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A GENERALIZATION OF THE WEAK AMENABILITY OF BANACH ALGEBRAS

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Abstract. Let A be a Banach algebra and let φ and ψ be continuous homomorphisms on A. We consider the following module actions on A,

$$a \cdot x = \varphi(a)x, \qquad x \cdot a = x\psi(a) \qquad\qquad (a, x \in A).$$

We denote by $A_{(\varphi,\psi)}$ the above A-module. We call the Banach algebra A, (φ,ψ)-weakly amenable if every derivation from A into $(A_{(\varphi,\psi)})^*$ is inner. In this paper among many other things we investigate the relations between weak amenability and (φ,ψ)-weak amenability of A. Some conditions can be imposed on A such that the (φ'',ψ'')-weak amenability of A^{**} implies the (φ,ψ)-weak amenability of A.

1. Introduction and preliminaries

Let A be a Banach algebra and let X be a Banach A-module. Then a derivation from A into X is a (bounded) linear map $D : A \longrightarrow X$ such that for every $a, b \in A$, $D(ab) = D(a) \cdot b + a \cdot D(b)$. If $x \in X$, the map $a \mapsto a \cdot x - x \cdot a$, $(a \in A)$ is a derivation. A derivation of this form is called an inner derivation. The set of all bounded linear operators from A into X is denoted by $\mathcal{B}(A, X)$. The set of all derivations from A into X is denoted by $Z^1(A, X)$, and the set of all inner derivations from A into X is denoted by $B^1(A, X)$. Then $H^1(A, X) = \frac{Z^1(A,X)}{B^1(A,X)}$ is the *first Hochschild cohomology group* of A with coefficients in X.

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Let A be a Banach algebra and X be a Banach A-module. Then X^* is the dual of Banach A-module X, and is also a Banach A-module as well, if for each $a \in A$, $x \in X$ and $x^* \in X^*$ we define

$$\langle a \cdot x^*, x\rangle = \langle x^*, x \cdot a\rangle, \qquad \langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle.$$

A Banach algebra A is amenable if every derivation from A into every dual Banach A-module is inner, equivalently if $H^1(A, X^*) = \{0\}$ for every Banach A-module X, this definition was introduced by Johnson in [12]. A is weakly amenable if $H^1(A, A^*) = \{0\}$; this definition generalizes that introduced by Bade, Curtis and Dales in [1]. We introduce the following new definition of amenability which is related to homomorphisms of Banach algebras.

Let A be a Banach algebra and let φ and ψ be continuous homomorphisms on A. We consider the following module actions on A,

$$a \cdot x := \varphi(a)x, \quad x \cdot a := x\psi(a) \quad (a, x \in A).$$

We denote the above A-module by $A_{(\varphi,\psi)}$.

Let X be an A-module. A bounded linear mapping $d : A \to X$ is called a (φ, ψ)-*derivation* if

$$d(ab) = d(a) \cdot \varphi(b) + \psi(a) \cdot d(b) \quad (a, b \in A).$$

A bounded linear mapping $d : A \to X$ is called a (φ, ψ)-*inner derivation* if there exists $x \in X$ such that

$$d(a) = x \cdot \varphi(a) - \psi(a) \cdot x \quad (a \in A).$$

Derivations of this form are studied in [14, 15, 16].

Definition 1.1. Let A be a Banach algebra and let φ and ψ be continuous homomorphisms on A. Then A is called (φ, ψ)-weakly amenable if $H^1(A, (A_{(\varphi,\psi)})^*) = \{0\}$.

Let A and B be Banach algebras. We denote by $Hom(A, B)$ the metric space of all bounded homomorphisms from A into B, with the metric derived from the usual linear operator norm $\| \cdot \|$ on $\mathcal{B}(A, B)$ and denote $Hom(A, A)$ by $Hom(A)$. The following assertions hold for any Banach algebra A.

(a) If A is amenable then A is an (φ, ψ)-weakly amenable for each φ and ψ in $Hom(A)$.

(b) A is weakly amenable if and only if A is an (id,id)-weakly amenable (id=identity homomorphism).

(c) Let A be a commutative weakly amenable Banach algebra. Then $Z^1(A, X) = \{0\}$ for each Banach A-module X [3, Theorem 2.8.63]. Therefore A is (φ, ψ)-weakly amenable for all $\varphi, \psi \in Hom(A)$ if and only if A is commutative and weakly amenable.

Definition 1.2. Let A be a Banach algebra, X be a Banach A-module and let $\varphi, \psi \in Hom(A)$. A derivation $D : A \to X$ is called approximately (φ, ψ)-inner if there exists a net (x_α) in X such that, for all $a \in A$, $D(a) = \lim_\alpha x_\alpha \cdot \varphi(a) - \psi(a) \cdot x_\alpha$ in norm.

Definition 1.3. A Banach algebra A is approximately (φ,ψ)-weakly amenable if every derivation $D: A \to (A_{(\varphi,\psi)})^*$ is approximately (φ,ψ)-inner.

Whenever $\varphi = \psi = \mathrm{id}$, this is just the definition of approximate weak amenability developed by Ghahramani and Loy in [9].

Definition 1.4. Let A be an algebra, and let $\varphi \in \Phi_A \cup \{0\}$ (Φ_A be the character space of A). A linear functional d on A is a point derivation at φ if

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in A).$$

Throughout this paper A denotes a Banach algebra and A^{**} is the second dual of A equipped with the first Arens product. This product can be characterized as the extension to $A^{**} \times A^{**}$ of the bilinear map $A \times A \to A : (a, b) \to ab$ with the following properties:

i) for fixed $b'' \in A^{**}$, $a'' \mapsto a''b''$ is w^*-continuous on A^{**}.

ii) for fixed $b \in A$, $a'' \mapsto ba''$ is w^*-continuous on A^{**}.

