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ON THE SOLUBILITY OF TRANSCENDENTAL EQUATIONS IN COMMUTATIVE C^*-ALGEBRAS

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ABSTRACT. It is known that $C(X)$ is algebraically closed if X is a locally connected, hereditarily unicoherent compact Hausdorff space. For such spaces, we prove that if $F : C(X) \to C(X)$ is an entire function in the sense of Lorch, i.e., is given by an everywhere convergent power series with coefficients in $C(X)$, and satisfies certain restrictions, then it has a root in $C(X)$. Our results generalizes the monic algebraic case.

1. INTRODUCTION

Let X be a compact Hausdorff space and let $C(X)$ be the Banach algebra of complex-valued continuous functions on X. We say that $F : C(X) \to C(X)$ is *entire* (in the sense of Lorch) if it is Fréchet differentiable at every point $w \in C(X)$ and its differential is given by a multiplication operator $L_w(h) = F'(w)h$, for some $F'(w) \in C(X)$ (see [6] for details). We denote the set of entire functions by $\mathcal{H}\big(C(X)\big)$ and make it into a unital algebra with the usual operations. It is well known that $F \in \mathcal{H}\big(C(X)\big)$ if and only if it admits a power series expansion

$$F(w) = \sum_{n=0}^{\infty} a_n w^n, \qquad w \in C(X),$$

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where $a_n \in C(X)$ for all $n \geq 0$, $\limsup_n \|a_n\|^{1/n} = 0$ and the series converges in norm for each fixed $w \in C(X)$.

To any entire function F, we may associate the map $X \times \mathbb{C} \to \mathbb{C}$ defined by

$$(x, z) \mapsto \sum_{n=0}^{\infty} a_n(x) z^n \ \Big(= F\left(z 1_{C(X)}\right)(x)\Big), \tag{1.1}$$

which is easily seen to be continuous on $X \times \mathbb{C}$ and holomorphic with respect to z for $x \in X$ fixed. On the other hand, it is obvious that the above map uniquely determines F. By a customary abuse of notation, we also write F for the map in (1.1); it should be clear from the context which case we are referring to.

We say that $F \in \mathcal{H}\big(C(X)\big)$ has a *root* in $C(X)$, if there exists $w \in C(X)$ such that $F(x, w(x)) = 0$ for all $x \in X$. If X is a locally connected compact Hausdorff space, it was observed by Miura and Niijima [7] that $C(X)$ is algebraically closed, i.e., every monic polynomial with coefficients in the algebra has at least one root in the algebra, if and only if X is hereditarily unicoherent (see also Honma and Miura [4]). We recall that X is said to be hereditarily unicoherent, if the intersection $A \cap B$ is connected for all closed connected subsets A, B of X. A short, but accurate introduction to the state of the art in monic algebraic equations can be found in Kawamura and Miura [5].

However, if we consider more general functions in $\mathcal{H}\big(C(X)\big)$, the existence of continuous roots is no longer guaranteed, even if X is as simple as the unit interval. For example, the function $F(x, z) = x^2 z - x$ does not have a root in $C([0, 1])$. We now introduce two phenomena that arise in the preceding example and have a strong relation with the existence of solutions of the equation $F(w) = 0$.

Definition 1.1. Let X be a compact Hausdorff space. A function $F \in \mathcal{H}\big(C(X)\big)$ is said to be *degenerate* at $x_0 \in X$ if the map $z \mapsto F(x_0, z)$ is constant; otherwise, it is said to be *nondegenerate* at x_0.

Definition 1.2. Let X be a compact Hausdorff space, let $Y \subset X$ be a connected subset and $x_0 \in \overline{Y} \setminus Y$. A function $w \in C(Y)$ is said to be an *asymptotic root* of $F \in \mathcal{H}\big(C(X)\big)$ if $F(x, w(x)) = 0$ for all $x \in Y$ and

$$\lim_{x \to x_0} w(x) = \infty, \quad x \in Y.$$

The aim of this paper is to prove that if X is a connected, locally connected, hereditarily unicoherent compact Hausdorff space, then any nowhere degenerate function $F \in \mathcal{H}\big(C(X)\big)$ with no asymptotic roots, satisfying $F(x_0, z_0) = 0$, has at least one root $w \in C(X)$ such that $w(x_0) = z_0$. It is easily seen that monic polynomials are nondegenerate at every point of X and do not have asymptotic roots. Consequently, our result generalizes that of Miura and Niijima [7].

