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COMPACT FAILURE OF MULTIPLICATIVITY FOR LINEAR MAPS BETWEEN BANACH ALGEBRAS

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Abstract. We introduce notions of compactness and weak compactness for multilinear maps from a product of normed spaces to a normed space, and prove some general results about these notions. We then consider linear maps $T : A \to B$ between Banach algebras that are "close to multiplicative" in the following senses: the failure of multiplicativity, defined by $S_T(a, b) = T(a)T(b) - T(ab)$ $(a, b \in A)$, is compact [respectively weakly compact]. We call such maps cf-homomorphisms [respectively wcf-homomorphisms]. We also introduce a number of other, related definitions. We state and prove some general theorems about these maps when they are bounded, showing that they form categories and are closed under inversion of mappings and we give a variety of examples. We then turn our attention to commutative C^*-algebras and show that the behaviour of the various types of "close-to-multiplicative" maps depends on the existence of isolated points in the maximal ideal space. Finally, we look at the splitting of Banach extensions when considered in the category of Banach algebras with bounded cf-homomorphisms [respectively wcf-homomorphisms] as the arrows. This relates to the (weak) compactness of 2-cocycles in the Hochschild-Kamowitz cohomology complex. We prove "compact" analogues of a number of established results in the Hochschild-Kamowitz cohomology theory.

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1. Introduction

For Banach algebras, A and B, and a linear map $T : A \to B$ we call the bilinear map $S_T : A \times A \to B$ given by $S_T(a,b) = T(a)T(b) - T(ab)$ the *failure of multiplicativity of* T. Our focus in this paper will be new notions of "smallness" for the failure of multiplicativity of a bounded linear map T, based on compactness. We will use these notions to produce new results in function theory and in the theory of Hochschild-Kamowitz cohomology of Banach algebras. In order to provide some motivation we will first give a brief discussion of a well-studied notion of smallness for the failure of multiplicativity: namely that the norm of S_T be less than some small $\delta > 0$.

Let A and B be Banach algebras and denote by $\mathcal{B}(A,B)$ the space of all bounded linear maps from A to B. Let $\delta > 0$. We call a bounded linear map $T : A \to B$ *δ-multiplicative* if $\|S_T\| < \delta$. Now let $M(A,B)$ be the set of all multiplicative, bounded linear maps from A to B and, for $T \in \mathcal{B}(A,B)$, let $d(T) = \inf\{\|T - S\| : S \in M(A,B)\}$. In [12], Johnson proved the following.

Proposition 1.1. *Let A and B be Banach algebras and let $T : A \to B$ be a bounded linear map. Then*

$$\|S_T\| \leq (1 + d(T) + 2\|T\|)d(T).$$

This implies that for every $\delta > 0$ there exists $\varepsilon > 0$, such that all linear maps $T : A \to B$ with $\|T\| < 1$ that are within distance ε of some multiplicative bounded linear map are δ-multiplicative. Research on δ-multiplicativity has focused on when the converse of this holds. We call (A,B) an *AMNM pair* (an "almost multiplicative bounded linear maps are near multiplicative bounded linear maps pair") if for every $\varepsilon > 0$ there exists $\delta > 0$ such that all δ-multiplicative bounded linear maps $T : A \to B$ with $\|T\| < 1$ are within distance ε of some multiplicative bounded linear map. If $(A,\mathbb{C})$ is an AMNM-pair we call A *AMNM*. AMNM algebras are studied in [11], [16], [9], [3] and [7], and a good source for AMNM pairs is [12].

1.1. **Notation.** Let X be a topological space and $S \subset X$. We write $\overline{S}$ for the closure of S. In the case that X is a normed space then the closure is taken in the norm topology unless specifically stated otherwise.

For a normed space E we write $\mathrm{ball}(E)$ for the open unit ball of E and $\overline{\mathrm{ball}}(E)$ for the closed unit ball of E (i.e. $\overline{\mathrm{ball}}(E) = \overline{\mathrm{ball}(E)}$).

For Banach spaces $E_1, \ldots, E_n, F$ we let $\mathcal{B}(E_1, \ldots, E_n : F)$ be the space of all bounded multilinear maps from $E_1 \times E_n$ to F. In the case that $E_1, \ldots, E_n := E$ we write $\mathcal{B}^n(E,F)$ for $\mathcal{B}(E_1, \ldots, E_n : F)$.

The notions of smallness on which we shall concentrate in this paper are based on concepts of compactness for multilinear maps which we shall define and discuss in the following section.

2. Compactness of multilinear maps

We start with the following definition, which will be important in this paper. In the special case of the norm topology the same condition is considered by Krikorian in [14]. We made the more general version below independently.

Definition 2.1. Let $E_1, \ldots, E_n$ be normed spaces and F a vector space with a topology $\mathcal{T}$ defined on it. An n-linear map $T : E_1 \times \cdots \times E_n \to F$ is *compact with respect to* $\mathcal{T}$ if the closure in $\mathcal{T}$ of

$$T(\{(x_1, \ldots, x_n) : x_i \in \mathrm{ball}(E_i)\})$$

is compact when considered as topological space with the subspace topology induced by $\mathcal{T}$. If $\mathcal{T}$ is the norm topology we call T *compact*. If $\mathcal{T}$ is the weak topology we call T *weakly compact*. Let $E_1, \ldots, E_n$ be normed spaces. We denote the set of all compact n-linear maps from $E_1 \times \cdots \times E_n$ to F by $\mathcal{K}^n(E_1, \ldots, E_n; F)$; in the case where $E_1 = \cdots = E_n = E$ we denote $\mathcal{K}^n(E_1, \ldots, E_n; F)$ by $\mathcal{K}^n(E, F)$. We denote the set of all weakly-compact n-linear maps from $E_1 \times \cdots \times E_n$ to F by $w\mathcal{K}^n(E_1, \ldots, E_n; F)$; in the case where $E_1 = \cdots = E_n = E$ we denote $w\mathcal{K}^n(E_1, \ldots, E_n; F)$ by $w\mathcal{K}^n(E, F)$.

We note that weakly compact multilinear maps are necessarily bounded.

We mention the following interesting source of examples of compact multilinear maps, due to Krikorian ([14]).

Example 2.2. Let E and F be Banach spaces, U be an open subset of E, $n \in \mathbb{N}$, and $f : U \to F$ be an n-times-continuously-differentiable function that maps bounded sets to relatively compact sets. Then, for $x \in U$ and $k \in \{1, \ldots, n\}$ the kth derivative of f at x is a compact k-linear map from $E \times \cdots \times E$ to F.

