}} V_2^*, \tag{4.2}$$

where the generalized Kantorovich constant $K(h,p)$ is defined as (2.5).

Proof. For $0 < q < p \leq 1$ and every unit vector x,

$$\begin{aligned}
(x,&(A^q \natural_\alpha B^q)x)^{\frac{1}{q}} = \Big(\frac{A^{\frac{q}{2}}x}{\|A^{\frac{q}{2}}x\|}, (A^{-\frac{q}{2}} B^q A^{-\frac{q}{2}})^\alpha \frac{A^{\frac{q}{2}}x}{\|A^{\frac{q}{2}}x\|}\Big)^{\frac{1}{q}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}} \\
&\leq \Big(\frac{A^{\frac{q}{2}}x}{\|A^{\frac{q}{2}}x\|}, A^{-\frac{q}{2}} B^q A^{-\frac{q}{2}} \frac{A^{\frac{q}{2}}x}{\|A^{\frac{q}{2}}x\|}\Big)^{\frac{\alpha}{q}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}} \quad \text{by } 0<\alpha<1 \text{ and Theorem A (ii)} \\
&= (x, B^q x)^{\frac{\alpha}{q}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}-\frac{2\alpha}{q}} \\
&\leq (x, Bx)^{\alpha} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}-\frac{2\alpha}{q}} \quad \text{by } 0<q<1 \text{ and Theorem A (ii)} \\
&= (x, Bx)^{\frac{p\alpha}{p}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}-\frac{2\alpha}{q}} \quad (\star) \\
&\leq \left(K(h,p)^{-1}(x, B^p x)\right)^{\frac{\alpha}{p}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}-\frac{2\alpha}{q}} \quad \text{by } 0<p\leq 1 \text{ and Theorem A (ii)} \\
&= K(h,p)^{-\frac{\alpha}{p}} (x, B^p x)^{\frac{\alpha}{p}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}-\frac{2\alpha}{q}} \\
&= K(h,p)^{-\frac{\alpha}{p}} \Big(\frac{A^{\frac{p}{2}}x}{\|A^{\frac{p}{2}}x\|}, (A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}}) \frac{A^{\frac{p}{2}}x}{\|A^{\frac{p}{2}}x\|}\Big)^{\frac{\alpha}{p}} \|A^{\frac{p}{2}}x\|^{\frac{2\alpha}{p}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}-\frac{2\alpha}{q}} \\
&\leq K(h,p)^{-\frac{\alpha}{p}} K(h^{2p},\alpha)^{-\frac{1}{p}} (A^p \natural_\alpha B^p x, x)^{\frac{1}{p}} \|A^{\frac{p}{2}}x\|^{\frac{2\alpha}{p}-\frac{2}{p}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}-\frac{2\alpha}{q}} \quad \text{by } 0<\alpha<1 \\
&\leq K(h,p)^{-\frac{\alpha}{p}} K(h^{2p},\alpha)^{-\frac{1}{p}} (A^p \natural_\alpha B^p x, x)^{\frac{1}{p}}.
\end{aligned}$$

The last inequality holds since it follows from $0 < q < p$ that

$$\begin{aligned}
\|A^{\frac{p}{2}}x\|^{\frac{2\alpha}{p}-\frac{2}{p}} \|A^{\frac{q}{2}}x\|^{\frac{2}{q}-\frac{2\alpha}{q}} &= (A^p x, x)^{\frac{\alpha-1}{p}} (A^q x, x)^{\frac{1-\alpha}{q}} \\
&= (A^p x, x)^{\frac{\alpha-1}{p}} ((A^p)^{\frac{q}{p}} x, x)^{\frac{1-\alpha}{q}} \\
&\leq (A^p x, x)^{\frac{\alpha-1}{p}} (A^p x, x)^{\frac{1-\alpha}{p}} = 1.
\end{aligned}$$

By the minimax principle, there exists a subspace F of codimension $k-1$ such that

$$\lambda_k \left((A^q \natural_\alpha B^q)^{\frac{1}{q}}\right) = \max_{y\in F, \|x\|=1} (x, (A^q \natural_\alpha B^q)^{\frac{1}{q}} x) = \max_{y\in F, \|x\|=1} (x, (A^q \natural_\alpha B^q)x)^{\frac{1}{q}}.$$

Therefore, we have

$$\begin{aligned}
\lambda_k\left((A^q \natural_\alpha B^q)^{\frac{1}{q}}\right) &= \max_{y\in F,\|x\|=1}(x,(A^q \natural_\alpha B^q)x)^{\frac{1}{q}} \\
&\le \max_{y\in F,\|x\|=1} K(h,p)^{-\frac{\alpha}{p}}K(h^{2p},\alpha)^{-\frac{1}{p}}(A^p\natural_\alpha B^p x,x)^{\frac{1}{p}} \\
&\le K(h,p)^{-\frac{\alpha}{p}}K(h^{2p},\alpha)^{-\frac{1}{p}}\lambda_k\left((A^p \natural_\alpha B^p)^{\frac{1}{p}}\right) \quad \text{by } \tfrac{1}{p}\ge 1 \text{ and Theorem A(i).}
\end{aligned}$$

Hence there exist a unitary matrix U_2 such that

$$(A^q \natural_\alpha B^q)^{\frac{1}{q}} \le K(h,p)^{-\frac{\alpha}{p}}K(h^{2p},\alpha)^{-\frac{1}{p}}U_2(A^p \natural_\alpha B^p)^{\frac{1}{p}}U_2^*.$$

Replacing A and B by their inverses, we have the left-hand side inequality of (4.1).

