Banach J. Math. Anal. 2 (2008), no. 2, 23–30

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

REVERSE OF THE GRAND FURUTA INEQUALITY AND ITS APPLICATIONS

MASATOSHI FUJII[1*], RITSUO NAKAMOTO[2] AND MASARU TOMINAGA[3]

This paper is dedicated to Professor J.E. Pečarić

Submitted by A. R. Villena

Abstract. We shall give a norm inequality equivalent to the grand Furuta inequality, and moreover show its reverse as follows: Let A and B be positive operators such that $0 < m \le B \le M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then

$$\begin{aligned}
&\| A^{\frac{1}{2}}\{A^{-\frac{t}{2}}(A^{\frac{r}{2}}B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}}A^{\frac{r}{2}})^{\frac{1}{s}}A^{-\frac{t}{2}}\}^{\frac{1}{p}}A^{\frac{1}{2}} \| \\
&\le \ K(h^{r-t}, \frac{(p-t)s+r}{1-t+r})^{\frac{1}{ps}} \| A^{\frac{1-t+r}{2}}B^{r-t}A^{\frac{1-t+r}{2}} \|^{\frac{(p-t)s+r}{ps(1-t+r)}}
\end{aligned}$$

for $0 \le t \le 1$, $p \ge 1$, $s \ge 1$ and $r \ge t \ge 0$, where $K(h,p)$ is the generalized Kantorovich constant. As applications, we consider reverses related to the Ando-Hiai inequality.

1. Introduction

The origin of reverse inequalities is the Kantorovich inequality. It says that if a positive operator A on a Hilbert space H satisfies $0 \le m \le A \le M$, then

$$\langle A^{-1}x, x\rangle \le \frac{(M+m)^2}{4Mm}\langle Ax, x\rangle^{-1} \quad \text{for all unit vectors } x \in H. \tag{K}$$

Date: Received: 15 April 2008; Accepted: 20 April 2008.

* Corresponding author.

2000 *Mathematics Subject Classification.* 47A63.

Key words and phrases. grand Furuta inequality, Furuta inequality, Löwner-Heinz inequality, Araki-Cordes inequality, Bebiano-Lemos-Providência inequality, norm inequality, positive operator, operator inequality, reverse inequality.

The point in (K) is the convexity of the function $t \to t^{-1}$. Mond and Pečarić turned their attention to the convexity of functions, and established the so called Mond-Pečarić method in the theory of reverse inequalities, see [13] in detail. The subject of this note is just on the line of Mond-Pečarić's idea, and our target is the grand Furuta inequality.

Let A and B be positive (bounded linear) operators acting on a Hilbert space. The grand Furuta inequality [10] says that

$$A \geq B \geq 0 \quad \Rightarrow \quad A^{1-t+r} \;\geq\; \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{GFI}$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$ and $r \geq t$.

The inequality (GFI) is considered as a parametric formula interpolating the Furuta inequality (FI) and Ando-Hiai one (1.1), respectively [9] and [1]:

$$A \geq B \geq 0 \quad \Rightarrow \quad A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad (r \geq 0,\ p \geq 1) \tag{FI}$$

and

$$A \geq B \geq 0 \quad \Rightarrow \quad A^r \geq \{A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}} \quad (p, r \geq 1). \tag{1.1}$$

Now the Furuta inequality appeared as a useful extension of the so-called Löwner-Heinz inequality (cf. [14]):

$$A \geq B \geq 0 \quad \Rightarrow \quad A^\alpha \geq B^\alpha \quad (0 \leq \alpha \leq 1). \tag{1.2}$$

This Löwner-Heinz inequality (1.2) is equivalent to the Araki-Cordes inequality ([2], [4]):

$$\| A^{\frac{p}{2}}B^pA^{\frac{p}{2}} \| \leq \| A^{\frac{1}{2}}BA^{\frac{1}{2}} \|^p \quad (0 \leq p \leq 1). \tag{1.3}$$