It is important to mention that Gorin and Sánchez Fernández [2] studied the case where X is a connected, locally connected, hereditarily unicoherent, compact metric space and showed that any nowhere degenerate function $F \in \mathcal{H}\big(C(X)\big)$ with no asymptotic arcs, satisfying the condition $F(x_0, z_0) = 0$, has at least one root $w \in C(X)$ such that $w(x_0) = z_0$ (for a definition of asymptotic arc, see [2]). In our work, we do not assume that X is a first-countable space.

2. Existence of Roots

We start by pointing out a very useful lemma, which arises naturally from Rouché's Theorem.

Lemma 2.1. *Let X be a compact Hausdorff space, $F \in \mathcal{H}\big(C(X)\big)$ and pick $x_0 \in X$ such that the map $z \mapsto F(x_0, z)$ has a zero z_0 of multiplicity n. Then, there exist an open disk $D_r(z_0)$ and a neighborhood V of x_0 such that*

$$F(x,z) = P(x,z)\, G(x,z), \qquad (x,z) \in V \times D_r(z_0),$$

where $P(x,z) = z^n + a_1(x) z^{n-1} + \ldots + a_n(x)$ is a monic polynomial with coefficients in $C(V)$ satisfying $P(x_0, z) = (z - z_0)^n$ and G never vanishes in $V \times D_r(z_0)$.

Proof. Set $r > 0$ such that the map $z \mapsto F(x_0, z)$ has no roots in $\overline{D_r(z_0)} \setminus \{z_0\}$ and write $\Gamma = \{z \in \mathbb{C} \;:\; |z - z_0| = r\}$. Also, write $m = \min_\Gamma |F(x_0, z)| > 0$. By a standard compactness argument, we can find a neighborhood V of x_0 such that $|F(x,z) - F(x_0,z)| < m$ for all $x \in V$ and $z \in \Gamma$. Then, an application of Rouché's Theorem shows that $z \mapsto F(x,z)$ has exactly n zeros in $D_r(z_0)$, counting multiplicities, whenever $x \in V$.

For any $x \in V$, we denote the zeros of $z \mapsto F(x,z)$ in $D_r(z_0)$ by $z_1(x), \ldots, z_n(x)$, taken in any order and we define

$$P(x,z) = \big(z - z_1(x)\big) \ldots \big(z - z_n(x)\big) = z^n + a_1(x) z^{n-1} + \ldots + a_n(x).$$

Obviously, we have $P(x_0, z) = (z - z_0)^n$. Now, consider the central symmetric functions

$$s_k(x) = \sum_{i=1}^{n} \big(z_i(x)\big)^k, \qquad k \geq 0.$$

Since $z_1(x), \ldots, z_n(x)$ are the zeros of $z \mapsto F(x,z)$ in the interior of Γ, it is well known (and easily verified) that

$$s_k(x) = \frac{1}{2\pi i} \int_\Gamma z^k \, \frac{\frac{\partial F}{\partial z}(x,z)}{F(x,z)}\, dz.$$

Consequently, $s_k \in C(V)$ for all $k \geq 0$. It is also well known that the functions s_k are connected to the functions a_k via the so-called Newton identities. Therefore, the continuity of a_k for $1 \leq k \leq n$ can be established by an easy induction.

Finally, for $(x,z) \in V \times D_r(z_0)$, define $G(x,z)$ as the quotient $F(x,z)/P(x,z)$ if $P(x,z) \neq 0$ and set $G(x,z) = 1$ otherwise. □

Before going any further, we need some topological remarks. A good exposition of such facts can be found in [7], a great deal of which we reproduce for completeness. Let X be a connected topological space. A point $p \in X$ separates the distinct points $a, b \in X \setminus \{p\}$ if there exist disjoint open sets A and B such that $a \in A$, $b \in B$ and $X \setminus \{p\} = A \cup B$. If the point p belongs to every connected closed subset of X containing a and b, we say that p cuts X between a and b. If X is a locally connected and connected compact Hausdorff space, then p cuts X between a and b if and only if p separates the points a and b (cf. [3, Theorem 3-6]).

