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ON THE POLYNOMIAL NUMERICAL HULL OF A NORMAL MATRIX

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This paper is dedicated to Professor Abbas Salemi

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ABSTRACT. Let A be any n-by-n normal matrix and let $k > 0$ be an integer. By using the concept of the joint numerical range $W(A, A^2, \cdots, A^k)$, an analytic description of $V^k(A)$ for normal matrices will be presented. Additionally, new proof for Theorem 2.2 of Davis, Li and Salemi [Linear Algebra Appl., 428 (2008), pp. 137-153] is given.

1. INTRODUCTION AND PRELIMINARIES

The notion of polynomial numerical hull of a matrix $A \in M_n$ of order k, was first introduced by O.Nevanlinna [9] in 1993 as follows.

$$V^k(A) = \{\xi \in \mathbb{C} : |p(\xi)| \le \|p(A)\| \text{ for all } p(z) \in \mathbf{P}_k[\mathbb{C}]\},$$

where $\mathbf{P}_k[\mathbb{C}]$ is the set of complex polynomials with degree at most k. By the result in [3] (see also [5, 6])

$$V^k(A) = \{\zeta \in \mathbb{C} : (\zeta, \dots, \zeta^k) \in \operatorname{conv} W(A, \dots, A^k)\},$$

where $\operatorname{conv} X$ denotes the convex hull of $X \subseteq \mathbb{C}^k$ and the *joint numerical range* of $(A_1, A_2, \dots, A_m) \in M_n \times \cdots \times M_n$ is denoted by

$$W(A_1, A_2, \dots, A_m) = \{(x^* A_1 x, x^* A_2 x, \dots, x^* A_m x) : x \in \mathbb{C}^n, x^* x = 1\}.$$

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Similar to some other kinds of numerical range (see [10]), polynomial numerical hull of non-normal matrices have applications in approximating spectrum. Moreover, it has uses in ideal GMRES (see [5, 6, 7, 11]), but in the case of normal matrices we could not find any remarkable application. By the result in [6]it is proved that when A is a normal matrix

$$V^k(A) = \{\zeta \in \mathbb{C} : (\zeta, \ldots, \zeta^k) \in W(A, \ldots, A^k)\}.$$

After that, $V^2(A)$ for some special normal matrices was discussed by C.Davis and A.Salemi[4] but in the next work as a joint effort with C.K.Li [3] they could completely characterized $V^2(A)$ for any normal matrix A.

Next, in [2], we characterized $V^3(A)$ for some special matrices, and the relationship between $V^k(A)$ and "k^{th} roots of a convex set". Recently, in [1], we present a way of characterizing polynomial numerical hull of any order of each normal matrix by using new curves "polynomial inverse image of order k ". In the following we state the definition.

Definition 1.1. Let q be a polynomial of degree k and let $S \subseteq \mathbb{C}$. The set $\{z \in \mathbb{C} : \operatorname{Im}(q(z)) \in S\}$ is called a *polynomial inverse image of order k of S* and is abbreviated by $\mathrm{PII}_k(S)$.

In the above definition if $S = \{0\}$, then $\mathrm{PII}_k(\{0\})$ is called *polynomial inverse image of order k*.

However, there is still an open problem in the notion of polynomial numerical hull, such as

Problem 1.2. Let $A \in M_n$ be a normal matrix with at least $2k$ distinct eigenvalues and $V^k(A)$ be finite. Is $V^k(A) = \sigma(A)$?

To extend the characterization method of $V^2(A)$ in [3], at first we prove an extended version of [3, Theorem 2.5]. By this theorem, the recent problem is simplified and it suffices to solve it for $A \in M_{2k}$. After that, we simplify finding of $V^k(A)$ when it is finite, $A \in M_{2k}$ and $\sigma(A)$ lies on exactly one polynomial inverse image of order k. finally, we present new algebraic proof for [3, Theorem 2.2] that can be useful if one wants to extend the method of characterizing in [3].