The image of A in A^{**} under the canonical embedding is denoted by $\widehat{A}$.

In section 2 we prove the main results for this new concept of amenability. In section 3, we develop the relation between the (φ,ψ)-weak amenability of a Banach algebra A and A^{**}. Finally in section 4 we give some examples to show that the new concept of amenability is different from amenability and weak amenability.

2. (φ,ψ)-weak amenability

Let A be a Banach algebra, and let $A^2 = \mathrm{span}\{ab:\ a,b \in A\}$.

Proposition 2.1. *Let A be Banach algebra and let $\varphi, \psi \in Hom(A)$ such that $\varphi(a)b = a\psi(b)$ for all $a, b \in A$. If A is (φ,ψ)-weakly amenable, then $\overline{A^2} = A$, where $\overline{A^2}$ is the closure of A^2 in A.*

Proof. Suppose $\overline{A^2} \neq A$. Take $a_0 \in A \backslash \overline{A^2}$ and $f \in A^*$ such that $f|_{A^2} = 0$ and $\langle f, a_0\rangle = 1$. Define $d : A \to (A_{(\varphi,\psi)})^*$ by $d(a) = \langle f, a\rangle f$. It is easy check that d is a (φ,ψ)-derivation. Since A is (φ,ψ)-weakly amenable, d is (φ,ψ)-inner, so that there is a $g \in (A_{(\varphi,\psi)})^*$ such that $d(a) = g \cdot \varphi(a) - \psi(a) \cdot g$, for all $a \in A$. So we have $\langle da_0, a_0\rangle = 1$. On the other hand

$$\langle d(a_0), a_0\rangle = \langle g, \varphi(a_0)a_0\rangle - \langle g, a_0\psi(a_0)\rangle = 0.$$

This is a contradiction. □

Corollary 2.2. *Let A be a Banach algebra. Then A is $(0,0)$-weakly amenable if and only if $\overline{A^2} = A$.*

Proof. Let A be $(0,0)$-weakly amenable. Then by the above theorem, $\overline{A^2} = A$. For the converse let $d : A \to (A_{(0,0)})^*$ be a $(0,0)$-derivation. Then we have $d(A^2) = \{0\}$. Since d is continuous, we have $d = 0$. So d is $(0,0)$-inner. □

Let A be a weakly amenable Banach algebra or A be a Banach algebra with a bounded left (right) approximate identity. Then A^2 is dense in A. Thus A is $(0,0)$-weakly amenable.

Proposition 2.3. *Let A be a Banach algebra and ψ, φ and λ are continuous homomorphisms from A into A. If φ is an epimorphism and A is $(\psi\circ\varphi, \lambda\circ\varphi)$-weakly amenable, then A is (ψ, λ)-weakly amenable.*

Proof. Let $d : A \to (A_{(\psi,\lambda)})^*$ be a continuous (ψ, λ)-derivation, and $D = d\circ\varphi$. We see that D is a $(\psi\circ\varphi, \lambda\circ\varphi)$-derivation. So there exists a $f \in (A_{(\psi\circ\varphi,\lambda\circ\varphi)})^*$ such that for each $a \in A$, $D(a) = f\cdot(\psi\circ\varphi)(a) - (\lambda\circ\varphi)(a)\cdot f$. Let $b \in A$. Then there exists a $a \in A$ such that $\varphi(a) = b$ and so

$$d(b) = d(\varphi(a)) = D(a) = f\cdot\psi(\varphi(a)) - \lambda(\varphi(a))\cdot f = f\cdot\psi(b) - \lambda(b)\cdot f.$$

Thus d is an (ψ, λ)-inner. □

Corollary 2.4. *Let A be a Banach algebra and let $\varphi \in Hom(A)$. If φ is an epimorphism and A is (φ^n, φ^n)-weakly amenable for some $n \in \mathbb{N}$. Then A is weakly amenable.*

There are Banach algebras which are (φ, φ)-weakly amenable where φ is not an epimorphism, and A is weakly amenable. This will be presented in Examples 4.3 and 4.4. The converse of the Corollary 2.4 is true when $\varphi^2 = 1_A$ or φ is an epimorphism such that $\varphi^2|_{[A]} = 1_A$ where $[A] = \{ab - ba|a, b \in A\}$. In the following theorems and corollaries we prove the above claims.

Theorem 2.5. *Let A be a Banach algebra and let $\psi, \lambda, \varphi \in Hom(A)$ and $\varphi^2 = 1_A$. If A is (ψ, λ)-weakly amenable, then A is $(\psi\circ\varphi, \lambda\circ\varphi)$-weakly amenable.*

Proof. Let $D : A \to (A_{(\psi\circ\varphi,\lambda\circ\varphi)})^*$ be a $(\psi\circ\varphi, \lambda\circ\varphi)$-derivation and let $d = D\circ\varphi^{-1}$. It can be shown that d is a (ψ, λ)-derivation. Thus there exist a $f \in (A_{(\psi,\lambda)})^*$ such that for all $a \in A$, $d(a) = f\cdot\psi(a) - \lambda(a)\cdot f$ and so we have $D(a) = D(\varphi^{-1}(\varphi(a)) = d(\varphi(a)) = f\cdot\psi(\varphi(a)) - \lambda(\varphi(a))\cdot f$, i.e., D is an $(\psi\circ\varphi, \lambda\circ\varphi)$-inner derivation. □

Corollary 2.6. *If A is weakly amenable and $\varphi \in Hom(A)$ such that $\varphi^2 = 1_A$, then A is (φ^n, φ^n)-weakly amenable for all $n \in \mathbb{N}$.*