We shall now prove that T being (weakly) compact is equivalent to the associated linear map from the n-fold projective tensor product of E with itself to F being (weakly) compact. We refer the reader unfamiliar with this construction to [5, Appendix A1] for definitions and notation. We shall need the following result. The case where $n = 2$ it is [5, A.3.69] and the general version is similar.

Proposition 2.3. *Let $E_1, \ldots E_n$ be normed spaces and F be a Banach space, and let $T \in \mathcal{B}(E_1, \ldots, E_n; F)$. Then there is a unique, bounded, linear map $\widetilde{T} : \widehat{\bigotimes}_{i=1}^n E_i \to F$ such that*

$$\widetilde{T}(x_1 \hat{\otimes} \ldots \hat{\otimes} x_n) = T(x_1, \ldots, x_n) \quad (x_j \in E_j, j \in \{1, \ldots, n\}).$$

Furthermore, the map $T \mapsto \widetilde{T}$, $\mathcal{B}^n(E_1, \ldots, E_n; F) \to \mathcal{B}\left(\widehat{\bigotimes}_{i=1}^n E_i, F\right)$, is an isometric Banach space isomorphism.

The following is elementary.

Lemma 2.4. *Let $E_1, \ldots, E_n$ be normed spaces. Then $\overline{\mathrm{ball}}\left(\widehat{\bigotimes}_{i=1}^n E_i\right)$ is the closed convex hull of $\{x_1 \hat{\otimes} \ldots \hat{\otimes} x_n : x_i \in \mathrm{ball}(E_i)\}$.*

The following is a special case of [2, Theorem IV.5] (Krein's Theorem).

Proposition 2.5. *Let F be a Banach space and $\mathcal{T}$ a topology on F for which a functional on F is $\mathcal{T}$-continuous if and only if it is norm continuous (equivalently $\mathcal{T}$ contains the weak topology and is contained in the norm topology). Let S be a subset of of F which is compact with respect to $\mathcal{T}$. Then the closed convex balanced hull (and therefore the closed convex hull) of S is compact with respect to $\mathcal{T}$.*

In particular if $\mathcal{T}$ is the norm topology or the weak topology the above holds.

In the special case where $n = 2$ and $\mathcal{T}$ is the norm topology, the following theorem and corollary were proven in [14]. We proved this more general version independently.

Theorem 2.6. *Let F and $\mathcal{T}$ be as in Proposition 2.5 and let $E_1, \ldots, E_n$ be normed spaces. A multilinear map $T \in \mathcal{B}(E_1, \ldots, E_n; F)$ is compact with respect to $\mathcal{T}$ if and only if the linear map $\widetilde{T} \in \mathcal{B}\left(\widehat{\bigotimes}_{i=1}^n E_i, F\right)$ with $\widetilde{T}(x_1 \hat{\otimes} \ldots \hat{\otimes} x_n) = T(x_1, \ldots, x_n)$ is compact with respect to $\mathcal{T}$.*

Proof. In this proof "$\mathcal{T}$-compact" will mean "compact with respect to $\mathcal{T}$". First assume that T is $\mathcal{T}$-compact. Then

$$\overline{T(\{(x_1, \ldots, x_n) : x_i \in \mathrm{ball}(E_i)\})} = \overline{\widetilde{T}(\{x_1 \hat{\otimes} \ldots \hat{\otimes} x_n : x_i \in \mathrm{ball}(E_i)\})}$$

is $\mathcal{T}$-compact. We call this set S. By Proposition 2.5, $\overline{\mathrm{co}}(S)$ is also $\mathcal{T}$-compact.

$$\begin{aligned} K := & \; \overline{\widetilde{T}(\overline{\mathrm{co}}(\{x_1 \hat{\otimes} \ldots \hat{\otimes} x_n : x_i \in \mathrm{ball}(E_i)\}}))\\ = & \; \overline{\widetilde{T}(\mathrm{co}(\{x_1 \hat{\otimes} \ldots \hat{\otimes} x_n : x_i \in \mathrm{ball}(E_i)\}}))\\ = & \; \overline{\mathrm{co}(\widetilde{T}(\{x_1 \hat{\otimes} \ldots \hat{\otimes} x_n : x_i \in \mathrm{ball}(E_i)\}))} \subseteq \overline{\mathrm{co}}(S), \end{aligned}$$

and so the K is $\mathcal{T}$-compact. By Lemma 2.4, we have $K = \overline{\widetilde{T}\left(\overline{\mathrm{ball}}(\widehat{\bigotimes}_{i=1}^n E_i)\right)}$ and hence $\widetilde{T}$ is $\mathcal{T}$-compact.

We now assume that $\widetilde{T}$ is compact. We have,

$$\begin{aligned} \overline{T\big(\{(x_1, \ldots, x_n) : x_i \in \mathrm{ball}(E_i)\}\big)} &= \overline{\widetilde{T}\big(\{(x_1 \hat{\otimes} \ldots \hat{\otimes} x_n) : x_i \in \mathrm{ball}(E_i)\}\big)}\\ &\subseteq \overline{\widetilde{T}\big(\{z \in \widehat{\bigotimes}_{i=1}^n E_i : \|z\| < 1\}}\big). \end{aligned}$$

The right-hand side of the above expression is compact and hence so is the left hand side; the result follows. □

In the following corollary and its proof "closed" means "closed with respect to the norm topology".

Corollary 2.7. *Let $E_1, \ldots, E_n$ and F be Banach spaces; then $\mathcal{K}^n(E_1, \ldots, E_n; F)$ and $w\mathcal{K}^n(E_1, \ldots, E_n; F)$ are closed subspaces of $\mathcal{B}^n(E_1, \ldots, E_n; F)$ such that $\mathcal{K}^n(E_1, \ldots, E_n; F)$ is isometrically isomorphic to $\mathcal{K}\left(\widehat{\bigotimes}_{i=1}^n E_i, F\right)$ and $w\mathcal{K}^n(E_1, \ldots, E_n; F)$ is isometrically isomorphic to $w\mathcal{K}\left(\widehat{\bigotimes}_{i=1}^n E_i, F\right)$.*

Proof. Let E and F be Banach spaces. Then it is standard that $\mathcal{K}(E,F)$ and $w\mathcal{K}(E,F)$ are closed subspaces of $\mathcal{B}(E,F)$. Thus $\mathcal{K}\left(\widehat{\bigotimes}_{i=1}^n E_i, F\right)$ and $w\mathcal{K}\left(\widehat{\bigotimes}_{i=1}^n E_i, F\right)$ are Banach spaces. By Proposition 2.3 and Theorem 2.6, $\mathcal{K}^n(E_1,\dots,E_n;F)$ is isometrically isomorphic to $\mathcal{K}\left(\widehat{\bigotimes}_{i=1}^n E_i, F\right)$ and $w\mathcal{K}^n(E_1,\dots,E_n;F)$ is isometrically isomorphic to $w\mathcal{K}\left(\widehat{\bigotimes}_{i=1}^n E_i, F\right)$. Hence, they are complete, and therefore closed, subspaces of $\mathcal{B}^n(E_1,\dots,E_n;F)$ and the result follows. □

2.1. **(Weakly) compact failure of multiplicativity for linear maps.** In this subsection we introduce a class of functions between Banach algebras which are, in a certain sense, "close" to being homomorphisms.

