



ON SOME VON NEUMANN TOPOLOGICAL ALGEBRAS

RACHID CHOUKRI¹, ABDELLAH EL KINANI² AND MOHAMED OUDADESS^{3*}

Communicated by M. Abel

ABSTRACT. We show that a regular von Neumann Q - m -convex Fréchet algebra is of finite dimension. We also show that a regular von Neumann m -convex Fréchet algebra is a projective limit of finite dimensional algebras. Finally, we prove that a bilateral Q - F -algebra is a regular von Neumann algebra if and only if it is isomorphic to a finite product of algebras which are also fields.

1. INTRODUCTION

C. Le Page [10] considered conditions implying the commutativity of a unital complex Banach algebra A , among those is

$$Ax = Ax^2 \quad (x \in A). \quad (C_1)$$

Duncan and Tullo showed that (C_1) implies finite dimensionality [7]. According to an Aupetit's comment [3, p. 56–57], this result has been known for a long time. In fact, one observes that (C_1) infers the following condition

$$\forall x, \exists y : x = xyx. \quad (C_2)$$

He referred to a book of I. Kaplansky [8, p. 111], where the latter states, without any proof, that a Banach algebra satisfying (C_2) is of finite dimension. Thus Aupetit proposed a proof (see [3, p. 57]) using a seminal idea of Duncan and Tullo [7]. But there is an error therein (see Remark 4.4).

Date: Received: 29 October 2008; Revised: 19 Jan 2009; Accepted: 29 May 2009.

* Corresponding author.

2000 *Mathematics Subject Classification.* Primary 39B82; Secondary 44B20, 46C05.

Key words and phrases. Regular von Neumann Q - m -convex Fréchet algebra, bilateral Q - F -algebra, topological algebra, .

Before coming back to the condition (C_2) , which is the subject of the present paper, let us mention that a detailed study of the condition (C_1) has been made in the general frame of topological algebras [6].

Here, we extend the result claimed by Kaplansky to the frame of B_0 -algebras with the Q -property (Theorem 4.3). It is also given a structure result without the last property in the case of locally m -convex Fréchet algebras. The particular case of locally C^* -algebras is worthwhile to mention (Theorem 4.8).

Also, it is shown that a bilateral Q - F -algebra (not necessarily locally convex) is a regular von Neumann algebra if and only if it is isomorphic algebraically and topologically to a finite product of F -algebras which are also fields (Theorem 5.2).

2. PRELIMINARIES

A unital algebra A is said to be regular in the sense of von Neumann (for short, $v.N$ - r -algebra) if it satisfies the condition (C_2) .

A nonzero idempotent $p \in A$ is called minimal if the algebra pAp is a field. Two idempotents p and q are said to be orthogonal if $pq = qp = 0$.

A unital algebra over K ($K = \mathbb{R}, \mathbb{C}$), with unit e , is said to be bilateral if all its ideals are bilateral. It is said to be noetherian if the family of its bilateral ideals satisfies the ascending condition chain, i.e., every non trivial family of bilateral ideals of A , ordered by inclusion, admits a maximal element. A proper bilateral ideal P of A is prime if for $a, b \in A$ with $aAb \subset P$, one has $a \in P$ or $b \in P$. Equivalently, for any pair (I, J) of bilateral ideals such that $IJ \subset P$, one has $I \subset P$ or $J \subset P$. A left ideal I of A is said to be of finite type if it is of the form $I = Ax_1 + \dots + Ax_r$, for some x_1, \dots, x_r in A . If I_1, \dots, I_n are left ideals, we denote by $I_1 \dots I_n$ the left ideal generated by all elements x_1, \dots, x_n with $x_i \in I_i$. If $I_i = I$, for every I , then $I_1 \dots I_n$ is denoted I^n . The spectrum of an element $x \in A$ is $Sp_A x = \{\lambda \in K : x - \lambda e \text{ is non invertible}\}$. The spectral radius of x is $\rho_A(x) = \sup\{|\lambda| : \lambda \in Sp_A x\}$. An element x is said to be quasi-nilpotent if $\rho_A(x) = 0$.