2. Main results

In the following lemma we give an extended version of [3, remark 2.4 (c)].

Lemma 2.1. *Let A be a normal matrix and $\mu \in \partial V^k(A)$. Then $\left(\mu, \mu^2, \cdots, \mu^k\right) \in \partial W\left(A, A^2, \cdots, A^k\right)$*

Proof. Assume if possible $\left(\mu, \mu^2, \cdots, \mu^k\right) \in \operatorname{int} W\left(A, A^2, \cdots, A^k\right)$, so there exists $d > 0$ such that

$$|\varepsilon_1|^2 + \cdots + |\varepsilon_k|^2 < d \Rightarrow \left(\mu + \varepsilon_1, \cdots, \mu^k + \varepsilon_k\right) \in W\left(A, \cdots, A^k\right) \tag{2.1}$$

Let

$$e=\min_{1\leqslant n\leqslant k}\min_{0\leqslant j\leqslant n-1}\left\{\left(\frac{\sqrt{\frac{d}{k}}}{n\left(\begin{array}{c}n\\ j\end{array}\right)\left(|\mu|^{j}+1\right)}\right)^{\frac{1}{n-j}}\right\}.$$

Suppose that $\varepsilon_{k+1}\in\mathbb{C}$ be such that $|\varepsilon_{k+1}|<e$, so for any $n\in\{1,\cdots,k\}$,

$$\begin{aligned}|\varepsilon_{k+1}| &< \min_{0\leq j\leq n-1}\left\{\left(\frac{\sqrt{\frac{d}{k}}}{n\left(\begin{array}{c}n\\ j\end{array}\right)\left(|\mu|^{j}+1\right)}\right)^{\frac{1}{n-j}}\right\}\\ &\Rightarrow \sum_{j=0}^{n-1}\left(\begin{array}{c}n\\ j\end{array}\right)\left(|\mu|^{j}+1\right)|\varepsilon_{k+1}|^{n-j}<\sqrt{\frac{d}{k}}\\ &\Rightarrow \left|(\mu+\varepsilon_{k+1})^{n}-\mu^{n}\right|<\sqrt{\tfrac{d}{k}}.\end{aligned}$$

Therefore

$$|(\mu+\varepsilon_{k+1})-\mu|^{2}+\cdots+\left|(\mu+\varepsilon_{k+1})^{k}-\mu^{k}\right|^{2}<d$$

so by (2.1):

$$\left(\mu+\varepsilon_{k+1},\cdots,(\mu+\varepsilon_{k+1})^{k}\right)\in W\left(A,\cdots,A^{k}\right)$$

and proof is completed. □

Remark 2.2. [8] Let $\{b_j\}_{j=1}^{m}\subset\mathbb{R}^{n}$ and x be a boundary point of conv $\left(\{b_j\}_{j=1}^{m}\right)$, then x is a convex combination of at most n points of $\{b_j\}_{j=1}^{m}$.

Now, we present the extended version of [3, theorem 2.5]).

Theorem 2.3. *Let $A=diag\left(a_1,a_2,\cdots,a_n\right)$ has distinct eigenvalues. Then, the following results emerge*

a) $\partial V^{k}(A)\subset S=\bigcup\left\{V^{k}\left(\mathrm{diag}\left(a_{j_1},\cdots,a_{j_{2k}}\right)\right):1\leqslant j_1\leqslant\cdots\leqslant j_{2k}\leqslant n\right\}$

b) $V^{k}(A)=S\cup\{x: x \text{ enclosed by the closed curves in } S\}$

Proof. a) Let $\mu\in\partial V^{k}(A)$. It follows from Lemma 2.1 that $\left(\mu,\cdots,\mu^{k}\right)\in\partial W\left(A,\cdots,A^{k}\right)$. We can deduce from Remark 2.2 that there exists $\{j_1,\cdots,j_{2k}\}\in\{1,\cdots,n\}$ such that