Definition 2.8. Let A and B be normed algebras and let $T : A \to B$ be a linear map. Recall the definition of the failure of multiplicativity S_T. We call T a *cf-homomorphism* (where "cf" stands for "compact from") if S_T is compact and a *wcf-homomorphism* (where "wcf" stands for "weakly compact from") if S_T is weakly compact. If S_T is finite-dimensional we call T an *fdf-homomorphism* and if for $n \in \mathbb{N}$ we have S_T at most n-dimensional we call T an *$n$df-homomorphism*.

If, for each $a \in A$, we have that $S_T(a,\cdot)$ and $S_T(\cdot,a)$ are compact linear maps, we say that T is a *semi-cf-homomorphism*. We define "*semi-wcf-homomorphism*", "*semi-fdf-homomorphism*" and "*semi-$n$df-homomorphism*" similarly.

We note some obvious relationships between these conditions. The set of cf-homomorphisms from A to B contains all homomorphisms from A to B and all compact linear maps from A to B. The set of wcf-homomorphisms from A to B contains all weakly compact linear maps from A to B. Adding "w" or "semi-" to a condition makes it weaker. An $n$df-homomorphism is an $(n+1)$df-homomorphism and a continuous fdf-homomorphism is a cf-homomorphism.

Let E and F be Banach spaces. We say a map $T : E \to F$ is *weak-weak* continuous, if it is continuous when considered as a map from E equipped with the weak topology to F equipped with the weak topology. The following is part of [8, 27.6].

Proposition 2.9. *Let E and F be Banach spaces and $T : E \to F$ a bounded linear map. Then T is weak-weak continuous.*

We shall need the following lemma.

Lemma 2.10. *Let E, F and G be Banach spaces, $T_1 \in \mathcal{B}(G,E), T_2 \in \mathcal{B}(F,G)$ and $S \in \mathcal{B}^2(E,F)$. We define a map $R \in \mathcal{B}^2(G,F)$ by $R(g_1,g_2) = S(T_1(g_1),T_2(g_1))$. Suppose that $S \in \mathcal{K}^2(E,F)$. Then $R \in \mathcal{K}^2(G,F)$ and $T_2 \circ S \in \mathcal{K}^2(E,G)$. Suppose instead that $S \in w\mathcal{K}^2(E,F)$. Then $R \in w\mathcal{K}^2(G,F)$ and $T_2 \circ S \in w\mathcal{K}^2(E,G)$.*

Proof. In this proof we shall write "$X^{(2)}$" for the Cartesian product of a set X with itself. Closures will initially be in the norm topology. Suppose $S \in \mathcal{K}^2(E,F)$.

$$\begin{aligned}\overline{R\,(\mathrm{ball}(G)^{(2)})} &= \overline{S(\{(T_1(g_1),T_1(g_2)) : g_1,g_2 \in \mathrm{ball}(G)\})}\\ &\subseteq \overline{S(\{(e_1,e_2) : e_1,e_2 \in \|T\|_1\,\mathrm{ball}(E)\})}\\ &= \|T_1\|^2\,\overline{S\,(\mathrm{ball}(E)^{(2)})}, \end{aligned} \tag{2.1}$$

but $\overline{S\,(\mathrm{ball}(E)^{(2)})}$ is compact and so $\overline{R(\mathrm{ball}(G)^{(2)})}$ is a closed subset of the compact set $\|T_1\|^2\,\overline{S\,(\mathrm{ball}(E)^{(2)})}$ and thus is compact. Hence $R \in \mathcal{K}^2(G,F)$. Also,

$$\overline{T_2 \circ S(\mathrm{ball}(E)^{(2)})} = \overline{T_2(S(\mathrm{ball}(E)^{(2)}))} = \overline{T_2\left(\overline{S(\mathrm{ball}(E)^{(2)})}\right)}, \tag{2.2}$$

but $\overline{S(\mathrm{ball}(E)^{(2)})}$ is compact, and so (since T_2 is bounded)

$$\overline{T_2 \circ S(\mathrm{ball}(E)^{(2)})} = \overline{T_2\left(\overline{S(\mathrm{ball}(E)^{(2)})}\right)} = T_2\left(\overline{S(\mathrm{ball}(E)^{(2)})}\right), \tag{2.3}$$

is compact. Hence, $T_2 \circ S \in \mathcal{K}^2(E,G)$.

Now suppose $S \in w\mathcal{K}^2(E,F)$. By Proposition 2.9, T_1 and T_2 are continuous when E, F and G are considered with the weak topology. Hence, each of (2.2), (2.3) and (2.1) holds with the closure taken in the weak topology and so the result follows as in the norm topology case. □

Theorem 2.11. *Let A, B and C be Banach algebras and let $T_1 : A \to B$ and $T_2 : B \to C$ be bounded cf-homomorphisms [respectively bounded wcf-homomorphisms]. Then $T_2 \circ T_1$ is a bounded cf-homomorphism [respectively bounded wcf-homomorphism].*

Proof. Let $a, b \in A$. Then a direct calculation gives

$$S_{T_2\circ T_1}(a,b) = T_2(S_{T_1}(a,b)) - S_{T_2}(T_1(a),T_1(b)).$$

Thence the result is immediate from Lemma 2.10 and Corollary 2.7. □

Hence we have that the class of Banach algebras together with bounded cf-homomorphisms and the class of Banach algebras together with bounded wcf-homomorphism form concrete categories.

Theorem 2.12. *Let A and B be Banach algebras and T be a bounded cf-homomorphism [respectively a bounded wcf-homomorphism] that is bijective. Then the inverse mapping $T^{-1} : B \to A$ is a bounded cf-homomorphism [respectively a bounded wcf-homomorphism].*

Proof. By the Banach isomorphism theorem, T^{-1} is a bounded linear map. Also, for $b, b' \in B$ a direct calculation yields,

$$S_{T^{-1}}(b,b') = -T^{-1} \circ S_T(T^{-1}(b),T^{-1}(b')).$$

Thus the result follows from Lemma 2.10. □

Hence, a morphism in the category of Banach algebras with bounded cf-homomorphisms or in the category of Banach algebras with bounded wcf-homomorphisms is an isomorphism if and only if it is bijective.

3. Some examples

We give an example to show that bounded semi-cf-isomorphisms need not be wcf-homomorphisms.