A topological algebra (A, τ) is a locally m -convex algebra ($l.m.c.a.$; cf. [11, 12]) if the topology τ is given by a family $(p_\lambda)_\lambda$ of submultiplicative seminorms i.e.,

$$p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y).$$

Such an algebra is said a locally m -convex Fréchet algebra (Fréchet $l.m.c.a.$) if it is moreover metrizable and complete. In that case, its topology is given by a denumerable family of submultiplicative seminorms. The algebra A is an F -algebra if it is endowed with an algebra topology which is metrizable and complete. If, moreover, it is locally convex, it is called a B_0 -algebra. A unital topological algebra is said to be a Q -algebra if the group of its invertible elements is open. An involutive $l.m.c.a.$ is called a locally C^* -algebra if the p_λ 's satisfy the C^* -equality i.e., $p_\lambda(x^*x) = [p_\lambda(x)]^2$, for every x and every λ .

3. GENERAL PROPERTIES

We begin by putting together some properties which are common to all topological algebras which are also regular in the sense of von Neumann. We will write a topological $v.N$ - r -algebra.

Proposition 3.1. *Let A be a topological $v.N$ - r -algebra. Then*

- (i) A is semi-simple,
- (ii) For every $x \in A$, the ideals Ax and xA are closed.
- (iii) Every left or right ideal of A which is of finite type is closed.
- (iv) If A is a normed algebra, then it is a Q -algebra.
- (v) If A is a Q -algebra, then every bilateral ideal, not necessarily maximal, of A is closed.

Proof. (i) Let $x \in \text{Rad}A$, where $\text{Rad}A$ is the Jacobson radical of A . Consider $y \in A$ such that $x = xyx$. Then one has $x(e - yx) = 0$. Whence $x = 0$, since $e - yx$ is invertible.

(ii) Let (a_n) be a sequence in A such that $(a_n x) \rightarrow z$ with z in the closure of Ax . Consider $y \in A$ such that $x = xyx$. Then one has $(a_n xyx) \rightarrow z$. Hence $z = zy x$ and so $z \in Ax$. The same for xA .

(iii) Let $I = Ax_1 + \dots + Ax_r$ be an ideal of finite type. For every i , there is $y_i \in A$ such that $x_i = x_i y_i x$. It ensues that $Ax_i = Ay_i x_i$. Moreover $y_i x_i$ is idempotent. Thus replacing if necessary x_i by $y_i x_i$ we may suppose that the x_i 's are idempotent. Examine first the case $r = 2$. Consider (a_n) and (b_n) two sequences in A such that

$$(a_n x_1 + b_n x_2) \rightarrow z, \text{ with } z \text{ in the closure of } I.$$

Multiplying on the right by $(x_1 - e)$, one obtains

$$(b_n x_2 (x_1 - e)) \rightarrow z(x_1 - e).$$

By (ii), $Ax_2(x_1 - e)$ is closed, hence

$$z(x_1 - e) = ax_2(x_1 - e);$$

whence

$$z = zx_1 - ax_2 x_1 + ax_2 \in Ax_1 + Ax_2.$$

Thus $Ax_1 + Ax_2$ is closed. The proof is finished, using the induction argument on r .

(iv) A is inverse closed in its completion \hat{A} , i.e. every element of A which is invertible in the completion \hat{A} is actually invertible in A . Hence it is a Q -algebra. Indeed, let $x \in A$ be invertible in \hat{A} . There is $y \in A$ such that $x = xyx$. It then follows that $xy = yx = e$.

(v) Let I be a bilateral ideal of A . It is clear that A/I is also a $v.N$ - r -algebra. So A/I is semisimple by (i). Now the maximal left ideals of A/I are of the form M/I , where M runs over the family of maximal left ideals of A , that contain I . Then the semisimplicity of A is equivalent to the fact that I is equal to the intersection of maximal left ideals of A , that contain it. The latter are closed, since A is a Q -algebra. The result then follows.

□

4. B_0 -ALGEBRAS

We begin with some entirely algebraic considerations. It is easy to prove the following result.

Lemma 4.1. *Let A be a $v.N$ - r -algebra and $p \in A$ a nonzero idempotent. The following assertions are equivalent.*

- (i) p is minimal.
- (ii) The algebra pAp admits only p and 0 as idempotents

The following lemma is essential in the sequel. It is the principal ingredient (though not entirely detailed) in the proof of Duncan and Tullo [7].