Theorem 2.7. *Let $\varphi, \psi \in Hom(A)$ and let A be (ψ, ψ)-weakly amenable. If $\varphi|_{[A]} = id$ and φ is an epimorphism, then A is $(\varphi\circ\psi, \varphi\circ\psi)$- weakly amenable.*

Proof. Suppose that $D : A \to (A_{(\varphi\circ\psi,\varphi\circ\psi)})^*$ is an $(\varphi\circ\psi, \varphi\circ\psi)$-derivation. Set $d : A \to (A_{(\psi,\psi)})^*$ as follows

$$\langle d(a), b\rangle := \langle D(a), \varphi(b)\rangle.$$

Then d is a (ψ, ψ)-derivation. Thus there exists a $f \in (A_{(\psi,\psi)})^*$ such that for every $a \in A$, $d(a) = f\cdot\psi(a) - \psi(a)\cdot f$. Let $b \in A$. Since φ is onto, there exists $b_1 \in A$ such that $b = \varphi(b_1)$. So

$$\begin{aligned}\langle D(a), b\rangle &= \langle d(a), b_1\rangle \\ &= \langle f\cdot\psi(a) - \psi(a)\cdot f, b_1\rangle \\ &= \langle f, \varphi(\psi(a)b_1 - b_1\psi(a))\rangle \\ &= \langle f\cdot\varphi\circ\psi(a) - \varphi\circ\psi(a)\cdot f, b\rangle.\end{aligned}$$

Therefore D is an $(\varphi\circ\psi, \varphi\circ\psi)$-inner. □

Corollary 2.8. *Let A be a Banach algebra and let $\varphi \in Hom(A)$. Suppose that A is weakly amenable and φ is an epimorphism such that $\varphi|_{[A]} = id$. Then A is (φ^n, φ^n)-weakly amenable for all $n \in \mathbb{N}$.*

Proposition 2.9. *Let A be a Banach algebra and $\varphi \in Hom(A)$. Suppose that A is (φ^n, φ^n)-weakly amenable for all $n \in \mathbb{N}$, and $\varphi^n \to 1_A$ in norm. Then A is approximately weakly amenable.*

Proof. Let $D : A \to A^*$ be a derivation. For every $n \in \mathbb{N}$ set $D_n : A \to (A_{(\varphi^n,\varphi^n)})^*$, $D_n(a) = D(\varphi^n(a))$. It is clear that D_n is an (φ^n, φ^n)-derivation. So there exists a sequence (f_n) in A^* such that $D_n(a) = f_n \cdot \varphi^n(a) - \varphi^n(a) \cdot f_n$. Since $\varphi^n(a) \to a$, $D_n(a) \to D(a)$. Therefore $D(a) = \lim_n(f_n \cdot a - a \cdot f_n)$. □

In the proof of the Proposition 2.9, if the sequence (f_n) has an accumulation point then A is weakly amenable.

Theorem 2.10. *Let A be a Banach algebra, $\varphi \in Hom(A)$ and $0 \neq \psi \in \Phi_A$. Let A be (φ, φ)-weakly amenable and $Im\varphi \not\subset \ker\psi$. Then there are no non-zero continuous point derivations at $\psi \circ \varphi$.*

Proof. Let $\psi \in \Phi_A$ and let $\varphi \in Hom(A)$. Then $\psi \circ \varphi \in \Phi_A$. Suppose that $d = d_{\psi\circ\varphi} : A \to \mathbb{C}$ is a point derivation at $\psi \circ \varphi$. We define $D : A \to (A_{(\varphi,\varphi)})^*$ by $D(a) := d(a)\psi$. Then clearly D is a (φ, φ)-derivation. Since A is (φ, φ)-weakly amenable, there exists a $f \in (A_{(\varphi,\varphi)})^*$ such that $D(a) = f \cdot \varphi(a) - \varphi(a) \cdot f$. On the other hand since $\mathrm{Im}\varphi \not\subset \ker\psi$, there exist $a_1 \in A$ such that $\psi(\varphi(a_1)) = 1$. If $d_{\psi\circ\varphi}$ is a non-zero point derivation, then $\ker\psi\circ\varphi \not\subset \ker d_{\psi\circ\varphi}$. In fact if $\ker\psi\circ\varphi \subset \ker d_{\psi\circ\varphi}$, then there is an $\alpha \in \mathbb{C}$ such that $d_{\psi\circ\varphi} = \alpha(\psi \circ \varphi)$. So

$$\begin{aligned} 2\alpha &= 2\alpha((\psi \circ \varphi)(a_1)) = 2d_{\psi\circ\varphi}(a_1) \\ &= 2d_{\psi\circ\varphi}(a_1)\psi \circ \varphi(a_1) \\ &= d_{\psi\circ\varphi}(a_1)\psi \circ \varphi(a_1) + \psi \circ \varphi(a_1)d_{\psi\circ\varphi}(a_1) \\ &= d_{\psi\circ\varphi}(a_1^2) = \alpha(\psi \circ \varphi)(a_1^2) = \alpha. \end{aligned}$$

Thus $\alpha = 0$, i.e. $d = 0$ which is a contradiction.

Therefore there exist $a_2 \in \ker\psi \circ \varphi$ such that $d(a_2) = 1$. Put $a_0 = a_1 + (1 - d(a_1))a_2$, then

$$\psi \circ \varphi(a_0) = \psi \circ \varphi(a_1) + (1 - d(a_1))\psi \circ \varphi(a_2) = 1,$$

and

$$d(a_0) = d(a_1) + (1 - d(a_1))d(a_2) = d(a_1) + 1 - d(a_1) = 1.$$

Therefore

$$\begin{aligned} 1 &= d(a_0)\psi(\varphi(a_0)) \\ &= \langle D(a_0), \varphi(a_0)\rangle = \langle f \cdot \varphi(a_0) - \varphi(a_0) \cdot f, \varphi(a_0)\rangle \\ &= \langle f, (\varphi(a_0))^2\rangle - \langle f, (\varphi(a_0))^2\rangle = 0. \end{aligned}$$

Which is a contradiction. □

By using the Theorem 2.10, if A is a weakly amenable Banach algebra, then there is no non-zero continuous point derivation on A. Therefore Theorem 2.10 could be considered as a generalization of [3, Theorem 2.8.63]. If A is approximately (φ,φ)-weakly amenable, then the Theorem 2.10 is also true.