If X is a connected compact Hausdorff space, there exists a minimal connected closed subset, with respect to set inclusion, containing both a and b (cf. [3, Theorem 2-10]). If X is hereditarily unicoherent, such a minimal set is unique and we denote it by $E[a,b]$. Clearly, every point in $E[a,b]\setminus\{a,b\}$ cuts X between a and b. Therefore, if we assume that X is also locally connected, such points also separate a and b. We define the separation order $\preceq$ in $E[a,b]$ the following way: for distinct points $p,q\in E[a,b]$, we say that $p\prec q$ if $p=a$ or p separates a and q. Then, we write $p\preceq q$ if $p=q$ or $p\prec q$. Such choice makes $E[a,b]$ into a totally ordered space (cf. [3, Theorem 2-21]). If we define the order topology in $E[a,b]$ the usual way, then it coincides with the induced topology in $E[a,b]$ (cf. [3, Theorem 2-25]). Also, by [3, Theorem 2-26], every non-empty subset of $E[a,b]$ has a least upper bound, i.e., $E[a,b]$ is order-complete.

To avoid repetitions, we assume henceforth that X is a connected, locally connected, hereditarily unicoherent compact Hausdorff space, unless stated otherwise.

Lemma 2.2. *The following two properties hold:*

i-) Any connected subset of X containing a and b, must contain $E[a,b]$.

ii-) An arbitrary intersection of connected subsets of X is either empty or connected.

Proof. The first part is a direct consequence of the fact that any point in the set $E[a,b]\setminus\{a,b\}$ separates a and b. For the second part, let $\{M_\alpha\}$ be a collection of connected subsets of X and suppose that $\cap_\alpha M_\alpha$ has at least two points. Given any pair of distinct points $a,b\in\cap_\alpha M_\alpha$, we must have $E[a,b]\subset M_\alpha$ for all α, whence we obtain $E[a,b]\subset\cap_\alpha M_\alpha$. The connectedness of $\cap_\alpha M_\alpha$ is now obvious. $\square$

The above lemma will be used very often later.

Lemma 2.3. *Let $D\subset X$ be connected and $x^*\in\overline{D}\setminus D$. Suppose that the function $F\in\mathcal{H}\big(C(X)\big)$ is nondegenerate at x^* and consider $w\in C(D)$ such that $F(x,w(x))=0$ for all $x\in D$. Then, there exists the limit*

$$\lim_{x\to x^*} w(x),\quad x\in D,$$

in the Riemann sphere.

Proof. Denote the Riemann sphere by $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ and let $\{U_\alpha\}_{\alpha\in I}$ be a local basis at x^* consisting of connected open sets. It is readily seen that the family $\mathcal{F}=\big\{\overline{w(D\cap U_\alpha)}\ :\ \alpha\in I\big\}$ is a filterbase in $\widehat{\mathbb{C}}$. Since the latter is compact, $\mathcal{F}$ has at least one accumulation point, i.e.,

$$\mathcal{F}_{ac}=\bigcap_{\alpha\in I}\overline{w(D\cap U_\alpha)}\neq\emptyset.$$

Next, by Lemma 2.2, it is easy to see that $D\cap U_\alpha$ is connected for all $\alpha\in I$ and the continuity of w implies that $\overline{w(D\cap U_\alpha)}$ is also connected. Suppose that $\mathcal{F}_{ac}$ is not connected, i.e., there exist disjoint open sets $A,B\subset\widehat{\mathbb{C}}$ such that $\mathcal{F}_{ac}\subset A\cup B$, $\mathcal{F}_{ac}\cap A\neq\emptyset$ and $\mathcal{F}_{ac}\cap B\neq\emptyset$. Note that we can write

$$\bigcap_{\alpha\in I}\overline{w(D\cap U_\alpha)}\ \cap\ \big(\widehat{\mathbb{C}}\setminus(A\cup B)\big)=\mathcal{F}_{ac}\ \cap\ \big(\widehat{\mathbb{C}}\setminus(A\cup B)\big)=\emptyset$$