Example 3.1. Following [15] we say a Banach algebra, A has *compact multiplication* if, for each $a \in A$, left and right multiplication by a, which we denote L_a and R_a respectively, are both compact operators on A. Let A and B be Banach algebras with compact multiplication and let $T : A \to B$ be a bounded linear map. Then it is clear that T is automatically a semi-cf-homomorphism. Now let $A = c_0$, with pointwise multiplication. Then A has compact multiplication since if we take $(a_n) \subset c_{00}$ such that $a_n \to a$ then $L_{a_n} \to L_a = R_a$ and each L_{a_n} has finite rank. Let $T : A \to A$ be given by $T(a) = 2a$; then S_T is surjective. Since the unit ball of c_0 is not relatively compact in the weak topology, it follows that T is not a wcf-homomorphism.

We now show that weakly compact linear maps need not be semi-cf-homomorphisms.

Example 3.2. Let A be an infinite-dimensional, unital Banach algebra, which is reflexive as a Banach space. For example, let A_0 be ℓ^2 with pointwise multiplication and let A be the one-dimensional unitisation of A_0. Denote the unit of A by e. Now let $T : A \to A$ be given by $T(a) = 2a$ for each $a \in A$. Then $S_T(e, \cdot) = T$ which is bijective, and so not compact.

We now give examples to show that two Banach algebras may be isomorphic in the category of Banach algebras with bounded cf-homomorphisms without being isomorphic in the usual category of Banach algebras.

Example 3.3. Let $n \in \mathbb{N}$ and let A and B be non-isomorphic Banach algebras, each with underlying vector space $\mathbb{C}^n$. Clearly, the identity map from $\mathbb{C}^n$ to itself defines a compact linear bijection from A to B and so it defines an isomorphism in the category of Banach algebras with bounded cf-homomorphisms. In particular we may take A to be $\mathbb{C}$ with the usual product and B to be $\mathbb{C}$ with zero product. If we let $A = C(\{1, 2, 3, 4\})$ and $B = \mathcal{B}(\ell^2(\{1, 2\}))$ (the algebra of 2×2 matrices over $\mathbb{C}$) we have an example where both algebras are unital C^*-algebras.

Example 3.4. Let $A = \ell^\infty$ with the pointwise product and let B be the vector space $\mathcal{B}(\ell^2(\{1, 2\})) \oplus A$ with the norm $\|(\mathbf{A}, a)\| = \max\{\|\mathbf{A}\|, \|a\|\}$ and the product $(\mathbf{A}, a)(\mathbf{B}, b) = (\mathbf{AB}, ab)$ $(\mathbf{A}, \mathbf{B} \in \mathcal{B}(\ell^2(\{1, 2\}), a, b \in A)$. Then B is a Banach algebra. We define a linear map $T : A \to B$ by

$$(a_n)_{n\in\mathbb{N}} \to \left(\begin{bmatrix} a_1, & a_2 \\ a_3, & a_4 \end{bmatrix}, (a_{n-4})_{n\in\mathbb{N}} \right).$$

It is easy to check that T is a bounded linear bijection. Also

$$S_T\left((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\right) = \left(\begin{bmatrix} a_1, & a_2 \\ a_3, & a_4 \end{bmatrix} \begin{bmatrix} b_1, & b_2 \\ b_3, & b_4 \end{bmatrix} - \begin{bmatrix} a_1b_1, & a_2b_2 \\ a_3b_3, & a_4b_4 \end{bmatrix}, 0 \right)$$

and so S_T is of finite rank. Thus T is an isomorphism in the category of Banach algebras and cf-homomorphisms. Clearly, A is commutative and B is not so they are not isomorphic as Banach algebras.

We now give an example of a bounded 1df-homomorphism (and hence of a cf-homomorphism) that is neither a homomorphism nor a compact linear map.

Example 3.5. Let A be ℓ^∞ with the pointwise product. For $k \in \mathbb{N}$ let $e_k \in A$ be the sequence with 1 in the kth place and 0 in all other places. Define a bounded linear map $T : A \to A$ by $T(a) = a + a_1 e_1$, $(a = (a_k)_{k\in\mathbb{N}} \in A)$. Then T is a linear bijection from ℓ^∞ onto itself and, hence, is not weakly-compact. Also, $e_1^2 = e_1$ and $T(e_1) = 2e_1$, so

$$T(2e_1e_1) = T(2e_1) = 4e_1 \neq 8e_1 = T(2e_1)T(e_1).$$

Hence, T is not a homomorphism. However, for $a = (a_k)_{k\in\mathbb{N}}$, $b = (b_k)_{k\in\mathbb{N}} \in A$,

$$\begin{aligned} S_T(a,b) &= T(ab) - T(a)T(b) \\ &= ab + a_1b_1e_1 - (a + a_1e_1)(b + b_1e_1) \\ &= ab + a_1b_1e_1 - (ab + 3a_1b_1e_1) = -2a_1b_1e_1. \end{aligned}$$

This has rank 1 and so is compact.

Since cf-homomorphisms are "a compact map away from being homomorphisms" one may conjecture that if A and B are Banach algebras, $T_1 : A \to B$ is a continuous homomorphism and $T_2 : A \to B$ is a compact linear map then $T := T_1 + T_2$ must be a cf-homomorphism. The following example shows that this is not true, even if T_2 is a rank 1 homomorphism.

Example 3.6. Let A and $e_k \in A$ be as in Example 3.5 and denote the identity element of A by 1. Let $T_1 : A \to A$ be the identity homomorphism $T_1(a) = a$, $a \in A$ and let $T_2 : A \to A$ be the bounded, rank 1, linear map given by $T_2(a) = a_1 1$. Then, if $T = T_1 + T_2$,

$$T(e_k) = \begin{cases} e_1 + 1 & \text{if } k = 1, \\ e_k & \text{otherwise.} \end{cases}$$

Hence, for $k > 1$,

$$S_T(e_1, e_k) = T(e_1)T(e_k) - T(e_1e_k) = e_k,$$

so

$$(e_k)_{k=2}^\infty \subseteq S_T\left(\overline{\mathrm{ball}(A)}^{(2)}\right) \subseteq \overline{S_T\left(\mathrm{ball}(A)^{(2)}\right)},$$

but $(e_k)_{k=2}^\infty$ has no convergent subsequence, so T is not a cf-homomorphism.

4. Commutative C^*-algebras

In this section we discuss how these notions relate to functions on locally-compact, Hausdorff topological spaces. Many of the results could be extended to more general classes of Banach function algebras, but to avoid having to give a large number of definitions we shall restrict to the case of commutative C^*-algebras.