Theorem 2.11. *Let $\varphi \in Hom(A)$ and $\varphi^2 = \varphi$. Suppose that A and $\ker\varphi$ are weakly amenable, $Im\varphi$ is an ideal of A. Then A is (φ,φ)-weakly amenable.*

Proof. Let $D : A \to (A_{(\varphi,\varphi)})^*$ be a (φ,φ)-derivation. Take $d : A \to A^*$ with $\langle d(a), b\rangle := \langle D(a), \varphi(b)\rangle$, and so d is a derivation. Then there exists a $f \in A^*$ such that $d(a) = f \cdot a - a \cdot f$, $(a \in A)$.
Since $\varphi : A \to \mathrm{Im}\varphi$ is a projection, $A=\mathrm{Im}\varphi\oplus\ker\varphi$ where $\mathrm{Im}\varphi$ and $\ker\varphi$ are closed ideals of A. Let $a \in A$. Then there exist $a_1, a_2 \in A$ such that $a = a_1 + a_2$ where $a_1 \in \mathrm{Im}\varphi$ and $a_2 \in \ker\varphi$. Since $\ker\varphi$ is weakly amenable, $\overline{(\ker\varphi)^2} = \ker\varphi$. So, there is a net $(t_\alpha s_\alpha)_\alpha \subset (\ker\varphi)^2$ such that $t_\alpha s_\alpha \to a_2$, and $D(a_2) = \lim_\alpha D(t_\alpha s_\alpha) = \lim_\alpha(D(t_\alpha)\cdot\varphi(s_\alpha) - \varphi(t_\alpha)\cdot D(s_\alpha)) = 0$.

Therefore

$$D(a) = D(a_1) = D(\varphi(a_1)) = D(\varphi(a)),$$

so we have

$$\begin{aligned}\langle D(a), \varphi(b)\rangle &= \langle D(\varphi(a)), \varphi(b)\rangle = \langle d(\varphi(a)), b\rangle\\ &= \langle f\cdot\varphi(a) - \varphi(a)\cdot f, b\rangle = \langle f, \varphi(a)b - b\varphi(a)\rangle\\ &= \langle d(-b), \varphi(a)\rangle = \langle D(-\varphi(b)), \varphi^2(a)\rangle\\ &= \langle d(-\varphi(b)), \varphi(a)\rangle = \langle f\cdot\varphi(a) - \varphi(a)\cdot f, \varphi(b)\rangle.\end{aligned}$$

On the other hand $b = b_1 + b_2$ such that $b_1 \in \mathrm{Im}\varphi$, $b_2 \in \ker\varphi$ we have $\varphi(b) = \varphi(b_1) = b_1$, hence

$$\begin{aligned}\langle D(a), b_2\rangle &= \lim_\alpha\lim_\beta\langle D(s_{1\beta}s_{2\beta}), t_{1\alpha}t_{2\alpha}\rangle\\ &= \lim_\alpha\lim_\beta\langle D(s_{1\beta})\cdot\varphi(s_{2\beta}) + \varphi(s_{1\beta})\cdot D(s_{2\beta}), t_{1\alpha}t_{2\alpha}\rangle\\ &= \lim_\alpha\lim_\beta\langle D(s_{1\beta}), \varphi(s_{2\beta})t_{1\alpha}t_{2\alpha}\rangle + \lim_\alpha\lim_\beta\langle D(s_{2\beta}), t_{1\alpha}t_{2\alpha}\varphi(s_{1\beta})\rangle\\ &= 0,\end{aligned}$$

since $s_{1\beta}, s_{2\beta} \in\ker\varphi$. $\varphi(s_{2\beta})t_{1\alpha}t_{2\alpha}$, $t_{1\alpha}t_{2\alpha}\varphi(s_{1\beta}) \in \mathrm{Im}\varphi\cap\ker\varphi = \{0\}$. Therefore A is (φ,φ)- weakly amenable. □

3. (φ,ψ)-weak amenability of the second dual

Let A be a Banach algebra. We consider A^{**} the second dual of A. It is known that the Banach algebra A inherits amenability from A^{**} [11]. No example is yet known whether this fails if one considers the weak amenability instead, but the property is known to hold for the Banach algebras A which are left ideals in A^{**}[10], the dual Banach algebras [8], the Banach algebras A which are Arens regular and every derivation from A into A^* is weakly compact [5], Banach algebras for which the second adjoint of each derivation $D : A \to A^*$ satisfies $D''(A^{**}) \subseteq \mathrm{WAP}(A)$, and the Banach algebras A which are right ideals in A^{**} and satisfy $A^{**}A = A^{**}$ [7]. Now let A be a Banach algebra and let $\varphi, \psi \in Hom(A)$ such that $\varphi(a)b = a\psi(b)$ for all $a, b \in A$. If A^{**} is (φ'', ψ'')-weakly amenable, then by Proposition 2.1, $\overline{A^{**2}} = A^{**}$. Thus we can show that $\overline{A^2} = A$ [8, Proposition 2.1]. So by Corollary 2.2, A is $(0,0)$-weakly amenable. A question remains whether the (φ'', ψ'')-weak amenability of A^{**} implies the (φ, ψ)-weak amenability of A and vice versa?