M.Fujii and Y.Seo [8] gave a reverse inequality of the Araki-Cordes inequality: If A and B are positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m}$ (> 1), then

$$K(h,p)\ \| A^{\frac{1}{2}}BA^{\frac{1}{2}} \|^p \leq \| A^{\frac{p}{2}}B^pA^{\frac{p}{2}} \| \quad (0 \leq p \leq 1) \tag{1.4}$$

where a generalized Kantorovich constant $K(h,p)$ is defined as follows:

$$K(h,p) := \frac{1}{h-1}\frac{h^p-h}{p-1}\left(\frac{p-1}{h^p-h}\frac{h^p-1}{p}\right)^p \tag{1.5}$$

for all $h(\neq 1), p \in \mathbb{R}$ and $K(h,0) = K(h,1) = 1$, see [11] and [13].

In this note, we first give a norm inequality equivalent to the grand Furuta inequality (GFI). Based on this, we show a reverse inequality of (GFI), in which the generalized Kantorovich constant (1.5) is used. As an application, we obtain reverses of a generalization of Ando-Hiai inequality (1.1).

2. Norm Inequality equivalent to the grand Furuta inequality

The grand Furuta inequality (GFI) is equivalent to the following norm inequality:

Lemma 2.1. *Let A and B be positive operators. Then the grand Furuta inequality (GFI) is equivalent to*

$$\| A^{\frac{1-t+r}{2}} B^{r-t} A^{\frac{1-t+r}{2}} \|^{\frac{(p-t)s+r}{ps(1-t+r)}} \leq \| A^{\frac{1}{2}} \{A^{-\frac{t}{2}}(A^{\frac{r}{2}} B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}} A^{\frac{r}{2}})^{\frac{1}{s}} A^{-\frac{t}{2}}\}^{\frac{1}{p}} A^{\frac{1}{2}} \| \tag{2.1}$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$ and $r \geq t$.

Proof. Replace A to A^{-1} and put

$$C = \{A^{\frac{t}{2}}(A^{-\frac{r}{2}} B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}} A^{-\frac{r}{2}})^{\frac{1}{s}} A^{\frac{t}{2}}\}^{\frac{1}{p}}$$

in (2.1). Since $B^{r-t} = \{A^{\frac{r}{2}}(A^{-\frac{t}{2}} C^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$, we have

$$\| A^{-\frac{1-t+r}{2}} \{A^{\frac{r}{2}}(A^{-\frac{t}{2}} C^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{-\frac{1-t+r}{2}} \|^{\frac{(p-t)s+r}{ps(1-t+r)}} \leq \| A^{-\frac{1}{2}} C A^{-\frac{1}{2}} \| .$$

This is equivalent to the inequality

$$A \geq C \quad \Rightarrow \quad A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}} C^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}},$$

that is, (2.1) is equivalent to the grand Furuta inequality (GFI). □

Corollary 2.2. *Let A and B be positive operators. Then*

$$\| A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}} \|^{\frac{p+s}{p(1+s)}} \leq \| A^{\frac{1}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}} \| \tag{2.2}$$

for $p \geq 1$ and $s \geq 0$.

Moreover

$$\| A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}} \| \leq \| A^{\frac{1}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}} \| \tag{2.3}$$

for $s \geq t \geq 0$.

Proof. Put $t = 0$, $s = 1$ in (2.1). Then replacing r and B to s and $B^{\frac{1+s}{s}}$, respectively, (2.1) implies (2.2).