and accordingly, the compactness of $\widehat{\mathbb{C}}$ implies the existence of a finite set of indices $\alpha_1, \ldots, \alpha_n \in I$ such that $\overline{w(D \cap U_{\alpha_1})} \cap \ldots \cap \overline{w(D \cap U_{\alpha_n})} \cap \big(\widehat{\mathbb{C}} \setminus (A \cup B)\big) = \emptyset$. Since $\mathcal{F}$ is a filterbase, we can find $\beta \in I$ such that $\overline{w(D \cap U_\beta)} \subset \overline{w(D \cap U_{\alpha_1})} \cap \ldots \cap \overline{w(D \cap U_{\alpha_n})}$ and thus, $\overline{w(D \cap U_\beta)} \subset A \cup B$. However, as $\mathcal{F}_{ac} \subset \overline{w(D \cap U_\beta)}$, we must have $\overline{w(D \cap U_\beta)} \cap A \neq \emptyset$ and $\overline{w(D \cap U_\beta)} \cap B \neq \emptyset$. Hence, $\overline{w(D \cap U_\beta)}$ cannot be connected, which is absurd.

We assume, towards contradiction that $\mathcal{F}_{ac}$ contains at least two points. Let $\epsilon > 0$ be arbitrary and let $z^* \in \mathcal{F}_{ac}$, $z^* \neq \infty$. Pick $\delta > 0$ and a neighborhood U_γ of x^* with $\gamma \in I$ such that $|F(x,z) - F(x^*, z^*)| < \epsilon$ whenever $x \in U_\gamma$ and $|z - z^*| < \delta$. Since $z^* \in \overline{w(D \cap U_\gamma)}$, there exists $x_\gamma \in D \cap U_\gamma$ such that $|w(x_\gamma) - z^*| < \delta$, whence we obtain that $|F(x_\gamma, w(x_\gamma)) - F(x^*, z^*)| < \epsilon$. Given that $F(x_\gamma, w(x_\gamma)) = 0$, we must have $|F(x^*, z^*)| < \epsilon$. Since ϵ is arbitrary, $F(x^*, z^*) = 0$. Therefore, any finite point of $\mathcal{F}_{ac}$ is a root of $z \mapsto F(x^*, z)$. Since F is nondegenerate at x^*, $z \mapsto F(x^*, z)$ is a non-constant entire function and therefore has at most countably many roots. As a result, $\mathcal{F}_{ac}$ is at most countable. Since it is also a non-empty, connected subset of $\widehat{\mathbb{C}}$, we get our desired contradiction and conclude that $\mathcal{F}_{ac}$ reduces to a single point. Then, it is straightforward to see that such point must be the limit of $w(x)$ as $x \to x^*$. □

We now prove the main result of the paper.

Theorem 2.4. *Let $F \in \mathcal{H}\big(C(X)\big)$ be a nowhere degenerate function, having no asymptotic roots and assume that there exist $x_0 \in X$ and $z_0 \in \mathbb{C}$ such that $F(x_0, z_0) = 0$. Then there exists $w \in C(X)$ such that $w(x_0) = z_0$ and $F(x, w(x)) = 0$ for all $x \in X$.*

Proof. Let $\mathfrak{D}$ be the set of pairs (D, w), where $D \subset X$ is a connected subset containing x_0, $w \in C(D)$, $w(x_0) = z_0$ and $F(x, w(x)) = 0$ for all $x \in X$. The family $\mathfrak{D}$ is not empty, as it contains the pair (D_0, w_0), where $D_0 = \{x_0\}$ and $w_0 : D_0 \to \mathbb{C}$ is defined by $w_0(x_0) = z_0$. We define a partial order in $\mathfrak{D}$ as follows: we write $(D_1, w_1) \leq (D_2, w_2)$ if $D_1 \subset D_2$ and $w_2|_{D_1} = w_1$.

Let $\{(D_\alpha, w_\alpha)\}_{\alpha \in I}$ be a chain in $\mathfrak{D}$. Set $\widetilde{D} = \bigcup_\alpha D_\alpha$ and define $\widetilde{w} : \widetilde{D} \to \mathbb{C}$ by $\widetilde{w}(x) = w_\alpha(x)$, if $x \in D_\alpha$. It is obvious that $\widetilde{D}$ is a connected subset of X containing x_0 and $\widetilde{w}$ is a well defined function such that $\widetilde{w}(x_0) = z_0$ and $F(x, \widetilde{w}(x)) = 0$ for all $x \in \widetilde{D}$.