Lemma 4.2. *Let A be a $v.N$ - r -algebra and $p \in A$ a nonzero idempotent which is not minimal. If it is not the sum of pairwise orthogonal minimal idempotents, then A admits an infinity of nonzero pairwise orthogonal minimal idempotents.*

Proof. The idempotent p being not minimal, there is (Lemma 4.1) an idempotent $q \in pAp$ with $q \neq 0$ and $q \neq p$. So q and $p - q$ are two orthogonal idempotents the sum of which is p . One of them, say q , is not minimal. By the preceding, there are two orthogonal idempotents $r, s \in qAq$ the sum of which is q . Thus r, s and $p - q$ are three nonzero idempotents which are pairwise orthogonal. Continuing the process, one obtains an infinite sequence of nonzero idempotents which are pairwise orthogonal. □

Now here is the first structure result. Recall that the notation $A \ker p_{k_0+1}$ is for all finite sums $\sum a_i x_i$, where $a_i \in A$ and $x_i \in \ker p_{k_0+1}$. It is a left ideal.

Theorem 4.3. *Let A be a unital $v.N$ - r -algebra which is also a Q - B_0 -algebra. Then A is of finite dimension.*

Proof. By (i) of Proposition 3.1, the algebra A is semi-simple. Let $(p_k)_{k \geq 0}$ be an increasing sequence of seminorms defining the topology of A and satisfying

$$p_k(xy) \leq p_{k+1}(x)p_{k+1}(y) \quad (x, y \in A).$$

As A is a Q -algebra, there is k_0 and a constant $K > 0$ such that

$$\rho_A(x) \leq K p_{k_0}(x), \quad x \in A \quad (\text{Tsertos inequality; [13]}).$$

Hence $A \ker p_{k_0+1}$ is a left ideal elements of which are quasi-nilpotent (hence quasi-invertible) by the previous inequality. So $A \ker p_{k_0+1}$ is contained in $\text{Rad}A = \{0\}$ [4, Proposition 16 (iii), p. 125]. Thus p_{k_0+1} is a vector space norm on A . It ensues that p_k is a norm for every $k \geq k_0 + 1$. Suppose now that A admits a sequence (e_n) of nonzero idempotents which are pairwise orthogonal. The series of general term

$$\frac{1}{n^2} \frac{e_n}{p_n(e_n)}, \quad n \geq k_0 + 1,$$

is absolutely convergent. Indeed, let $r \in N$, which can be supposed larger than $k_0 + 1$ because the sequence $(p_k)_{k \geq 0}$ is increasing. One then has

$$\begin{aligned}
\sum_{k_0+1 \leq n} p_r\left(\frac{1}{n^2} \frac{e_n}{p_n(e_n)}\right) &= \sum_{k_0+1 \leq n} \frac{1}{n^2} \frac{1}{p_n(e_n)} p_r(e_n) \\
&= \sum_{k_0+1 \leq n \leq r-1} \frac{1}{n^2} \frac{1}{p_n(e_n)} p_r(e_n) + \sum_{r \leq n} \frac{1}{n^2} \frac{1}{p_n(e_n)} p_r(e_n) \\
&\leq \sum_{k_0+1 \leq n \leq r-1} \frac{1}{n^2} \frac{1}{p_n(e_n)} p_r(e_n) + \sum_{r \leq n} \frac{1}{n^2} < \infty.
\end{aligned}$$

Put then

$$x = \sum_{k_0+1 \leq n} \frac{1}{n^2} \frac{e_n}{p_n(e_n)}.$$

Consider $y \in A$ such that $x = xyx$. Multiplying the members of this equality, on the left and on the right by e_n and remarking that $e_n x = x e_n = \lambda_n e_n$ where $\lambda_n = \frac{1}{n^2} \frac{1}{p_n(e_n)}$, one obtains

$$\lambda_n^2 e_n y e_n = \lambda_n e_n$$

for all n . Whence $\lambda_n e_n y e_n = e_n$. Now it is clear that $S p_A(e_n) = \{0, 1\}$, hence $\rho_A(e_n) = 1$. On the other hand, since every e_n is idempotent and $\rho_A(ab) = \rho_A(ba)$ one has

$$\begin{aligned}
1 &= \rho_A(e_n) \leq \rho_A(\lambda_n e_n y e_n) = \rho_A(\lambda_n e_n^2 y) = \rho_A(\lambda_n e_n y) \\
&\leq K p_{k_0}(\lambda_n e_n y) \\
&\leq K p_{k_0+1}(\lambda_n e_n) p_{k_0+1}(y).
\end{aligned}$$

But $\lambda_n e_n$ is the general term of a convergent series. Whence $1 \leq 0$, which is absurd.