4.1. **Nowhere-zero preserving maps.** The following is the Gleason-Kahane-Żelazko theorem, which may be found as [10, Theorem 2.3]. The author would like to thank Joel Feinstein for pointing him towards this result.

Proposition 4.1. *Let A be a Banach algebra with unit denoted 1 and let ϕ be a linear functional on A. Assume that for each invertible $f \in A$, $\phi(f) \neq 0$. Then $\phi/\phi(1)$ is multiplicative.*

This has the following corollary.

Corollary 4.2. *Let A be a Banach algebra with unit denoted 1, B be a commutative, unital, semisimple Banach algebra and $T : A \to B$ be a linear map. Assume that for each invertible $f \in A$, $T(f)$ is invertible in B. Then $T/T(1)$ is multiplicative.*

Proof. Via the Gel′fand transform, we may assume that B is a Banach function algebra on its character space. Therefore, $b \in B$ is invertible if $\phi(b) \neq 0$ for each multiplicative linear functional ϕ on B, and T is multiplicative if and only if $T \circ \phi$ is multiplicative for each multiplicative linear functional ϕ on B. Thus, the result follows from Proposition 4.1. □

Definition 4.3. A topological space is *perfect* if it is non-empty and has no isolated points.

Theorem 4.4. *Let X and Y be infinite, locally-compact, Hausdorff spaces. If X is perfect and $T : C_0(X) \to C_0(Y)$ is a bounded semi-wcf-homomorphism, then T is multiplicative and thus is of the form $T(f) = f \circ \psi$ for some homeomorphism $\psi : Y \to X$.*

Proof. First, we reduce this to the unital case. Assume for now that the result holds in the case where X and Y are both compact (i.e. that $C_0(X) = C(X)$ and $C_0(Y) = C(Y)$ are unital). Let X and Y be locally-compact, Hausdorff spaces and let $T : C_0(X) \to C_0(Y)$ be a bounded semi-wcf-homomorphism. Let $\widetilde{X}$ and $\widetilde{Y}$ be the unconditional one-point compactification of X and Y respectively (that is, the compact space obtained by adjoining an extra point whether or not the original space was compact). Let $f \in C_0(X)$. We define a map as follows:

$$\widetilde{T} : C(\widetilde{X}) \to C(\widetilde{Y}), \quad f + \alpha 1_{\widetilde{X}} \mapsto T(f) + \alpha 1_{\widetilde{Y}}.$$

Now, let $g \in C_0(X)$ and $(f_i)_i \subset C_0(X)$ be a bounded net. Then $(S_{\widetilde{T}}(f_i, g))_i = (S_T(f_i, g))_i$ has a weakly convergent subnet. By symmetry it follows that $\widetilde{T}$ is a semi-wcf-homomorphism. Hence, by assumption, $\widetilde{T}$ is multiplicative. Now let $f, g \in C_0(X)$ and $\alpha, \beta \in \mathbb{C}$. We have

$$\begin{aligned} S_{\widetilde{T}}((f+\alpha 1_{\widetilde{X}}),(g+\beta 1_{\widetilde{X}})) &= \widetilde{T}((f+\alpha 1_{\widetilde{X}})(g+\beta 1_{\widetilde{X}})) - \widetilde{T}(f+\alpha 1_X)\widetilde{T}(g+\beta 1_Y) \\ &= \widetilde{T}(fg+\beta f+\alpha g+\alpha\beta 1_X) - \\ &\quad\ \widetilde{T}(f)\widetilde{T}(g) - \widetilde{T}(\alpha g) - \widetilde{T}(\beta f) - \alpha\beta 1_X \\ &= \widetilde{T}(fg) - \widetilde{T}(f)\widetilde{T}(g) \\ &= S_T(f,g) = 0, \end{aligned}$$

and so T is multiplicative.

Henceforth we assume that X and Y are compact. Let $T : C_0(X) \to C_0(Y)$ be a bijective bounded semi-wcf-homomorphism. We show that $T(1) = 1$. Assume otherwise; then

$$S_T(1, \cdot) = T(1)T(\cdot) - T(\cdot) = (T(1) - 1)T(\cdot) \neq 0.$$

Since T is a bounded linear bijection, it follows that multiplication by $T(1) - 1$ is weakly compact. Assume, towards contradiction, that $(T(1) - 1)(y)$ is not uniformly zero. Since Y is perfect there is an infinite, compact $K \subset Y$ and $\varepsilon > 0$ such that $|(T(1) - 1)(y)| > \varepsilon$ for all $y \in K$. We take $(y_k)_{k\in\mathbb{N}} \subset K$ to be a sequence of distinct points converging to a limit y_0. By the Tietze Extension Theorem, we may take a bounded sequence $(g_n)_{n\in\mathbb{N}} \subset C(Y)$ such that

$$g(y_k)(T(1)(y_k) - 1) = \begin{cases} 1 & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases}.$$

Clearly $(g_n(T(1) - 1))_{n\in\mathbb{N}}$ has no weakly convergent subnet since the limit, g, of any such net would have to have $g(y_k) = 1$ for all k and $g(y_0) = 0$. This contradicts the weak compactness of multiplication by $(T(1)-1)$. Thus $T(1) = 1$.

We now assume, towards a contradiction, that T is not multiplicative. Then T^{-1} is also not multiplicative. By Corollary 4.2 it follows that there exists a non-invertible $f \in C(X)$ such that $T(f)$ is invertible. Since the set of invertibles in $C(Y)$ is open, and since for any $x \in X$,

$$\{f \in C(X) : \text{there is a neighbourhood } U \text{ of } x \text{ with } f(U) = \{0\}\}$$

is dense in $\{f \in C(X) : f(x) = 0\}$, we may assume without loss of generality that there exists some non-empty, open subset $U \subset X$ such that $f(U) = \{0\}$. We can then take $(g_n) \subset C(X)$ such that, for each n, we have that $g_n(X \setminus U) \subseteq \{0\}$ and (g_n) has no weakly convergent subsequence. Thus, we have that $fg_n = 0$ and so

$$S_T(f, g_n) = T(fg_n) - T(f)T(g_n) = -T(f)T(g_n).$$

Since T is a Banach space isomorphism and $T(f)$ is invertible, the map $g \mapsto -T(f)T(g)$ is a Banach space isomorphism. Thus $S_T(f, g_n)$ has no weakly convergent subsequence and so $S_T(f, \cdot)$ is not weakly compact, which contradicts our original assumption that T is a semi-wcf-homomorphism.

It is standard that Banach algebra isomorphisms from $C_0(X)$ to $C_0(Y)$ are of the form $f \mapsto f \circ \psi$ for some homeomorphism $\psi : Y \to X$. □

For locally-compact Hausdorff spaces with isolated points the situation is quite different. Indeed, we have the following.