We subsequently prove that $\widetilde{w}$ is continuous on $\widetilde{D}$. Let $\widetilde{x} \in \widetilde{D}$ be arbitrary and consider a local basis $\{U_\beta\}_{\beta \in J}$ at $\widetilde{x}$ consisting of connected open sets. The family $\mathcal{F} = \Big\{\overline{\widetilde{w}(\widetilde{D} \cap U_\beta)} \ : \ \beta \in J\Big\}$ may be regarded as a filterbase in $\widehat{\mathbb{C}}$. If we denote its set of accumulation points by $\mathcal{F}_{ac} = \bigcap_\beta \overline{\widetilde{w}(\widetilde{D} \cap U_\beta)}$, it is obvious that $\widetilde{w}(\widetilde{x}) \in \mathcal{F}_{ac}$, since $\widetilde{x} \in \widetilde{D} \cap U_\beta$ for all $\beta \in J$.

We show that $\widetilde{w}(\widetilde{D} \cap U_\beta)$ is connected for all $\beta \in J$. Suppose on the contrary that there exist two disjoint open sets $A, B \subset \widehat{\mathbb{C}}$ such that $\widetilde{w}(\widetilde{D} \cap U_\beta) \subset A \cup B$, $\widetilde{w}(\widetilde{D} \cap U_\beta) \cap A \neq \emptyset$ and $\widetilde{w}(\widetilde{D} \cap U_\beta) \cap B \neq \emptyset$. Pick $\xi_A \in \widetilde{w}(\widetilde{D} \cap U_\beta) \cap A$ and $\xi_B \in \widetilde{w}(\widetilde{D} \cap U_\beta) \cap B$. Then, we can find $x_A, x_B \in \widetilde{D} \cap U_\beta$ such that $\widetilde{w}(x_A) = \xi_A$

and $\widetilde{w}(x_B) = \xi_B$. Note that

$$x_A \in \left(\bigcup_{\alpha\in I} D_\alpha\right) \cap U_\beta = \bigcup_{\alpha\in I}(D_\alpha \cap U_\beta)$$

and accordingly, there exists an index $\alpha_1 \in I$ such that $x_A \in D_{\alpha_1} \cap U_\beta$. Similarly, there exists $\alpha_2 \in I$ such that $x_B \in D_{\alpha_2} \cap U_\beta$. Since $\{(D_\alpha, w_\alpha)\}_{\alpha\in I}$ is a chain, we may assume $D_{\alpha_1} \subset D_{\alpha_2}$. In that case, $x_A, x_B \in D_{\alpha_2} \cap U_\beta$, whence we derive that $E[x_A, x_B] \subset D_{\alpha_2} \cap U_\beta$, by an application of Lemma 2.2. Observe that $\widetilde{w}(E[x_A, x_B]) = w_{\alpha_2}(E[x_A, x_B])$ is connected; however, $\widetilde{w}(E[x_A, x_B]) \subset A \cup B$, $\xi_A \in \widetilde{w}(E[x_A, x_B]) \cap A$ and $\xi_B \in \widetilde{w}(E[x_A, x_B]) \cap B$, which is clearly impossible. We have reached a contradiction, which proves the connectedness of $\widetilde{w}(\widetilde{D} \cap U_\beta)$ for all $\beta \in J$. Therefore, $\overline{\widetilde{w}(\widetilde{D} \cap U_\beta)}$ is also connected and an analogous argument to that of Lemma 2.3 shows that $\mathcal{F}_{ac}$ must be connected as well.

Also, by reviewing the techniques introduced in the proof of Lemma 2.3, it is straightforward to see that any finite point of $\mathcal{F}_{ac}$ is a zero of the non-constant entire function $z \mapsto F(\widetilde{x}, z)$, which shows that $\mathcal{F}_{ac}$ is at most countable. Since it is also non-empty and connected, it must reduce to a single point, which in this case is obviously $\widetilde{w}(\widetilde{x})$. Then, it is easy to conclude that $\widetilde{w}$ is continuous at $\widetilde{x}$.

A standard application of Zorn's Lemma shows that $\mathfrak{D}$ has a maximal element, which we denote by (D^*, w^*). We wish to prove that $D^* = X$.