Now take the collection \mathcal{F} of the families of pairwise orthogonal idempotents. Due to the preceding, all the elements of \mathcal{F} are finite. The order is defined as follows

$$\{e_1, \dots, e_r\} \leq \{e'_1, \dots, e'_s\} \iff \{e_1, \dots, e_r\} \subset \{e'_1, \dots, e'_s\}.$$

Then \mathcal{F} endowed with this order is an inductive family. Indeed, if $(I_\lambda)_\lambda$ is a totally ordered subfamily of \mathcal{F} , then it is majorized (it admits an upper bound). Put $I = \cup I_\lambda$. By the preceding, I is necessarily finite. It is clear that $I \in \mathcal{F}$ and that it is an upper bound of $(I_\lambda)_\lambda$. By Zorn's lemma, \mathcal{F} admits a maximal element $\{f_1, \dots, f_r\}$. One then necessarily has $f_1 + \dots + f_r = e$, where e is the unit element of A . Otherwise the family $\{f_1, \dots, f_r, f\}$, with $f = e - (f_1 + \dots + f_r)$, would be an element of \mathcal{F} which is larger than $\{f_1, \dots, f_r\}$; but this contradicts the maximality of the latter. Now by Lemma 4.2, every f_i can be written

$$f_i = \sum_{1 \leq j \leq r_i} f_{i,j},$$

where the family $(f_{i,j})_{1 \leq j \leq r_i}$ is made of minimal idempotents which are pairwise orthogonal; if f_i is not minimal, take $r_i = 1$ and $f_{i,1} = f_i$. Moreover one has

$$f_{i,j} = f_i f_{i,j} = f_{i,j} f_i; \quad j = 1, \dots, r_i.$$

Hence for $i \neq i', 1 \leq k \leq r_i, 1 \leq k' \leq r_{i'}$, we have

$$f_{i,k}f_{i',k'} = f_{i',k'}f_{i,k} = 0.$$

Thus the family $(f_{i,j})_{1 \leq i \leq r, 1 \leq j \leq r_i}$ is made of pairwise orthogonal minimal idempotents. Finally it is clear that

$$\sum_{1 \leq i \leq r, 1 \leq j \leq r_i} f_{i,j} = e.$$

This family of now minimal idempotent elements will be denoted (e_1, \dots, e_n) in the sequel. To finish, let $i, j \in \{1, \dots, n\}$. Since e_i is minimal, $e_i A e_i$ is finite dimensional by Gelfand -Mazur theorem. It is also actually the case for $e_i A e_j$. Indeed let $a \in A$ be such that $e_i a e_j \neq 0$. Since every e_i is minimal, the ideal $A e_j$ is a left minimal ideal ([4, Proposition 6, p.155]). But $\{0\} \neq A e_i a e_j \subset A e_j$. Whence

$$A e_j = A e_i a e_j.$$

Hence

$$e_i A e_j = e_i A e_i a e_j$$

which is finite dimensional since $e_i A e_i$ is. Finally, one has

$$A = \sum_{i,j} e_i A e_j,$$

due to $e = e_1 + \dots + e_n$. Whence the result. \square

Remark 4.4. The proof of Aupetit [3, p. 57], in the Banach case, contains an error. Indeed, the family $\{p_1, \dots, p_{i-1}\} \cup \{p, p_i - p\} \cup \{p_{i+1}, \dots, p_k\}$ is not larger than $\{p_1, \dots, p_k\}$ with respect to the order considered there.

As a consequence we do have the result (claimed by Kaplansky) announced in the introduction.

Corollary 4.5. *A Banach algebra is a v.N-r-algebra if and only if it is semisimple and finite dimensional.*

For the second structure result, we will need the following fact which has an interest in its own.

Theorem 4.6. *Let A be a Fréchet l.m.c.a. (not necessarily a Q -algebra). If A a v.N-r-algebra, then it is a projective limit of finite dimensional algebras.*