Corollary 4.5. *Let X be an infinite, locally-compact Hausdorff space. Then the following are equivalent:*

(a) *every bounded semi-wcf-isomorphism from $C_0(X)$ to $C_0(X)$ is multiplicative;*
(b) *every bounded 1df-isomorphism is nowhere-zero preserving;*
(c) *X is perfect.*

Proof. Clearly (a) implies (b) and it follows from Theorem 4.4 that (c) implies (a). It remains to show that (b) implies (c). Assume that X is not perfect, let $x_0 \in X$ be an isolated point and $x_1 \in X \setminus \{x_0\}$. Define $e_{x_0} \in C_0(X)$ by $e_{x_0}(x_0) = 1$ and $e_{x_0}(X \setminus \{x_0\}) = \{0\}$. Then we can define a Banach space isomorphism $T : C_0(X) \to C_0(X)$ by

$$T(f)(x) = \begin{cases} f(x_0) - f(x_1) & \text{if } x = x_0, \\ f(x) & \text{otherwise.} \end{cases}.$$

Now $S_T(A \times A) \in e_{x_0}\mathbb{C}$, and so T is a 1df-isomorphism. However, if $f \in C_0(X)$ is nowhere zero, then $g := f + (f(x_1) - f(x_0))e_{x_0} \in A$ is also nowhere zero but has $T(g)(x_0) = 0$. The result follows. □

5. [W]cf-splitting of Banach extensions and compactness of Kamowitz cocycles

These notions of compact failure of multiplicativity fit together nicely with theory of Banach extensions and Kamowitz's cohomology theory for Banach algebras (see [13] and [1]). For definitions of terms from this theory and for the notation we shall use, as well as for background, we point the reader towards [5, Section 2.8]. Let A be a Banach algebra and Σ a Banach extension of A. We denote the equivalence class, with respect to equivalence, of Σ by $[\Sigma]$ and with respect to strong equivalence by $[\Sigma]_s$. The following is a slightly rewritten version of [5, 2.8.12].

Proposition 5.1. *Let A be a Banach algebra, E a Banach A-bimodule, and $T, T' \in \mathcal{Z}^2(A, E)$. Then:*

(a) *if $T - T' \in \widetilde{N}^2(A, E)$, Σ_T is equivalent to $\Sigma_{T'}$. Moreover,*
$$T + \widetilde{N}^2(A, E) \mapsto [\Sigma_T]$$
is a bijection from $\widetilde{H}^2(A, E)$ to the family of equivalence classes, with respect to equivalence, of singular, admissible Banach extensions of A by E;

(b) *if $T - T' \in \mathcal{N}^2(A, E)$, Σ_T is strongly equivalent to $\Sigma_{T'}$. Moreover,*
$$T + \mathcal{N}^2(A, E) \mapsto [\Sigma_T]_s$$
is a bijection from $\mathcal{H}^2(A, E)$ to the family of equivalence classes, with respect to strong equivalence, of singular, admissible Banach extensions of A by E.

Definition 5.2. We define the following vector spaces as analogues of the groups appearing in the Kamowitz cohomology theory:

$$\begin{aligned} Z^n_K(A, E) &:= \mathcal{Z}^n(A, E) \cap \mathcal{K}^n(A, E); \\ N^n_K(A, E) &:= \mathcal{N}^n(A, E) \cap \mathcal{K}^n(A, E) (= \mathcal{N}^n(A, E) \cap Z^n_K(A, E)); \\ \widetilde{N}^n_K(A, E) &:= N^n(A, E) \cap \mathcal{K}^n(A, E) (= N^n(A, E) \cap Z^n_K(A, E)); \\ H^n_K(A, E) &:= Z^n_K(A, E)/N^n_K(A, E); \\ \widetilde{H}^n_K(A, E) &:= Z^n_K(A, E)/\widetilde{N}^n_K(A, E). \end{aligned}$$

Similarly, we define the following weakly compact versions:

$$\begin{aligned} Z^n_w(A,E) &:= \mathcal{Z}^n(A,E) \cap w\mathcal{K}^n(A,E); \\ &\textit{etcetera.} \end{aligned}$$

Note that $H^1_K(A,E) = \widetilde{H}^1_K(A,E)$, and is zero if and only if all compact derivations from A to E are inner. If A is a commutative Banach algebra and E is a symmetric Banach A-bimodule, this is equivalent to there being no non-zero, compact derivations from A to E. Similarly, $H^1_w(A,E) = \widetilde{H}^1_w(A,E)$ is zero if and only if all weakly compact derivations from A to E are inner and, if A is a commutative Banach algebra and E a symmetric Banach A-bimodule, this is equivalent there being no non-zero weakly compact derivations from A to E.

As an example showing that these groups may be different to the usually continuous cohomology groups, we note that, in [6, Theorem 5.7.3], the present author showed that if we let A be the convolution algebra $\ell^1(\mathbb{Z}_+)$, then $H^1_K(A,A^*)$ is isometrically isomorphic (via $D \mapsto (D(t^k)(1))_{k\in\mathbb{N}}$) to c_0, while it is known that $\mathcal{H}^1(A,A^*)$ is isometrically isomorphic to ℓ^∞ via the same map. Related to this is [4], in which Choi and the current author calculated the image of the weakly compact derivations under this same map.

Definition 5.3. We say that a Banach extension

$$\Sigma : 0 \to I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} A \to 0$$

cf-splits if there is a cf-homomorphism Q such that $q \circ Q = \mathrm{id}_A$. We call $Q : A \to \mathfrak{A}$ a *splitting cf-homomorphism.* We say Σ *wcf-splits* if there is a wcf-homomorphism $Q : A \to \mathfrak{A}$ such that $q \circ Q = \mathrm{id}_A$. In this case we call Q a *splitting wcf-homomorphism.* If the extension Σ cf-splits [respectively wcf-splits] with a bounded splitting cf-homomorphism [respectively wcf-homomorphism], we say that Σ *cf-splits strongly* [respectively *wcf-splits strongly*].

Note that "(w)cf-splitting strongly" can be thought of as "splitting in the category of Banach algebras and bounded (w)cf-homomorphisms". The statements in the following lemma that refer to splitting and splitting strongly are well known but do not seem to be explicitly stated in the standard textbooks. The statements relating to (w)cf-splitting are new. The proofs are trivial and are omitted.