We first show that D^* is closed. Conversely, suppose that there exists $x^* \in \overline{D^*} \setminus D^*$. A direct application of Lemma 2.3 shows that $w^*(x)$ has a limit in the Riemman sphere as $x \to x^*$ ($x \in D^*$), which cannot be infinity by the assumption on the non-existence of asymptotic roots for F. Therefore, w^* has a continuous extension $\widetilde{w}^*$ to $D^* \cup \{x^*\}$. Note that the map $x \mapsto F(x, \widetilde{w}^*(x))$ vanishes on D^* and is continuous on the connected set $D^* \cup \{x^*\}$, whence we deduce that $F(x, \widetilde{w}^*(x)) = 0$ for all $x \in D^* \cup \{x^*\}$. Consequently, we have proven that $(D^*, w^*) < (D^* \cup \{x^*\}, \widetilde{w}^*)$, which contradicts the maximality of (D^*, w^*).

Finally, suppose that $D^* \neq X$, i.e., there exists $y \in X \setminus D^*$. Since, as noted in page 4, $E[x_0, y]$ is order-complete with respect to the separation order, there exists a least upper bound m of $E[x_0, y] \cap D^*$. Since D^* is closed, it is easy to see that $m \in D^*$; moreover, we have the inclusions $E[x_0, m] \subset D^*$ (by Lemma 2.2) and $E[m, y] \setminus \{m\} \subset X \setminus D^*$. By taking into account that $F(m, w^*(m)) = 0$ and F is nowhere degenerate, we can use Lemma 2.1 to find an open disk $D_r(w^*(m))$ and a neighborhood V of m such that $F(x, z) = P(x, z)\, G(x, z)$ for all $(x, z) \in V \times D_r(w^*(m))$, where P is a monic polynomial with coefficients in $C(V)$ and G is free of zeros in $V \times D_r(w^*(m))$. Without loss of generality, we may assume that V is connected and then, we select $y_1 \in E[m, y] \setminus \{m\}$ such that $E[m, y_1] \subset V$. Since $E[m, y_1]$ is a totally ordered and order-complete space, we can find $w_1 \in C(E[m, y_1])$ such that $P(x, w_1(x)) = 0$ for all $x \in E[m, y_1]$, by [1, Theorem 3]. Also, given that $P(m, z)$ is a power of $(z - w^*(m))$ (see Lemma 2.1), we must have $w_1(m) = w^*(m)$. By the continuity of w_1, we can pick $\bar{y} \in E[m, y_1] \setminus \{m\}$ such that $w_1(E[m, \bar{y}]) \subset D_r(w^*(m))$. Now, we write

$\widetilde{D} = D^* \cup E[m, \bar{y}]$ and consider the function $\widetilde{w} : \widetilde{D} \to \mathbb{C}$ defined by

$$\widetilde{w}(x) = \begin{cases} w^*(x), & x \in D^*; \\ w_1(x), & x \in E[m, \bar{y}]. \end{cases}$$

It is easy to see that $D^* \setminus \{m\}$ and $E[m, \bar{y}] \setminus \{m\}$ are both open in $\widetilde{D}$, whence it may be inferred that $\widetilde{w}$ is continuous on $\widetilde{D}$. We prove that $F(x, \widetilde{w}(x)) = 0$ for all $x \in \widetilde{D}$. The result is obvious for $x \in D^*$. On the other hand, if $x \in E[m, \bar{y}]$, then it is straightforward to see that $\widetilde{w}(x) \in D_r(w^*(m))$ (recall the choice of $\bar{y}$) and consequently, we have $F(x, \widetilde{w}(x)) = P(x, \widetilde{w}(x))\, G(x, \widetilde{w}(x)) = 0$. Thus, we have shown that $(D^*, w^*) < (\widetilde{D}, \widetilde{w})$, which contradicts the maximality of (D^*, w^*). The proof is now complete. □

Remark 2.5. Note that we have assumed that X is connected in the preceding theorem, while Miura and Niijima [7] have shown that such restriction is unnecessary for $C(X)$ to be algebraically closed. Can we drop the connectedness hypothesis in Theorem 2.4? Not completely. The connected components of a locally connected space are open. Hence, if we can find a root of F in $C(X_\lambda)$ for every connected component X_λ of X, we easily conclude that F has a root in $C(X)$. If F is nowhere degenerate and has no asymptotic roots, this can be done by Theorem 2.4, *provided that* $F(x_0, z_0) = 0$ *for some* $x_0 \in X_\lambda$ *and* $z_0 \in \mathbb{C}$. Such condition is not always met for arbitrary functions $F \in \mathcal{H}\big(C(X)\big)$ (e.g., take F to be a suitable exponential function in one connected component of X). However, if F is a non-constant monic polynomial, it is trivially fulfilled and we may recover the results from [7].