Proof. Let $(p_k)_{k \geq 0}$ be a directed sequence of submultiplicative seminorms defining the topology of A . The standard normed algebra $(A/\ker p_k, \tilde{p}_k)$ is also a v.N-r-algebra. It is a Q -algebra, by (iv) of Proposition 3.1. It then follows that the algebra $A/\ker p_k$, endowed with the quotient Fréchet topology, is also a Q -algebra. So, by Theorem 4.3, it is of finite dimension. It remains only to use the Arens-Michael decomposition [12, Theorem 5.1, p. 20],

$$A = \varprojlim A/\ker p_k.$$

\square

Remark 4.7. Theorem 4.6 is not valid without metrizable. Indeed consider the algebra of stationary complex sequences that is which are constant from an integer on. For any $k \in \mathbb{N}$, put

$$A_k = \{(x_n)_n \in A : x_n = x_k, \forall n \geq k\}.$$

The A_k 's constitute an increasing sequence of finite dimensional subalgebras of A , the union of which is A . Actually A is, in a standard way, the algebraic inductive limit of $(A_k)_k$. Endow A with the associated inductive limit topology τ . It is a *l.m.c.a.* [2]. It is in fact the finest locally convex topology on A , since the A_k 's are of finite dimension. Moreover, as every A_k is barrelled it is true for A , [5, Corollaire 3, p. III.23]. The spectrum of every element of A is finite (hence bounded). Thus A is a Q -algebra [14, Corollary 3, p. 296]. Now suppose that A is a projective limit of finite dimensional (commutative) algebras. The latter are semisimple. Thus all the factors of the projective limit are of the form \mathbb{C}^l with $l \in \mathbb{N}$. But A is a Q -algebra and so τ must be the coarsest locally convex topology on A . It ensues that there is only one locally convex topology on A , which is not the case.

It is known, by a result of Apostol [1], that if $(A, (p_\lambda)_\lambda)$ is a locally C^* -algebra then the factors $A/\ker p_k$ are Banach algebras; even C^* -algebras. So, using Theorem 4.3, we obtain the following.

Theorem 4.8. *If a unital locally C^* -algebra is also a $v.N$ - r -algebra, then it is a projective limit of finite dimensional algebras.*

5. BILATERAL \mathbf{F} -ALGEBRAS

Assertion (v) of Proposition 3.1 suggests to look at the case where all the ideals of an algebra A are bilateral. When A is unital, it is equivalent to say that it is itself bilateral i.e.,

$$\forall x, y \in A, \exists u, v \in A : xy = ux = yv.$$

We obtain the following structure theorem in the frame of F -algebras as indicated in the heading of this section. But first an algebraic result which is well known in the commutative case [9, Lemme 2, p. 69]. We have never met it in the non commutative one, so we are providing a proof.

Lemma 5.1. *Let A be a noetherian algebra. Then every bilateral ideal of A contains a finite product of prime ideals.*

Proof. Denote by \mathcal{F} the family of bilateral ideals, of A , which do not satisfy the desired conclusion. Suppose that \mathcal{F} is not void. Consider I a maximal element of \mathcal{F} , the existence of which is assured by the noetherianity of A . In particular, I is not a prime ideal. So there are two bilateral ideals J and K of A , not contained in I , such that I contains JK . As $I + J$ and $I + K$ contain strictly I , each one of them contains a finite product of prime ideals. The same for I , since $(I + J)(I + K) \subset I$. But this should not be the case. \square

Theorem 5.2. *Let A be a bilateral Q - F -algebra. Then A is a $v.N$ - r -algebra if and only if it is algebraically and topologically isomorphic to a finite product of F -algebras which are also fields.*

Proof. Sufficiency is clear. For necessity, we first show that A is noetherian. Let $(I_n)_n$ be an increasing sequence of ideals of A and put $I = \cup I_n$ which is also an ideal of A . By (v) of Proposition 3.1, I is closed. But the I_n 's are closed in A , hence also in I . Then by the Baire theorem, there is an n_0 such that I_{n_0} is of non void interior. Whence $I = I_{n_0}$ for I_{n_0} is a subspace of I . It follows that $I_n = I_{n_0}$, for $n \geq n_0$. Now A being noetherian, there are by Lemme 5.1 prime ideals P_1, \dots, P_r which are pairwise distinct and integers $\alpha_1, \dots, \alpha_r$ such that

$$P_1^{\alpha_1} \dots P_r^{\alpha_r} = \{0\}.$$

But $P_i^{\alpha_i} = P_i$. Indeed if $x \in P_i$, there is $y \in A$ such that $x = xyx \in P_i^2$. Hence $P_i^2 = P_i$. Whence the claim. Thus, we actually have