Lemma 5.4. *Let A be an algebra and let $\Sigma(\mathfrak{A}; I)$ and $\Sigma(\mathfrak{B}; I)$ be equivalent extensions of A. If $\Sigma(\mathfrak{A}; I)$ splits, then $\Sigma(\mathfrak{B}; I)$ splits.*

Let A be a Banach algebra and let $\Sigma(\mathfrak{A}; I)$ and $\Sigma(\mathfrak{B}; I)$ be strongly equivalent Banach extensions of A. Then the following hold:

- *if $\Sigma(\mathfrak{A}; I)$ splits strongly, then $\Sigma(\mathfrak{B}; I)$ splits strongly;*
- *if $\Sigma(\mathfrak{A}; I)$ cf-splits [respectively cf-splits strongly], then $\Sigma(\mathfrak{B}; I)$ cf-splits [respectively cf-splits strongly];*
- *if $\Sigma(\mathfrak{A}; I)$ wcf-splits [respectively wcf-splits strongly], then $\Sigma(\mathfrak{B}; I)$ wcf-splits [respectively wcf-splits strongly].*

For the remainder of this section we will refer only to the norm-topology case. In all cases the weak-topology version of any result holds and the proof is basically identical.

The following is an analogue of 5.1 in this new setting.

Theorem 5.5. *Let A be a Banach algebra, E a Banach A-bimodule, and $T, T' \in Z^2_K(A, E)$. Then:*

(a) *Σ_T cf-splits strongly.*

(b) *if $T - T' \in \widetilde{N}^2_K(A, E)$, Σ_T is equivalent to $\Sigma_{T'}$. Moreover,*

$$T + \widetilde{N}^2_K(A, E) \mapsto [\Sigma_T]$$

is a bijection from $\widetilde{H}^2_K(A, E)$ to the family of equivalence classes C, with respect to equivalence, of singular, admissible Banach extensions of A by E such that C contains an extension that cf-splits (or equivalently contains an extension that cf-splits strongly);

(c) *if $T - T' \in N^2_K(A, E)$, Σ_T is strongly equivalent to $\Sigma_{T'}$. Moreover,*

$$T + N^2_K(A, E) \mapsto [\Sigma_T]_s$$

is a bijection from $H^2_K(A, E)$ to the family of equivalence classes, with respect to strong equivalence, of singular, admissible Banach extensions of A by E that cf-split strongly.

Proof. First, we prove part (a). Let $T \in Z^2_K(A, E)$ be arbitrary and define a bounded linear map $Q : A \to \mathfrak{A}_T$ by $Q(a) = (a, 0)$, $(a \in A)$. Then, for $a, b \in A$,

$$\begin{aligned} S_Q(a, b) : &= Q(a)Q(b) - Q(ab) = (a, 0)(b, 0) - (ab, 0) \\ &= (ab, T(a, b)) - (ab, 0) = (0, T(a, b)), \end{aligned}$$

and so S_Q is compact. Clearly, $q \circ Q = \mathrm{id}_A$ so Σ_T cf-splits strongly with bounded splitting cf-homomorphism Q and so part (a) holds.

Now we prove parts (b) and (c). Let $T, T' \in Z^2_K(A, E)$. Suppose that $T - T' \in \widetilde{N}^2_K(A, E)$. Then $T - T' \in \widetilde{N}^2(A, E)$, and so Σ_T is equivalent to $\Sigma_{T'}$ by part (a) of Proposition 5.1.

Suppose further that $T - T' \in N^2_K(A, E)$. Then $T - T' \in \mathcal{N}^2(A, E)$, and so Σ_T is strongly equivalent to $\Sigma_{T'}$ by part (a) of Proposition 5.1.

Now suppose instead that $[\Sigma_T] = [\Sigma_{T'}]$. Then $T - T' \in \widetilde{N}^2(A, E)$ by part (a) of Proposition 5.1. Also, $T - T' \in Z^2_K(A, E)$ by assumption so $T - T' \in \widetilde{N}^2_K(A, E)$ and $T + \widetilde{N}^2_K(A, E) \mapsto [\Sigma_T]$ is injective.

Suppose further that $[\Sigma_T]_s = [\Sigma_{T'}]_s$. Then $T - T' \in \mathcal{N}^2(A, E)$ by part (a) of Proposition 5.1. Also, $T - T' \in Z^2_K(A, E)$ by assumption so $T - T' \in N^2_K(A, E)$ and $T + N^2_K(A, E) \mapsto [\Sigma_T]_s$ is injective.

That the two maps are into the collection of equivalence classes (with respect to the relevant relation) of extensions that cf-split follows from part (a), proven above.

It only remains to show that the maps are surjective, i.e. that, for each singular Banach extension Σ of A by E that cf-splits, there exists $T \in Z^2_K(A, E)$ with Σ_T equivalent to Σ and that if Σ cf-splits strongly we may take Σ_T to be strongly equivalent to Σ. Let

$$\Sigma = \Sigma(\mathfrak{A}; E) : 0 \to E \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} A \to 0$$

be a singular Banach extension of A by E with splitting cf-homomorphism Q. By the definition of a cf-homomorphism, we have that $S_Q \in \mathcal{K}^2(A, \mathfrak{A})$. Now, since q is a homomorphism,

$$\begin{aligned} q \circ S_Q(a,b) &= q(Q(a)Q(b) - Q(ab)) \\ &= q \circ Q(a) q \circ Q(b) - q \circ Q(ab) \\ &= ab - ab = 0, \quad (a, b \in A). \end{aligned}$$

Hence, $S_Q(A^{(2)}) \subseteq \ker(q) = E$ and so we can define $T \in \mathcal{K}^2(A, E)$ by $T(a,b) = S_Q(a,b)$, $(a, b \in A)$. Furthermore, a direct calculation yields,

$$\delta^2(T)(a,b,c) = 0$$

Hence, $T \in Z^2_K(A, E)$. We claim that the Banach extension Σ_T is equivalent to Σ. For $a \in \mathfrak{A}$, $a - Q(q(a)) \in E$ so we may define a map

$$\psi : \mathfrak{A} \to \mathfrak{A}_T, \; a \mapsto (q(a), a - Q(q(a))).$$

It is clear that ψ is linear. Furthermore, if Q is bounded (which we may assume if Σ cf-splits strongly) then ψ is also bounded. Also, if we define a map $\phi : \mathfrak{A}_T \to \mathfrak{A}$ by $\phi((b,e)) = Q(b) + e$ it is easily checked that ϕ and ψ are mutually inverse. Further, $q_T \circ \psi(a) = q(a)$, $(a \in \mathfrak{A})$. It remains only to show that ψ is an algebra homomorphism. Let $a, b \in \mathfrak{A}$; then a direct calculation yields

$$\psi(a)\psi(b) = \big(q(ab), ab - Q(q(ab)) + (a - Q(q(a)))(b - Q(q(b)))\big),$$

but $a - Q(q(a)), b - Q(q(b)) \in E$ so $(a - Q(q(a)))(b - Q(q(b))) = 0$ and so

$$\psi(a)\psi(b) = (q(ab), ab - Q(q(ab))) = \psi(ab).$$

Thus the result holds. □

Note that part (b) of the above theorem implies that, if a singular, admissible Banach extension of A by E cf-splits, then it is equivalent to a singular, admissible Banach extension of A by E that cf-splits strongly.