$$\begin{aligned}&\left(\Re\mu,\Im\mu,\cdots,\Re\left(\mu^{k}\right),\Im\left(\mu^{k}\right)\right)\\ &\in\mathrm{conv}\left(\begin{array}{l}\left(\Re\left(a_{j_1}\right),\Im\left(a_{j_1}\right),\cdots,\Re\left(a_{j_1}^{k}\right),\Im\left(a_{j_1}^{k}\right)\right),\\ \left(\Re\left(a_{j_2}\right),\Im\left(a_{j_2}\right),\cdots,\Re\left(a_{j_2}^{k}\right),\Im\left(a_{j_2}^{k}\right)\right),\\ \vdots\\ \left(\Re\left(a_{j_{2k}}\right),\Im\left(a_{j_{2k}}\right),\cdots,\Re\left(a_{j_{2k}}^{k}\right),\Im\left(a_{j_{2k}}^{k}\right)\right)\end{array}\right)\end{aligned}$$

and so $\mu\in V^{k}\left(\mathrm{diag}\left(a_{j_1},a_{j_2},\cdots,a_{j_{2k}}\right)\right)$.

b) By [4, Lemma 3.5] it suffices to prove that

$$\mathrm{int}V^{k}(A)\subset\{x: x \text{ enclosed by the closed curves in } S\}.$$

We know that $\mathbb{C}$ is partitioned by S into some connected regions. Since $S \subset V^k(A) \subset W(A)$ there is one unbounded region, U. Suppose that $v \in U \cap \operatorname{int}V^k(A)$ and let $v \neq w \in \left(V^k(A)\right)^C$. Assume that there is a path $M = \{(x, f(x)) : f : [0,1] \to \mathbb{C}\}$ from v to w, that $M \subset U$.
Let $\alpha = \sup\left\{z \in [0,1] : f(z) \in V^k(A)\right\}$. By continuity of f and that $V^k(A)$ is closed, $f(\alpha) \in V^k(A)$. Again, consider continuity of f; so we have $f(\alpha) \in \partial V^k(A) \subset S$ that contradicts with $M \subset U$. □

By the recent theorem, we see that in order to solve Problem 1.2 it suffices to concentrate on matrices that have $2k$ distinct eigenvalues. By the following theorem we simplify finding polynomial numerical hull of order k of $A \in M_{2k}$ when $V^k(A)$ is finite, in one of its special cases.

Theorem 2.4. *Assume that $A = diag(a_1, \cdots, a_{2k})$ be such that exactly one polynomial inverse image of order k passes through $\sigma(A)$. Therefore, if $V^k(A)$ be a finite set, then $V^k(A) = \bigcup_{i=1}^{2k} V^k(A_i)$ in which $A_i = diag(\sigma(A) \setminus \{a_i\})$.*

Proof. Suppose that $\mu \in V^k(A) \backslash \sigma(A)$. Then there exist $\lambda_i \geq 0, i = 1, \ldots, 2k$ such that

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_{2k} = 1 \\ \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_{2k} a_{2k} = \mu \\ \vdots \\ \lambda_1 a_1^k + \lambda_2 a_2^k + \cdots + \lambda_{2k} a_{2k}^k = \mu^k \end{cases}$$

Assume if possible $\lambda_i > 0, 1 \leq i \leq 2k$, then by [1, Theorem 3.2] there exists non constant polynomials $p_1, \cdots, p_{2k}$ such that $\forall j, \lambda_j = \operatorname{Im}(p_j(\mu))$ and $V^k(A) = \bigcap_{i=1}^{2k} \left\{z : (\operatorname{Im}p_i)^{-1}[0,\infty)\right\}$. But for any i, $(\operatorname{Im}p_i)^{-1}(0,\infty)$ is a nonempty open set, and hence $\bigcap_{i=1}^{2k} \left\{z : (\operatorname{Im}p_i)^{-1}(0,\infty)\right\}$ is a nonempty open set, which is a contradiction. □

In [3, Theorem 2.2] Davis et al. proved a key theorem for determining $V^2(A)$ for normal matrices. Their proof was based on geometric view. In the following, we present an Algebraic proof for it.