Moreover, let t be a real number satisfying $s \geq t \geq 0$. Then (2.2) implies

$$\| A^{\frac{1+t}{2}} B^{1+t} A^{\frac{1+t}{2}} \|^{\frac{p+s}{p(1+t)}} \leq \| A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}} \|^{\frac{p+s}{p(1+s)}} \leq \| A^{\frac{1}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}} \|$$

by $\frac{1+t}{1+s} \in [0, 1]$ and the Araki-Cordes inequality (1.3). Furthermore, replacing B to $B^{\frac{t}{1+t}}$ and putting $p = \frac{s}{t}$, we have (2.3). □

Remark 2.3. The inequality (2.3) is originated by Bebiano-Lemos-Providência in [3]. In our previous note [7], we call it the BLP inequality and we showed (2.2) as a generalization of the BLP inequality (2.3). Incidentally it is equivalent to (FI). For convenience, we give a proof of (2.2) $\Rightarrow$ (FI). The inequality (2.2) is rephrased by replacing A to A^{-1} as follows:

$$\| A^{-\frac{1+t}{2}} B^t A^{-\frac{1+t}{2}} \|^{\frac{p+s}{p(1+t)}} \leq \| A^{-\frac{1}{2}} (A^{-\frac{s}{2}} B^{\frac{t(p+s)}{1+t}} A^{-\frac{s}{2}})^{\frac{1}{p}} A^{-\frac{1}{2}} \| .$$

Moreover, putting

$$C = (A^{-\frac{s}{2}} B^{\frac{t(p+s)}{1+t}} A^{-\frac{s}{2}})^{\frac{1}{p}}, \text{ or } B^t = (A^{\frac{s}{2}} C^p A^{\frac{s}{2}})^{\frac{1+t}{p+s}},$$

it is also rephrased as

$$\| A^{-\frac{1+t}{2}}(A^{\frac{s}{2}}C^pA^{\frac{s}{2}})^{\frac{1+t}{p+s}}A^{-\frac{1+t}{2}} \|^{\frac{p+s}{p(1+t)}} \leq \| A^{-\frac{1}{2}}CA^{-\frac{1}{2}} \|$$

which obviously implies the Furuta inequality (FI) by taking $s = t = r$.

Remark 2.4. In [12], Furuta gave a similar inequality to (2.1).

3. A reverse grand Furuta inequality and its applications

In this section, we give a reverse inequality of (2.1) by using the generalized Kantorovich constant (1.5).

Theorem 3.1. *Let A and B be positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then*

$$\begin{aligned} &\| A^{\frac{1}{2}}\{A^{-\frac{t}{2}}(A^{\frac{r}{2}}B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}}A^{\frac{r}{2}})^{\frac{1}{s}}A^{-\frac{t}{2}}\}^{\frac{1}{p}}A^{\frac{1}{2}} \| \\ &\leq\ K\left(h^{\frac{1-t+r'}{1-t+r}(r-t)}, \frac{(p-t)s+r}{1-t+r'}\right)^{\frac{1}{ps}} \| A^{\frac{1-t+r'}{2}}B^{\frac{1-t+r'}{1-t+r}(r-t)}A^{\frac{1-t+r'}{2}} \|^{\frac{(p-t)s+r}{ps(1-t+r')}} \end{aligned} \tag{3.1}$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$ and $1+r \geq 1+r' > t$, where $K(h,p)$ is the generalized Kantorovich constant defined by (1.5).

Proof. For $p \geq 1$ and $s \geq 1$, the Araki-Cordes inequality (1.3) implies that

$$\begin{aligned} &\| A^{\frac{1}{2}}\{A^{-\frac{t}{2}}(A^{\frac{r}{2}}B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}}A^{\frac{r}{2}})^{\frac{1}{s}}A^{-\frac{t}{2}}\}^{\frac{1}{p}}A^{\frac{1}{2}} \| \\ &\leq \| A^{\frac{p}{2}}\{A^{-\frac{t}{2}}(A^{\frac{r}{2}}B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}}A^{\frac{r}{2}})^{\frac{1}{s}}A^{-\frac{t}{2}}\}A^{\frac{p}{2}} \|^{\frac{1}{p}} \\ &= \| A^{\frac{p-t}{2}}(A^{\frac{r}{2}}B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}}A^{\frac{r}{2}})^{\frac{1}{s}}A^{\frac{p-t}{2}} \|^{\frac{1}{p}} \\ &\leq \| A^{\frac{(p-t)s}{2}}(A^{\frac{r}{2}}B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}}A^{\frac{r}{2}})A^{\frac{(p-t)s}{2}} \|^{\frac{1}{ps}} \\ &= \| A^{\frac{(p-t)s+r}{2}}B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}}A^{\frac{(p-t)s+r}{2}} \|^{\frac{1}{ps}} . \end{aligned}$$