$L^1(\mathbb{R})$ is (id,id)-weakly amenable but $L^1(\mathbb{R})^{**}$ is not (id,id)-weakly amenable. In general if G is a nondiscrete locally compact group then $L^1(G)^{**}$ is not (id,id)-weakly amenable [4], but $L^1(G)$ is (id,id)-weakly amenable [13].

For an infinite compact metric space X, $\mathrm{lip}_\alpha(X)$ is (id,id)-weakly amenable,for $0 < \alpha < 1/2$, but $\mathrm{lip}_\alpha(X)^{**}$ is not (id,id)-weakly amenable [1].

Proposition 3.1. *Let A be Banach algebra and $\varphi \in Hom(A)$, if A^{**} is $(\varphi'', 0)$-weakly amenable, then A is $(\varphi, 0)$-weakly amenable.*

Proof. Suppose that $D : A \to (A_{(\varphi,0)})^*$ is a continuous $(\varphi, 0)$-derivation. Take $a'', b'' \in A^{**}$ and take bounded nets (a_α) and (b_β) in A with $\widehat{a_\alpha} \to a''$, $\widehat{b_\beta} \to b''$ in the w^*-topology of A^{**}. Then $D'' : A^{**} \to (A^{**}_{(\varphi'',0)})^*$ is an $(\varphi'', 0)$-derivation because

$$\begin{aligned} D''(a''b'') &= w^* - \lim_\alpha \lim_\beta D''(\widehat{a}_\alpha \widehat{b}_\beta) \\ &= w^* - \lim_\alpha \lim_\beta (\widehat{D(a_\alpha) \cdot \varphi}(b_\beta)) \\ &= D''(a'') \cdot \varphi''(b''). \end{aligned}$$

Therefore there exists $a_0''' \in A^{***}$ such that $D''(a'') = a_0'''\varphi''(a'')$ for all $a'' \in A^{**}$. We obtain $D(a) = a_0\varphi(a)$for all $a \in A$, where a_0' is the restriction of a_0''' to A. Thus A is an $(\varphi, 0)$-weakly amenable. □

If A^{**} is $(0, \psi'')$-weakly amenable, then A is $(0, \psi)$-weakly amenable if and only if $D''(a''b'') = \psi''(a'') \cdot D''(b'')$ if and only if $\psi''(a'') \cdot D''(b'') = w^* - \lim_\alpha \psi''(\widehat{a}_\alpha) \cdot D''(b'')$ [9]. The last equality is true if A^{**} is a Banach algebra under the second Arens product. Let A be a Banach algebra with a bounded approximate identity, then A is $(\varphi, 0)$ and $(0, \psi)$-weakly amenable (see Example 4.2).

For a Banach algebra A, let A^{op} be the Banach algebra with underlying Banach space A and with product $\circ$ given by $a \circ b = ba$. We have the following simple observation.

Proposition 3.2. *Let A be Banach algebra and $\varphi, \psi \in Hom(A)$. Then A is (φ, ψ)-weakly amenable if and only if A^{op} is (ψ, φ)-weakly amenable.*

For a Banach algebra A, A^{**} is $(0,0)$-weakly amenable with the first Arens product if and only if A^{**} is $(0,0)$-weakly amenable with the second Arens product. We immediately observe that the $(0,0)$-weak amenability of A^{**} implies that A is $(0,0)$-weakly amenable. However, the $(0,0)$-weak amenability of A does not imply that A^{**} is $(0,0)$-weakly amenable unless every derivation from A^{**} to A^{***} is w^*-continuous. Some conditions can be imposed on A such that the (φ'', ψ'')-weak amenability of A^{**} implies the (φ, ψ)-weak amenability of A where $\varphi, \psi \neq 0$.

Theorem 3.3. *Let A be a Banach algebra and $\varphi, \psi \in Hom(A)$. Let A^{**} be (φ'', ψ'')-weakly amenable,and suppose $\widehat{A}$ is a left ideal in A^{**}. Then A is (φ, ψ)-weakly amenable.*

Proof. It is known that $\varphi''(\widehat{a}) = \widehat{\varphi(a)}$ and $\psi''(\widehat{a}) = \widehat{\psi(a)}$, for all $a \in A$. The proof of the Theorem is similar to [10, Theorem 2.3]. □

A Banach algebra A is said to be dual if there is a closed submodule A_* of A^* such that $A = A_*^*$. Let $i : A_* \to A^*$ be the canonical embedding and i' be the adjoint of i. If $a \in A$, we have $i'(\widehat{a}) = \widehat{a}$. Obviously i is norm-continuous, hence i' is w^*-continuous. Let $a'', b'' \in A^{**}$ and take nets (a_α) and (b_β) in A such that $\widehat{a}_\alpha \xrightarrow{w^*} a''$ and $\widehat{b}_\beta \xrightarrow{w^*} b''$. Then

$$\begin{aligned} i'(a''b'') &= i'(w^* - \lim_\alpha \lim_\beta \widehat{a}_\alpha \widehat{b}_\beta) = w^* - \lim_\alpha \lim_\beta i'(\widehat{a}_\alpha \widehat{b}_\beta) \\ &= w^* - \lim_\alpha \lim_\beta (\widehat{a}_\alpha \widehat{b}_\beta) = (w^* - \lim_\alpha \widehat{a}_\alpha)(w^* - \lim_\beta \widehat{b}_\beta) \\ &= i'(w^* - \lim_\alpha \widehat{a}_\alpha) i'(w^* - \lim_\beta \widehat{b}_\beta) = i'(a'')i'(b''). \end{aligned}$$