This gives us the following corollaries.

Corollary 5.6. *Let A be a Banach algebra, and let E be a Banach A-bimodule.*

(1) *The following are equivalent:*
   (a) $\widetilde{H}^2_K(A, E) = \{0\}$;
   (b) *each singular Banach extension of A by E, which cf-splits, does split.*

(2) *The following are equivalent:*
   (a) $H^2_K(A, E) = \{0\}$;
   (b) *each singular Banach extension of A by E, which cf-splits strongly, does split strongly.*

Proof. (1) To show that (a) implies (b), let $\widetilde{H}^2_K(A, E) = \{0\}$ and let

$$\Sigma : 0 \to I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} A \to 0$$

be a Banach extension of A by E which cf-splits. By Theorem 5.5, Σ is equivalent to the Banach extension Σ_0 (that is the Banach extension Σ_T where T is the zero map):

$$\Sigma_0 : 0 \to I \xrightarrow{\iota_0} \mathfrak{A} \xrightarrow{q_0} A \to 0$$

where, for $x \in E$ and $a \in A$, $\iota_0(x) = (0, x)$ and $q_0((a, x)) = a$. The extension Σ_0 splits strongly with the continuous splitting homomorphism $\theta : A \to \mathfrak{A}_0$, $\theta(a) = (a, 0)$, $(a \in A)$. By Lemma 5.4, Σ splits.

To show that (b) implies (a), we assume that each singular Banach extension of A by E, which cf-splits, splits and let $T \in Z^2_K(A, E)$. Then, by Theorem 5.5, Σ_T splits; let θ be a splitting homomorphism for Σ_T. Since, $q_T \circ \theta = \mathrm{id}_A$ it follows that there exists a linear map $S : A \to E$ with $\theta(a) = (a, S(a))$ $(a \in A)$. Hence,

$$\begin{aligned} (ab, S(ab)) &= \theta(ab) = \theta(a)\theta(b) \\ &= (a, S(a))(b, S(b)) = (ab, a \cdot S(b) + S(a) \cdot b + T(a, b)), \end{aligned}$$

so

$$S(ab) = a \cdot S(b) + S(a) \cdot b + T(a, b),$$

and

$$T(a, b) = a \cdot (-S(b)) + (-S(a)) \cdot b - (-S(ab)) = \delta^1(-S)(a, b).$$

Thus $T = \delta^1(-S) \in N^2(A, E)$ and so $\widetilde{H}^2_K(A, E) = \{0\}$.

(2) To show (a) implies (b) let $H^2_K(A, E) = \{0\}$ and Σ be a singular Banach extension of A by E, which cf-splits strongly. By Theorem 5.5, Σ is strongly equivalent to Σ_0 and so, by Lemma 5.4, Σ splits strongly.

To show that (b) implies (a), assume that each singular Banach extension of A by E, which cf-splits strongly, splits strongly and let $T \in Z^2_K(A, E)$. Then Σ_T splits strongly; let θ be a continuous splitting homomorphism for Σ_T. Since, $q_T \circ \theta = \mathrm{id}_A$ it follows that there exists $S \in \mathcal{B}(A, E)$ with $\theta(a) = (a, S(a))$ $(a \in A)$. As in the proof of the first part of this result, $T = \delta^1(-S)$ and so $T \in \mathcal{N}^2(A, E)$. Hence $H^2_K(A, E) = \{0\}$. □

The following result gives us a new way of showing that bounded cf-homomorphisms need not be homomorphisms or compact linear maps (which we showed directly in Example 3.5).

Corollary 5.7. *Let A be an infinite-dimensional Banach algebra and E a Banach A-bimodule such that $H^2_K(A, E) \neq \{0\}$. Then there exists a Banach algebra $\mathfrak{A}$ with underlying Banach space isomorphic to $A \oplus_1 E$ and a bounded cf-homomorphism $Q : A \to \mathfrak{A}$ which is neither a homomorphism nor a compact linear map.*

Proof. By Corollary 5.6 there exists a Banach extension,

$$\Sigma : 0 \to E \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} A \to 0,$$

of A by E which does not split strongly but such that there is a bounded cf-homomorphism, $Q : A \to \mathfrak{A}$ with $q \circ Q = \mathrm{id}_A$. Since Σ does not split strongly, Q is not a homomorphism, and since $q \circ Q = \mathrm{id}_A$ is not a compact linear map, neither is Q. By Theorem 5.5, Σ is strongly equivalent to Σ_T for some $T \in N^2_K(A, E)$ and so $\mathfrak{A}_T$ is isomorphic as a Banach space to $A \oplus_1 E$. □

Below is an example of a choice of A and $E(= A^*)$ satisfying the hypotheses of Corollary 5.7. The author would like to thank Yemon Choi for sending him some notes, which helped with the construction.

Example 5.8. Let A be ℓ^2 equipped with the pointwise product. A direct calculation gives that the map $\gamma^1 : \mathcal{B}(A, A^*) \to \mathcal{B}^2(A, A^*)$ is injective. For $N \in \mathbb{N}$ we set $G_N : A \to A^*$ to be the bounded linear map given by

$$G_N(e_k)(e_j) = g_{N,j,k} = \begin{cases} \frac{1}{3\sqrt{2N+1}} & \text{if } |j|, |k| \leq N \\ 0 & \text{otherwise} \end{cases}$$

and set $F_N := \gamma^1(G_N)$. We have that F_N is finite rank and bounded (and thus compact).

A direct calculation gives that, for each $N \in \mathbb{N}$, $\|F_N\| \leq 1$ and $\|G_N\| \to \infty$ as $N \to \infty$. It follows from an application of the Banach isomorphism theorem that $N_K^2(A; A^*)$ cannot be complete. In particular $N_K^2(A; A^*) \neq Z_K^2(A; A^*)$ i.e. $H_K^2(A, A^*) \neq 0$.

6. Open questions

We finish by listing some questions relating to the material in this paper.

Question 6.1. *For well known examples of Banach algebras, A, what are the automorphisms of A in the category of Banach algebras with cf-homomorphisms; what are the automorphisms of A in the category of Banach algebras with wcf-homomorphisms?*

Question 6.2. *Does there exist a wcf-homomorphism which is neither a weakly compact linear map nor a cf-homomorphism?*

Question 6.3. *What can we say about (weakly) compact failure of other algebraic identities: for example, commutativity?*

Question 6.4. *What do the groups $H_K^n(A, E)$ (and the others introduced in Definition 5.2) tell us about A and E when $n > 2$?*

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