Moreover, since $(p-t)s+r \geq 1-t+r' > 0$, it follows from the reverse Araki-Cordes inequality (1.4) that

$$\begin{aligned} &\| A^{\frac{(p-t)s+r}{2}}B^{\frac{(r-t)\{(p-t)s+r\}}{1-t+r}}A^{\frac{(p-t)s+r}{2}} \|^{\frac{1}{ps}} \\ &\leq \| A^{\frac{(p-t)s+r}{2}}B^{(r-t)\frac{1-t+r'}{1-t+r}\frac{(p-t)s+r}{1-t+r'}}A^{\frac{(p-t)s+r}{2}} \|^{\frac{1}{ps}} \\ &\leq\ K\left(h^{\frac{1-t+r'}{1-t+r}(r-t)}, \frac{(p-t)s+r}{1-t+r'}\right)^{\frac{1}{ps}} \| A^{\frac{1-t+r'}{2}}B^{\frac{1-t+r'}{1-t+r}(r-t)}A^{\frac{1-t+r'}{2}} \|^{\frac{(p-t)s+r}{ps(1-t+r')}} . \end{aligned}$$

Combining them, we have the desired inequality (3.1). □

From the reverse grand Furuta inequality (3.1) we have the following reverse Furuta inequality (see [7]):

Corollary 3.2. *Let A and B be positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then*

$$\| A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{p+s}A^{\frac{s}{2}})^{\frac{1}{p}}A^{\frac{1}{2}} \| \quad \leq \quad K\left(h^{1+t}, \frac{p+s}{1+t}\right)^{\frac{1}{p}} \| A^{\frac{1+t}{2}}B^{1+t}A^{\frac{1+t}{2}} \|^{\frac{p+s}{p(1+t)}} \quad (3.2)$$

for all $p \geq 1$ and $s \geq t > -1$.

Proof. In (3.1), if we put $t = 0$, $s = 1$, and replace r, r', B and h to s, t, $B^{\frac{1+s}{s}}$ and $h^{\frac{1+s}{s}}$, respectively, then the desired inequality (3.2) holds. □

On the other hand, Ando and Hiai [1] proved

$$A\sharp_\alpha B \leq 1 \;\Rightarrow\; A^r\sharp_\alpha B^r \leq 1 \quad \text{for } 0 \leq \alpha \leq 1,\; r \geq 1$$

where $A\sharp_\alpha B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. This inequality is equivalent to

$$\| A^r\sharp_\alpha B^r \| \leq \| A\sharp_\alpha B \|^r . \quad \text{(AH)}$$

M.Fujii and E.Kamei [6] proved that (AH) is equivalent to (FI). Also they extended (AH) as follows:

$$\| A^r\sharp_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \|^{\frac{(1-\alpha)s+\alpha r}{sr}} \leq \| A\sharp_\alpha B \| \quad \text{(GAH)}$$

for $r, s \geq 1$ and $0 \leq \alpha \leq 1$. It is easy to see that the inequality (2.1) equivalent to the grand Furuta inequality is rewritten as follows:

$$\| A^{\frac{1-t+r}{2}}(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{1-t+r}{2}} \|^{\frac{(p-t)s+r}{ps(1-t+r)}} \leq \| A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\frac{1}{p}}A^{\frac{1}{2}} \|$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$ and $r \geq t \geq 0$. Here if we put $\alpha = \frac{1}{p}$, then we have

$$\| A^{\frac{1-t+r}{2}}(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}}A^{\frac{1-t+r}{2}} \|^{\frac{(1-\alpha t)s+\alpha r}{s(1-t+r)}} \leq \| A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\alpha}A^{\frac{1}{2}} \| . \quad (3.3)$$

This inequality (3.3) implies (GAH) by $t = 1$.