Remark 2.6. The restrictions imposed to F in the hypotheses of Theorem 2.4 are not necessary for the existence of roots. For example, consider the algebra $C([0,1])$ and define $F_1(x, z) = \exp(xz) - 1$. It is clearly degenerate at $x_0 = 0$. Moreover, the function $\omega : (0, 1] \to \mathbb{C}$ defined by $\omega(x) = 2\pi i x^{-1}$ is an asymptotic root of F_1. However, it obviously has the zero function as a root.

To finish this paper, we introduce two examples showing how the presence of degeneracy and asymptotic roots can interfere with the existence of roots.

Example 2.7. Recall that F is degenerate at $x_0 \in X$ if $z \mapsto F(x_0, z)$ is a constant map. Obviously, if it is not the zero map, F cannot have any root. On the other hand, let $X = [0, 1]$ and write $h(x) = \sin(1/x)$. Consider the function

$$F(x, z) = \begin{cases} x\big(\exp z - \exp h(x)\big), & 0 < x \le 1; \\ 0, & x = 0. \end{cases}$$

It can be easily verified that $F \in \mathcal{H}\big(C(X)\big)$. Also, note that F is degenerate at $x_0 = 0$ and $z \mapsto F(0, z)$ is the zero function. Suppose that $w \in C(X)$ is a root of F. Then, $F(x, w(x)) = 0$ for all $x \in [0, 1]$ implies that $w(x) = h(x) + 2k(x)\pi i$ for $x \in (0, 1]$, where $k(x) \in \mathbb{Z}$. By continuity, $k(x)$ must be constant, which yields $w(x) = \sin(1/x) + 2k\pi i$ for all $x \in (0, 1]$. Since this function does not have a continuous extension to the interval $[0, 1]$, we have reached a contradiction.

Moreover, although the function $g(x) = \sin(1/x) + 2k\pi i$ satisfies $F(x, g(x)) = 0$ for all $x \in (0, 1]$, it does not have a limit in the Riemann sphere as $x \to 0$. Therefore, the hypothesis of nondegeneracy is also essential for Lemma 2.3.

Example 2.8. Let $X = [0, 1]$. Consider the function $\varphi(z) = z\exp(-z)$ and any continuous curve $\omega : [0, 1) \to \mathbb{C}$ such that $\omega(0) = 0$, $\omega(x) = (1 - x)^{-1}$ for $1/2 \le x < 1$ and its image avoids the point 1 (the zero of φ'). Define the function

$$F(x, z) = \begin{cases} \varphi(z) - \varphi(\omega(x)), & 0 \le x < 1; \\ \varphi(z), & x = 1. \end{cases}$$

It can be easily seen that $F \in \mathcal{H}\big(C(X)\big)$ and is nowhere degenerate; however, ω is an asymptotic root of F. Suppose that $w \in C(X)$ is a root of F. Then, we must have $\varphi(w(x)) = \varphi(\omega(x))$ for all $x \in [0, 1)$. We prove that the set $A = \{x \in [0, 1) \mid w(x) = \omega(x)\}$ is open and closed in $[0, 1)$. The second assertion is obvious from the continuity of $w-\omega$. On the other hand, if $w(x_0) = \omega(x_0) = z_0$, we have that φ is locally injective at z_0 (since $\varphi'(\omega(x)) \neq 0$ for all $x \in [0, 1)$). Since $\varphi(w(x)) = \varphi(\omega(x))$, the continuity of w and ω implies that such functions must coincide in a neighborhood of x_0, proving that A is open in $[0, 1)$. Next, note that $0 \in A$. Since $[0, 1)$ is connected, we conclude that $A = [0, 1)$. However, this means that $w(x) \to \infty$ as $x \to 1$, which is clearly absurd.

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