Hence i' is an algebra homomorphism from A^{**} onto A. Let $\varphi : A \longrightarrow A$ be a continuous homomorphism. Then the second conjugate φ'' is w^*-continuous and

$$\langle i'(\varphi''(\widehat{a})), b\rangle = \langle \varphi''(\widehat{a}), i(b)\rangle = \langle \varphi''(\widehat{a}), b\rangle = \langle \varphi''(i'(\widehat{a})), b\rangle.$$

Hence $i'(\varphi''(\widehat{a})) = \varphi''(i'(\widehat{a}))$. We know that $\varphi''|_A = \varphi$, if $a'' \in A^{**}$, $(a_\alpha) \subset A$, $\widehat{a}_\alpha \xrightarrow{w^*} a''$, then

$$\begin{aligned} \varphi(i'(a'')) &= \varphi''(i'(a'')) = \varphi''(i'(w^* - \lim_\alpha \widehat{a}_\alpha)) = w^* - lim_\alpha \varphi''(i'(\widehat{a}_\alpha)) \\ &= w^* - \lim_\alpha i'(\varphi''(\widehat{a}_\alpha)) = i'(\varphi''(w^* - \lim_\alpha \widehat{a}_\alpha)) = i'(\varphi''(a'')). \end{aligned}$$

Theorem 3.4. *Let A be a dual Banach algebra and let $\varphi, \psi \in Hom(A)$. If A^{**} is (φ'', ψ'')-weakly amenable then A is (φ, ψ)-weakly amenable.*

Proof. Let i be as above. Suppose that $d : A \longrightarrow (A_{(\varphi,\psi)})^*$ is an (φ, ψ)-derivation. Set $D = i'' \circ d \circ i' : A^{**} \longrightarrow (A^{**}_{(\varphi'',\psi'')})^*$, then for every $a'', b'', c'' \in A^{**}$ we have

$$\begin{aligned} \langle D(a''b''), c''\rangle &= \langle d(i'(a'')i'(b'')), i'(c'')\rangle \\ &= \langle d(i'(a'')) \cdot \varphi(i'(b'')) + \psi(i'(a'')) \cdot d(i'(b'')), i'(c'')\rangle \\ &= \langle d(i'(a'')), \varphi''(i'(b''))i'(c'')\rangle + \langle d(i'(b'')), i'(c'')\psi''(i'(a''))\rangle \\ &= \langle d(i'(a'')), i'(\varphi''(b''))i'(c'')\rangle + \langle d(i'(b'')), i'(c'')i'(\psi''(a''))\rangle \\ &= \langle (i'' \circ d \circ i'(a'')), \varphi''(b'')c''\rangle + \langle (i'' \circ d \circ i'(b'')), c''\psi''(a'')\rangle \\ &= \langle D(a'') \cdot \varphi''(b'') + \psi''(a'') \cdot D(b''), c''\rangle. \end{aligned}$$

Therefore D is an (φ'', ψ'')-derivation. Since A^{**} is (φ'', ψ'')-weakly amenable, there exists $a_0''' \in A^{***}$ such that

$$D(a'') = a_0''' \cdot \varphi''(a'') - \psi''(a'') \cdot a_0''', \quad a'' \in A^{**}.$$

Now let $R : A^{***} \longrightarrow A^*$ be the restriction map. Set $a_0' = R(a_0''')$. For every $a, b \in A$ we have

$$\begin{aligned}\langle d(a), b\rangle &= \langle d(i'(\widehat{a}), i'(\widehat{b})\rangle = \langle i'' \circ d \circ i'(\widehat{a}), \widehat{b}\rangle \\ &= \langle D(\widehat{a}), \widehat{b}\rangle = \langle a_0''' \cdot \varphi''(\widehat{a}), \widehat{b}\rangle - \langle \psi''(\widehat{a}) \cdot a_0''', \widehat{b}\rangle \\ &= \langle a_0''', \varphi''(\widehat{a})\widehat{b}\rangle - \langle a_0''', \widehat{b}\psi''(\widehat{a})\rangle = \langle a_0''', \varphi(a)b\rangle - \langle a_0''', b\psi(a)\rangle \\ &= \langle R(a_0'''), \varphi(a)b\rangle - \langle R(a_0'''), b\psi(a)\rangle = \langle a_0' \cdot \varphi(a) - \psi(a) \cdot a_0', b\rangle.\end{aligned}$$

So $d(a) = a_0' \cdot \varphi(a) - \psi(a) \cdot a_0'$. Therefore d is an (φ, ψ)-inner. □

4. Examples

Example 4.1. Let A be a commutative weakly amenable Banach algebra. A Banach A-module X is called symmetric if $a.x = x.a$, for $a \in A$ and $x \in X$. Then for every symmetric Banach A-module X we have $H^1(A, X) = \{0\}$ [1]. On the other hand for every $\varphi \in Hom(A)$, $(A_{(\varphi,\varphi)})^*$ is a symmetric Banach A-module. Thus A is (φ, φ)-weakly amenable.

Example 4.2. Let A be a Banach algebra with a bounded right approximate identity (e_α). Let $D : A \to (A_{(0,\psi)})^*$ be a derivation. Then for every $a, b \in A$, we have $D(ab) = \psi(a) \cdot D(b)$. Since D is bounded, $(D(e_\alpha))$ is a bounded net in $(A_{(0,\psi)})^*$. Let $f \in (A_{(0,\psi)})^*$ be a cluster point of $(D(e_\alpha))$. We can suppose that $w^* - \lim_\alpha D(e_\alpha) = f$ in $(A_{(0,\psi)})^*$. Then for every $a \in A$, we have $w^* - \lim_\alpha aD(e_\alpha) = af$ in $(A_{(0,\psi)})^*$. Thus we have

$$D(a) = \lim_\alpha D(ae_\alpha) = \lim_\alpha \psi(a) \cdot D(e_\alpha) = \psi(a) \cdot f.$$

This means that D is $(0, \psi)-$inner. So A is $(0, \psi)$-weakly amenable. Similarly every Banach algebra with a bounded left approximate identity is $(\varphi, 0)$-weakly amenable. So every group algebra and C^*-algebra are $(\varphi, 0)$ and $(0, \psi)-$weakly amenable.