From the viewpoint of the Ando-Hiai inequality, we consider the following inequality related to a reverse inequality of (3.3) which is equivalent to (3.1).

Theorem 3.3. *Let A and B be positive operators such that $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then*

$$\begin{aligned} &K\left(h^{r+s}, \frac{\alpha(1-t+r')}{(1-\alpha t)s+\alpha r}\right) \| A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\alpha}A^{\frac{1}{2}} \|^{\frac{s(1-t+r')}{(1-\alpha r)s+\alpha r}} \\ &\leq \| A^{\frac{1-t+r'}{2}}(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{\alpha(1-t+r')}{(1-\alpha t)s+\alpha r}}A^{\frac{1-t+r'}{2}} \| \end{aligned} \quad (3.4)$$

for $0 \leq t \leq 1$, $s \geq 1$, $1 + r \geq 1 + r' \geq t$ and $0 \leq \alpha \leq 1$ where $K(h, p)$ is the generalized Kantorovich constant defined by (1.5).

Proof. In (3.1), we replace B^{r-t}, h^{r-t} and p to $(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}}$, $h^{\frac{\alpha(r+s)(1-t+r)}{(1-\alpha t)s+\alpha r}}$ and $\frac{1}{\alpha}$, respectively. Then we have

$$\begin{aligned}\| A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\alpha}A^{\frac{1}{2}} \| \ &\leq \ K\left(h^{\frac{\alpha(r+s)(1-t+r')}{(1-\alpha t)s+\alpha r}}, \frac{(1-\alpha t)s+\alpha r}{\alpha(1-t+r')}\right)^{\frac{\alpha}{s}} \\ &\times \| A^{\frac{1-t+r'}{2}}(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{\alpha(1-t+r')}{(1-\alpha t)s+\alpha r}}A^{\frac{1-t+r'}{2}} \|^{\frac{(1-\alpha t)s+\alpha r}{s(1-t+r')}} .\end{aligned}$$

By the inversion formula (i.e., $K(h^r, \frac{1}{r}) = K(h,r)^{-\frac{1}{r}}$ for all $r \neq 0$) [5], it implies

$$K\left(h^{\frac{\alpha(r+s)(1-t+r')}{(1-\alpha t)s+\alpha r}}, \frac{(1-\alpha t)s+\alpha r}{\alpha(1-t+r')}\right)^{\frac{\alpha}{s}} \ = \ K\left(h^{r+s}, \frac{\alpha(1-t+r')}{(1-\alpha t)s+\alpha r}\right)^{-\frac{(1-\alpha t)s+\alpha r}{s(1-t+r')}},$$

and hence (3.4) holds. □

Remark 3.4. If $r = r'$ in (3.4), then we have the following reverse inequality of (3.3):

$$\begin{aligned}&K\left(h^{r+s}, \frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}\right) \| A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\alpha}A^{\frac{1}{2}} \|^{\frac{s(1-t+r)}{(1-\alpha r)s+\alpha r}} \\ &\leq \| A^{\frac{1-t+r}{2}}(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}}A^{\frac{1-t+r}{2}} \|\end{aligned}$$

for $0 \leq t \leq 1$, $s \geq 1$, $1 + r \geq t$ and $0 \leq \alpha \leq 1$. Moreover, let $t = 1$ in Theorem 3.3. As a reverse inequality of (GAH), we have

$$\begin{aligned}&K\left(h^{r+s}, \frac{\alpha r}{(1-\alpha)s+\alpha r}\right) \| A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \|^{\frac{sr}{(1-\alpha)s+\alpha r}} \\ &\leq \| A^{\frac{r}{2}}(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{\alpha r}{(1-\alpha)s+\alpha r}}A^{\frac{r}{2}} \|,\end{aligned}$$

that is,

$$K\left(h^{r+s}, \frac{\alpha r}{(1-\alpha)s+\alpha r}\right) \| A\sharp_{\alpha}B \|^{\frac{sr}{(1-\alpha)s+\alpha r}} \leq \| A^r \sharp_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \|$$

for $s \geq 1$, $r \geq 0$ and $0 \leq \alpha \leq 1$.