Suppose that $p\ge 1$. In the part $(\star)$, we have $(x,Bx)^p \le (x,B^px)$ by Theorem A(i). Therefore it follows that the inequality (4.2) holds by a similar method. □

By Lemma 4.1, we have another reverse of the Golden–Thompson type inequalities due to Ando, Hiai and Petz.

Theorem 4.2. *Let H and K be Hermitian matrices such that $mI \le H,K \le MI$ for some scalars $m\le M$, and let $\alpha\in[0,1]$. Then*

$$|||e^{(1-\alpha)H+\alpha K}||| \;\le\; K(e^{M-m},p)^{-\frac{\alpha}{p}}K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}}\;|||(e^{pH} \natural_\alpha e^{pK})^{\frac{1}{p}}||| \qquad \textit{for all } 0<p\le 1$$

and

$$|||e^{(1-\alpha)H+\alpha K}||| \;\le\; K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}}\;|||(e^{pH} \natural_\alpha e^{pK})^{\frac{1}{p}}||| \quad \textit{for all } p\ge 1,$$

where the generalized Kantorovich constant $K(h,p)$ is defined as (2.5). In particular,

$$|||e^{H+K}||| \le \frac{e^{2M}+e^{2m}}{2e^Me^m}|||e^{2H} \natural e^{2K}|||. \tag{4.3}$$

Proof. Replacing A and B by e^H and e^K in Lemma 4.1, we have this theorem. □

Remark 4.3. (1) In Theorem 4.2, the constant $K(e^{M-m},p)^{-\frac{\alpha}{p}}K(e^{2p(M-m)},\alpha)^{-\frac{1}{p}} = 1$ in the cases of $(\alpha,p)=(0,1)$ and $(1,1)$.

(2) Comparison of the constants (3.9) in Theorem 3.4 and (4.3) in Theorem 4.2: If $\alpha=\frac{1}{2}$, then for each $p>0$ it follows from Specht theorem (2.1) that

$$K(h^{2p},\frac{1}{2})^{-\frac{1}{p}} = \left(\frac{h^{\frac{p}{2}}+h^{-\frac{p}{2}}}{2}\right)^{\frac{1}{p}} \le \left(S(h^p)\sqrt{h^{\frac{p}{2}}h^{-\frac{p}{2}}}\right)^{\frac{1}{p}} = S(h^p)^{\frac{1}{p}}.$$

Hence for each $p\ge 1$

$$\begin{aligned}
|||e^{H+K}||| &\le K(e^{4p(M-m)},\frac{1}{2})^{-\frac{1}{p}}\;|||\left(e^{2pH} \natural e^{2pK}\right)^{\frac{1}{p}}||| \\
&\le S(e^{2p(M-m)})^{\frac{1}{p}}\;|||\left(e^{2pH} \natural e^{2pK}\right)^{\frac{1}{p}}|||
\end{aligned}$$

In particular, if we put $p = 1$, then

$$\|\|e^{H+K}\|\| \leq \frac{e^{2M}+e^{2m}}{2e^M e^m} \|\|e^{2H} \sharp e^{2K}\|\| \leq S(e^{2(M-m)}) \|\|e^{2H} \sharp e^{2K}\|\|.$$

Finally, in the case of $0 < p \leq 1$, if we put $h = 2$, $\alpha = \frac{1}{2}$, then we graph two functions $S(2^p)^{\frac{1}{p}}$ and $K(2,p)^{-\frac{1}{2p}} K(2^{2p}, \frac{1}{2})^{-\frac{1}{p}}$ on p as follows:

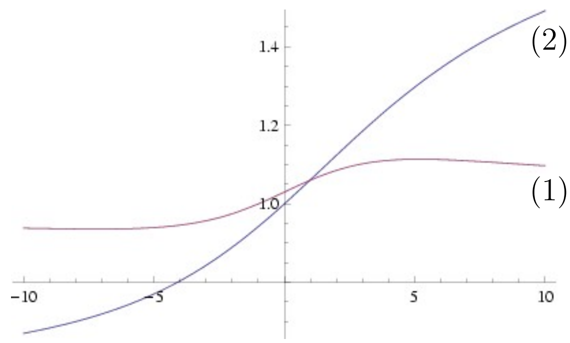

FIGURE 1. Graphs of $y = S(2^p)^{\frac{1}{p}} \cdots (1)$ and $y = K(2,p)^{-\frac{1}{2p}} K(2^{2p}, \frac{1}{2})^{-\frac{1}{p}} \cdots (2)$

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