Theorem 2.5. *Let $A = \operatorname{diag}(1, -1, x_3 + iy_3, x_4 + iy_4)$, $x_3 < x_4$, $0 < y_3 \leq y_4$ be such that $\sigma(A)$ is not contained in two perpendicular lines. Suppose $R \subseteq C \equiv R^2$ is a rectangular hyperbola that is a union of 2 branches ,$R = R_1 \cup R_2$, such that $-1, 1 \in R_1$ and $a_3 = x_3 + iy_3, a_4 = x_4 + iy_4 \in R_2$. Then $V^2(A) \cap R_1$ can be determined as follows.*

$$V^2(A) \cap R_1 = \{(x,y) \in R_1 : x \in (-1,1) \cap [x_3, x_4], y > 0\} \cup \{(-1,0), (1,0)\}$$

Proof. Step (I)- left-to-right inclusion. Assume that $(x,y) \in V^2(A) \cap R_1$ then by [3, Theorem 2.1]

$$\exists \lambda_3, \lambda_4 \geq 0 \ s.t. \ \begin{cases} (3) : \lambda_3 y_3 + \lambda_4 y_4 = y \\ (4) : \lambda_3 x_3 y_3 + \lambda_4 x_4 y_4 = xy \end{cases} \tag{2.2}$$

and hence $y \geq 0$. If $y = 0$, $(x, y) \in R_1$ shows that $x = \pm 1$ but if $y > 0$ by (2.2) at least one of λ_3, λ_4 are positive, and if one of them is positive and another is zero then $x \in \{x_3, x_4\}$. So assume that both of λ_3, λ_4 are positive. Then (2.2) shows that

$$\lambda_4 y_4 (x_4 - x) = \lambda_3 y_3 (x - x_3)$$

and so $x \in (x_3, x_4)$.

Finally, note that any straight line intersects non-degenerate hyperbola in at most 2 points, so $R_1 \cap W(A) = \{(x, y) \in R_1 : x \in [-1, 1]\}$ and proof of step(I) is completed.

Step (II)- right-to-left inclusion. Assume that $(x, y) \in R_1$, $x \in (-1, 1) \cap [x_3, x_4]$ and $y > 0$. By [3, Theorem 2.1] it suffices to find nonnegative solution $\{\lambda_i\}_{i=1}^4$ for the following system of equations:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ \lambda_1 - \lambda_2 + \lambda_3 x_3 + \lambda_4 x_4 = x \\ \lambda_3 y_3 + \lambda_4 y_4 = y \\ \lambda_3 x_3 y_3 + \lambda_4 x_4 y_4 = xy \end{cases}$$

We have $\lambda_3 = \frac{y(x_4 - x)}{y_3(x_4 - x_3)} \geq 0$, $\lambda_4 = \frac{y(x - x_3)}{y_4(x_4 - x_3)} \geq 0$ and

$$\begin{aligned} 2\lambda_1 &= x + 1 - \lambda_3 (x_3 + 1) - \lambda_4 (x_4 + 1) \\ &= \tfrac{1}{y_3 y_4 (x_4 - x_3)} \big(y_3 y_4 (x_4 - x_3)(x + 1) - y y_4 (x_4 - x)(x_3 + 1) \\ &\quad - y y_3 (x - x_3)(x_4 + 1) \big) \\ &= \tfrac{1}{(x_4 - x_3) y_3 y_4} \big((x - x_3)(y_4 - y)(1 + x_4) y_3 + (x_4 - x)(y_3 - y)(1 + x_3) y_4 \big) \\ &= \tfrac{(x - x_3)(x_4 - x)}{(x_4 - x_3) y_3 y_4} \left(\left(\tfrac{y_4 - y}{x_4 - x} \right)(1 + x_4) y_3 - \left(\tfrac{y_3 - y}{x_3 - x} \right)(1 + x_3) y_4 \right) \\ &= \tfrac{(x - x_3)(x_4 - x)}{(x_4 - x_3) y_3 y_4} \left(\tfrac{(1 + x) y_3 - (1 + x_3) y}{x - x_3} y_4 - \tfrac{(1 + x_4) y - (1 + x) y_4}{x_4 - x} y_3 \right) \\ &\geq \tfrac{(x - x_3)(x_4 - x)}{(x_4 - x_3) y_3 y_4} (y y_4 - y y_3) \geq 0 \end{aligned}$$