Under the conditions of $0 \leq s \leq 1$ and $r' = r$, we prove the following inequality as in Theorem 3.3:

Theorem 3.5. *Let A and B be positive operators on a Hilbert space H such that $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then*

$$\begin{aligned}&\| A^{\frac{1-t+r}{2}}(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}}A^{\frac{1-t+r}{2}} \| \\ &\leq \ K(h^{1+t}, \alpha)^{-\frac{s(1-t+r)}{(1-\alpha t)s+\alpha r}} \ \| A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\alpha}A^{\frac{1}{2}} \|^{\frac{s(1-t+r)}{(1-\alpha t)s+\alpha r}}\end{aligned} \tag{3.5}$$

for $0 \leq s, t \leq 1$, $1 + r \geq t$ and $0 \leq \alpha \leq 1$ with $\alpha(1-t) \leq (1-\alpha t)s$ where $K(h, p)$ is the generalized Kantorovich constant defined by (1.5).

Proof. We use the Hölder-McCarthy inequality and its reverse: Let A be a positive operator with $0 < m \le A \le M$. Then for every vector $y \in H$

$$K(h,\beta)\langle Ay,y\rangle^{\beta} \parallel y \parallel^{2(1-\beta)} \le \langle A^{\beta}y,y\rangle \le \langle Ay,y\rangle^{\beta} \parallel y \parallel^{2(1-\beta)} \quad \text{for } 0 \le \beta \le 1.$$

Since $\frac{m}{M^t} \le mA^{-t} \le A^{-\frac{t}{2}}BA^{-\frac{t}{2}} \le MA^{-t} \le \frac{M}{m^t}$ and $\parallel A^{\gamma}x \parallel \le \parallel A^{\gamma} \parallel = \parallel A \parallel^{\gamma} \le M^{\gamma}$ for all unit vectors $x \in H$ and $\gamma > 0$, we have for any $0 \le s \le 1$

$$\begin{aligned}
&\langle A^{\frac{1-t+r}{2}}(A^{-\frac{r}{2}}B^sA^{-\frac{r}{2}})^{\frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}}A^{\frac{1-t+r}{2}}x,x\rangle \\
&\le \langle A^{\frac{1-t}{2}}B^sA^{\frac{1-t}{2}}x,x\rangle^{\frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}} \parallel A^{\frac{1-t+r}{2}}x \parallel^{2\{1-\frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}\}} \\
&\le \langle A^{\frac{1-t}{2}}BA^{\frac{1-t}{2}}x,x\rangle^{\frac{s\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}} \parallel A^{\frac{1-t}{2}}x \parallel^{\frac{2(1-s)\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}} M^{\frac{1-t+r}{(1-\alpha t)s+\alpha r}(s-\alpha st-\alpha+\alpha t)} \\
&\le (K(h^{1+t},\alpha)^{-1}\langle A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\alpha}A^{\frac{1}{2}}x,x\rangle)^{\frac{s(1-t+r)}{(1-\alpha t)s+\alpha r}} \parallel A^{\frac{1}{2}}x \parallel^{-\frac{2(1-\alpha)s(1-t+r)}{(1-\alpha t)s+\alpha r}} \\
&\quad \times M^{\frac{1-t+r}{(1-\alpha t)s+\alpha r}(\alpha(1-s)(1-t))}M^{\frac{1-t+r}{(1-\alpha t)s+\alpha r}(s-\alpha st-\alpha+\alpha t)} \\
&\le K(h^{1+t},\alpha)^{-\frac{s(1-t+r)}{(1-\alpha t)s+\alpha r}} \parallel A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\alpha}A^{\frac{1}{2}} \parallel^{\frac{s(1-t+r)}{(1-\alpha t)s+\alpha r}} \\
&\quad \times M^{-\frac{1-t+r}{(1-\alpha t)s+\alpha r}(1-\alpha)s}M^{\frac{1-t+r}{(1-\alpha t)s+\alpha r}(s-\alpha s)} \\
&= K(h^{1+t},\alpha)^{-\frac{s(1-t+r)}{(1-\alpha t)s+\alpha r}} \parallel A^{\frac{1}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^{\alpha}A^{\frac{1}{2}} \parallel^{\frac{s(1-t+r)}{(1-\alpha t)s+\alpha r}} .
\end{aligned}$$