Example 4.3. Let $A = l^1(\mathbb{N})$ with the product $ab := a(1)b$ $(a, b \in l^1(\mathbb{N}))$. For every $\varphi, \psi \in Hom(A)$, A is (φ, ψ)-weakly amenable [16]. It is easy to check that A dose not have a bounded right approximate identity, thus A is not amenable.

Example 4.4. Let $S = \{x_1, x_2, x_3, x_4, x_5\}$ be a semigroup with $x_1^2 = x_1$, $x_1x_2 = x_2$, $x_3x_1 = x_3$, $x_3x_2 = x_4$ and all other products equal to x_5. We identify the elements of S with the point masses on $S(\delta_x := x)$. We know that, for any semigroup S,

$$l^1(S) = \left\{\sum_{s\in S} \alpha_s\delta_s; \quad \sum_{s\in S} \mid \alpha_s \mid < \infty, s \in S, \alpha_s \in \mathbb{C}\right\}$$

is a Banach algebra with the norm $\| \sum_{s\in S} \alpha_s\delta_s \| = \sum_{s\in S} \mid \alpha_s \mid$ and the convolution reduced to $\delta_s * \delta_t = \delta_{st}$, for $s, t \in S$ (see citeDal for details). In our case

$$l^1(S) = \left\{\lambda = \sum_{n=1}^{5} \alpha_n x_n; \quad \{\alpha_n\}_{n=1}^5 \subset \mathbb{C}, \ \{x_n\}_{n=1}^5 \subset S, \quad \|\lambda\| = \sum_{n=1}^{5} |\alpha_n|\right\},$$

and $l^1(S)$ is weakly amenable [2]. Since S is not regular semigroup, $l^1(S)$ is not amenable [6]. Let $\varphi, \psi : l^1(S) \to l^1(S)$ be continuous homomorphisms and $D : l^1(S) \to (l^1(S)_{(\varphi,\psi)})^*$ be a (φ,ψ)-derivation we show that $D = 0$. Therefore D is an (φ,ψ)-inner derivation for each $\varphi,\psi \in Hom(l^1(S))$. If $x, y \in S$ we show that $\langle Dx, y\rangle = 0$.

Suppose that $\varphi(x_j) = \sum_{k=1}^{5} \alpha_{jk}x_k$, $\psi(x_j) = \sum_{k=1}^{5} \beta_{jk}x_k$, $(1 \le j \le 5,\ \alpha_{jk}, \beta_{jk} \in \mathbb{C})$. Since $\varphi(x_1^2) = \varphi(x)$, $\alpha_{11}^2 = \alpha_{11}$, $\alpha_{11}\alpha_{12} = \alpha_{12}$, $\alpha_{13}\alpha_{11} = \alpha_{13}$, $\alpha_{13}\alpha_{12} = \alpha_{14}$.

I) If $\alpha_{11} = 0$, then $\alpha_{12} = \alpha_{13} = \alpha_{14} = 0$, $\alpha_{15}^2 = \alpha_{15}$. It is easy to show that above (φ,ψ)-derivation is zero.

II) If $\alpha_{11} = \beta_{11} = 1$ then

$$\varphi(x_j) = x_1 + \sum_{k=2}^{5} \alpha_{jk}x_k, \quad \psi(x_j) = x_1 + \sum_{k=2}^{5} \beta_{jk}x_k \quad (2 \le j \le 5). \tag{4.1}$$

We put $\langle Dx_i, x_j\rangle = t_{ij}$, $(i, j \in \{1, 2, 3, 4, 5\},\ x_i, x_j \in S,\ t_{ij} \in \mathbb{C})$. Also

$$\langle Dx_ix_k, x_j\rangle = \langle Dx_i, \varphi(x_k)x_j\rangle + \langle Dx_k, x_j\psi(x_i)\rangle \tag{4.2}$$

for all $i, j, k \in \{1, 2, 3, 4, 5\}$.
Since $x_1^2 = x_1$, $t_{1j} = \langle Dx_1, \varphi(x_1)x_j\rangle + \langle Dx_1, x_j\psi(x_1)\rangle$. Therefore

$$t_{14} = t_{15} = (2 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15})t_{15}.$$

If

$$\alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15} + 1 = 0 \tag{4.3}$$

since $\varphi(x_1^2) = \varphi(x_1)$ and $\psi(x_1^2) = \psi(x_1)$, we have $\alpha_{14} = \alpha_{12}\alpha_{13}$, $\beta_{14} = \beta_{12}\beta_{13}$ and

$$\begin{aligned}\alpha_{12}^2 + \alpha_{13}^2 + \alpha_{12}^2\alpha_{13}^2 + \alpha_{15}^2 + 2\alpha_{12}^2\alpha_{13} + 2\alpha_{12}\alpha_{13}^2 + 2\alpha_{12}\alpha_{13}\alpha_{15}\\ +3\alpha_{12}\alpha_{13} + 2\alpha_{12}\alpha_{15} + 2\alpha_{13}\alpha_{15} + \alpha_{12} + \alpha_{13} + \alpha_{15} = 0,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\beta_{12}^2 + \beta_{13}^2 + \beta_{12}^2\beta_{13}^2 + \beta_{15}^2 + 2\beta_{12}^2\beta_{13} + 2\beta_{12}\beta_{13}^2 + 2\beta_{12}\beta_{13}\beta_{15}\\ +3\beta_{12}\beta_{13} + 2\beta_{12}\beta_{15} + 2\beta_{13}\beta_{15} + \beta_{12} + \beta_{13} + \beta_{15} = 0.\end{aligned} \tag{4.5}$$