$$P_1 \cap \dots \cap P_r = \{0\}.$$

On the other hand A/P_i is without zero divisors. Indeed, let $x, y \in A$ such that $xy \in P$. As A is bilateral, one has $xAy = xyA$. Hence $xAy \subset P$. So $x \in P$ or $y \in P$. Thus A/P_i being also a $v.N$ - r -algebra, it is a division algebra. So the P_i 's are maximal and $P_1 \cap \dots \cap P_r = \{0\}$. Now, for every i , denote by s_i the canonical surjection of A onto A/P_i . The map

$$\varphi : A \longrightarrow A/P_1 \times \dots \times A/P_r$$

defined by

$$\varphi(x) = (s_1(x), \dots, s_r(x))$$

is an algebraic and a topological isomorphism. \square

As a consequence, we have the following result, where \mathbb{H} is the field of quaternions.

Corollary 5.3. *Let A be a bilateral Q - B_0 -algebra. Then A is a $v.N$ - r -algebra if and only if it is isomorphic to $\mathbb{R}^r \times \mathbb{C}^s \times \mathbb{H}^t$, with r, s, t are positive integers.*

As an outcome we have the following characterizations of the standard algebras $C^{\mathbb{N}}$ and C^n .

Proposition 5.4. *$C^{\mathbb{N}}$ is, up to an algebraic and topological isomorphism, the unique complex bilateral Fréchet l.m.c.a. A which is a $v.N$ - r -algebra of infinite dimension.*

Proof. By Theorem 4.6, A is a projective limit of finite dimensional algebras A_n . The latter are bilateral and $v.N$ - r -algebras. By Corollary 5.3, A_n is isomorphic to some \mathbb{C}^{s_n} . Whence the result \square

Proposition 5.5. *C^n is, up to an algebraic and topological isomorphism, the unique complex bilateral Fréchet l.m.c.a. A which is a $v.N$ - r -algebra of finite dimension.*

Acknowledgement. Thanks are offered to the referee for a three times very careful checking of the manuscript. He has detected many misprints and asked for numerous clarifications.

REFERENCES

- [1] C. Apostol, *b^{*}-algebras and their representation*, J. London Math. Soc. (2) **3** (1971), 30–38.
- [2] A. Arosio, *Locally convex inductive limits of normed algebras*, Rend. Sem. Mat. Univ. Padova **51** (1974), 339–356.
- [3] B. Aupetit, *Propriétés Spectrales des Algèbres de Banach*, Lecture Notes in Mathematics 735, Springer-Verlag, Berlin Heidelberg, New York, 1979.
- [4] F.F. Bonsall, J. Duncan, *Complete Normed Algebra*, Springer-Verlag, New York-Heidelberg, 1973.
- [5] N. Bourbaki, *Espaces vectoriels topologiques*, Chapitres 1–5, Masson, Paris, 1970.
- [6] R. Choukri, A. El Kinani, M. Oudadess, *Etude des algèbres A vérifiant $xA = Ax$ ou $xAx = x^2Ax^2$* . General topological algebras (Tartu, 1999), 59–71, Math. Stud. (Tartu), 1, Est. Math. Soc., Tartu, 2001.
- [7] J. Duncan and A.W. Tullo, *Finite dimensionality, nilpotents and quasi-nilpotents in Banach algebras*, Proc. Edinburgh Math. Soc. Ser. **21** (1974), 45–49.
- [8] I. Kaplansky, *Fields and rings*, Univ. Chicago Press, Chicago, 1959.
- [9] J.P. Lafon, *Algèbre commutative*, Hermann, Paris, 1979.
- [10] C. Le Page, *Sur quelques conditions entraînant la commutativité dans les algèbres de Banach*, C. R. Acad. Sci. Paris Ser A-B **265** (1967), 235–237.
- [11] A. Mallios, *Topological algebras. Selected topics*, North-Holland, Amsterdam, 1986.
- [12] E.A. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc. **1952** (1952), no. 11, 79 pp.
- [13] Y. Tsertos, *A characterization of Q -algebras*, Funct. analysis, approximation theory and numerical analysis, 277–280, World Sci. Publ., River Edge, NJ, 1994.
- [14] W. Zelazko, *On maximal ideals in commutative m -convex algebras*, Studia Math. **58** (1976), 290–298.

^{1,2,3} DÉPARTEMENT DE MATHÉMATIQUES, B.P. 5118, RABAT, 10105, MAROC.

E-mail address: rachoukri@yahoo.fr, abdellah_elkinani@yahoo.fr, oudadessm@yahoo.fr