Hence we obtain the desired inequality (3.5). □

Putting $t = 1$ in (3.5), we have an inequality given in [15]:

$$\parallel A^r \sharp_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \parallel \le K(h^2,\alpha)^{-\frac{rs}{(1-\alpha)s+\alpha r}} \parallel A \sharp_{\alpha} B \parallel^{\frac{rs}{(1-\alpha)s+\alpha r}}$$

for $0 \le s \le 1$, $r \ge 0$ and $0 \le \alpha \le 1$.

References

[1] T. Ando and F. Hiai, *Log majorization and complementary Golden-Thompson type inequality*, Linear Algebra Appl. **197** (1994), 113–131.

[2] H. Araki, *On an inequality of Lieb and Thirring*, Lett. Math. Phys., **19** (1990), 167–170.

[3] N. Bebiano, R. Lemos and J. Providência, *Inequalities for quantum relative entropy*, Linear Algebra Appl. **401** (2005), 159–172.

[4] H.O. Cordes, *Spectral Theory of Linear Differential Operators and Comparison Algebras*, Cambridge University Press, 1987.

[5] J.I. Fujii, M. Fujii, Y. Seo and M. Tominaga, *On generalized Kantorovich inequalities*, Banach and Function Spaces (2004), 205–213, Yokohama Publisher.

[6] M. Fujii and E. Kamei, *Ando-Hiai inequality and Furuta inequality*, Linear Algebra Appl. **416** (2006), 541–545.

[7] M. Fujii, R. Nakamoto and M. Tominaga, *Generalized Bebiano-Lemos-Providência inequalities and their reverses*, Linear Algebra Appl. **426** (2007), 33-39.

[8] M. Fujii and Y. Seo, *Reverse inequalities of Cordes and Löwner-Heinz inequalities*, Nihonkai Math. J., **16** (2005), 145–154.

[9] T. Furuta, *$A \ge B \ge 0$ assures $(B^rA^pB^r)^{1/q} \ge B^{(p+2r)/q}$ for $r \ge 0, p \ge 0, q \ge 1$ with $(1+2r)q \ge p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.

[10] T. Furuta, *Extension of the Furuta inequality and Ando-Hiai log majorization*, Linear Algebra Appl. **219** (1995), 139–155.

[11] T. Furuta, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*, J. Inequal. Appl., **2** (1998), 137–148.

[12] T. Furuta, *Operator inequality implying generalized Bebiano-Lemos-Providência one*, Linear Algebra Appl. **426** (2007), 342–348.

[13] T. Furuta, J. Mićić, J.E. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities **1**, Element, Zagreb, 2005.

[14] G.K. Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc., **36** (1972), 309–310.

[15] Y. Seo and M. Tominaga, *A reverse inequality of a generalized Ando-Hiai inequality*, preprint.

[1,2] Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316-0033, Japan.
E-mail address: [1]mfujii@cc.osaka-kyoiku.ac.jp, [2]nakamoto@mx.ibaraki.ac.jp

[3] Toyama National College of Technology, Hongo-machi, Toyama-shi 939-8630, Japan.
E-mail address: mtommy@toyama-nct.ac.jp