From (4.4)and (4.5) the following relation is obtained

$$\begin{cases}\alpha_{15} = -\alpha_{12}\alpha_{13} - \alpha_{12} - \alpha_{13} + 1\\ or\\ \alpha_{15} = -\alpha_{12}\alpha_{13} - \alpha_{12} - \alpha_{13}\end{cases} \tag{4.6}$$

and

$$\begin{cases}\beta_{15} = -\beta_{12}\beta_{13} - \beta_{12} - \beta_{13} + 1\\ or\\ \beta_{15} = -\beta_{12}\beta_{13} - \beta_{12} - \beta_{13}\end{cases} \tag{4.7}$$

by inserting these solutions in (4.3), we get the contradictions: $3 = 0, 2 = 0, 1 = 0$. Therefore

$$t_{14} = t_{15} = 0. \tag{4.8}$$

Since $\varphi(x_5^2) = \varphi(x_5)$ and $\psi(x_5^2) = \psi(x_5)$, similar to the above

$$t_{54} = t_{55} = 0. \tag{4.9}$$

From (4.1), (4.2) and (4.8), we deduce that

$$t_{24} = t_{25} = (1 + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15})t_{25}.$$

Since $Dx_5 = Dx_2x_1$,

$$t_{5j} = \langle Dx_2, \varphi(x_1)x_j\rangle + \langle Dx_1, x_j\psi(x_2)\rangle. \tag{4.10}$$

From (4.8), (4.9) and (4.10), we have

$$(1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15})t_{25} = 0.$$

If $\beta_{12} + \beta_{13} + \beta_{14} + \beta_{15} = 0$ and $1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} = 0$, from the relations $\alpha_{14} = \alpha_{12}\alpha_{13}$, $\beta_{14} = \beta_{12}\beta_{13}$, (4.6) and (4.7), we conclude the contradictions: $1 = -1$ and $1 = 0$. Therefore

$$t_{24} = t_{25} = 0. \tag{4.11}$$

From (4.1), (4.2) and (4.8), we have $t_{34} = t_{35} = (1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15})t_{35}$. Since $1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} \neq 0$,

$$t_{34} = t_{35} = 0. \tag{4.12}$$

Since $Dx_4 = Dx_3x_2$,

$$t_{4j} = \langle Dx_3, \varphi(x_2)x_j\rangle + \langle Dx_2, x_j\psi(x_3)\rangle. \tag{4.13}$$

From (4.1) and (4.13), we have

$$t_{44} = t_{45} = (1 + \sum_{k=2}^{5} \alpha_{2k})t_{25} + (1 + \sum_{k=2}^{5} \beta_{3k})t_{35}. \tag{4.14}$$

From (4.11), (4.12) and (4.14), we conclude that

$$t_{44} = t_{45} = 0.$$

If $t_{11} \neq 0$, we can conclude that $\alpha_{43} = 0$ and $\alpha_{43} = 2$, which is a contradiction. Hence $t_{11} = 0$. By using of the above relations $t_{ij} = 0$ for every i and j. Therefore $D = 0$.

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References

1. W.G. Bade, P.C. Curtis and H.G. Dales, *Amenability and weak amenability for Beurling and Lipschits algebra* , Proc. London Math. Soc. (3) **55** (1987), 359–377.
2. S. Bowling and J. Duncan, *The first cohomology of semigroup Banach algebras* , Semigroup Forum (3) **56** (1988), 130–145.
3. H.G. Dales, *Banach algebra and Automatic continuity*, Oxford University Press, 2000.
4. H.G. Dales, F. Ghahramani and A.Ya. Helemskii,*The amenability of measure algebras*, J. London Math. Soc. (2) **66** (2002), 213–226.
5. H.G. Dales, A. Rodriguez-Palacios and M.V. Velasco,*The second transpose of a derivation*, J. London Math. Soc. (2) **64** (2001),707–721.
6. J. Duncan and A.L.T. Paterson, *Amenability for discrete convolution semigroup algebras*, Math. Scand. **66** (1990), 141–146.

7. M. Eshaghi Gordji and M. Filali,*Weak amenability of the second dual of a Banach algebra*, Studia Math. **182**, no. 3, (2007), 205–213.
8. F. Ghahramani and J. Laali, *Amenability and topological center of the second duals of Banach algebras* Bull. Austral. Math. Soc. **65** (2002), 191–197.
9. F. Ghahramani and R.J. Loy, *Generalized notions of amenability*, J. Funct. Analysis, **208** (2004), 229–260.
10. F. Ghahramani, R.J. Loy and G.A. Willis, *Amenability and weak amenability of second cojugate Banach algebras*, Proc. Amer. Math. Soc. **124** (1996), 1489–1497.
11. F. Gourdeau, *Amenability and the second dual of a Banach algebras*, Studia Math. **125** (1997), 75–81.
12. B.E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972).
13. B.E. Johnson, *Weak amenability of group algebras*, Bull. London Math. Soc. **23** (1991), 281–284.
14. M. Mirzavaziri and M.S. Moslehian, *Automatic continuity of σ-derivations in C^*-algebras*, Proc. Amer. Math. Soc. **134** (2006), no. 11, 3319–3327.
15. M. Mirzavaziri and M.S. Moslehian, *σ-derivations in Banach algebras*, Bull. Iranian Math. Soc., (2006), 65–78.
16. M.S. Moslehian and A.N. Motlagh, *Some notes on (σ, τ)-amenability of Banach algebras*, Stud. Univ. Babes-Bolyai Math. **53** (2008), no. 3, 57–68.

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