and similarly

$$\begin{aligned} 2\lambda_2 &= \tfrac{(x - x_3)(x_4 - x)}{(x_4 - x_3) y_3 y_4} \left(\left(\tfrac{y_4 - y}{x_4 - x} \right)(1 - x_4) y_3 - \left(\tfrac{y_3 - y}{x_3 - x} \right)(1 - x_3) y_4 \right) \\ &= \tfrac{(x - x_3)(x_4 - x)}{(x_4 - x_3) y_3 y_4} \left(\tfrac{(1 - x) y_4 - (1 - x_4) y}{x_4 - x} y_3 - \left\{ \left(\tfrac{y_3 - y}{x_3 - x} \right)(1 - x_3) + y_3 \right\} y_4 \right) \\ &\overset{(*)}{\geq} \tfrac{(x - x_3)(x_4 - x)}{(x_4 - x_3) y_3 y_4} (y y_3 - 0 y_4) \\ &\geq 0 \end{aligned}$$

For inequality (*) notice that if we let

$$\beta = \left(\frac{y_3 - y}{x_3 - x} \right)(1 - x_3) + y_3$$

we can see that $\beta < y$. So the point (x, y) is in the segment whose vertices are (x_3, y_3) and $(1, \beta)$. But we know that R_1 creates 2 regions in the Cartesian plane, so the point $(1, \beta)$ has to be in the lower region and so we have $\beta < 0$. □

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References

1. H.R. Afshin, M.A. Mehrjoofard and A. Salemi, *Polynomial inverse images and polynomial numerical hulls of normal matrices*, Oper. Matrices (to appear).
2. H.R. Afshin, M.A. Mehrjoofard and A. Salemi, *Polynomial numerical hulls of order 3*, Electron. J. Linear Algebra **18** (2009), 253–263.
3. Ch. Davis, C.-K. Li and A. Salemi, *Polynomial numerical hulls of matrices*, Linear Algebra Appl. **428** (2008), 137–153.
4. Ch. Davis and A. Salemi, *On polynomial numerical hulls of normal matrices*, Linear Algebra Appl. **383** (2004), 151–161.
5. V. Faber, W. Joubert, M. Knill and T. Manteuffel, *Minimal residual method stronger than polynomial preconditioning*, SIAM J. Matrix Anal. Appl. **17** (1996), 707–729.
6. A. Greenbaum, *Generalizations of the field of values useful in the study of polynomial functions of a matrix*, Linear Algebra Appl. **347** (2002), 233–249.
7. A. Greenbaum and L.N. Trefethen, *GMRES/CR and Arnoldi/Lanczos as matrixapproximation problems*, SIAM J. Sci. Comput. **15** (1994), no. 2, 359-368.
8. B.V. Limaye, *Functional Analysis*, Second edition. New Age International Publishers Limited, New Delhi, 1996.
9. O. Nevanlinna, *Convergence of Iterations for Linear Equations*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993.
10. A. Salemi, *Total decomposition and block numerical range*, Banach. J. Math. Anal. **5** (2011), no. 1, 51–55.
11. P. Tichý, J. Liesen and V. Faber, *On worst-case GMRES, ideal GMRES, and the polynomial numerical hull of a Jordan block*, Electron. Trans. Numer. Anal. **26** (2007), 453–473.

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