Banach J. Math. Anal. 5 (2011), no. 2, 106–121

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

FUNCTIONAL DECOMPOSITION OF STATE INDUCED C^*-MATRIX SPACES

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Communicated by I. B. Jung

ABSTRACT. A theorem of Dixmier states that each bounded linear functional f on the algebra of bounded linear operators on a separable Hilbert space is a direct sum of a trace functional g and a singular functional h, vanishing on the compact operators, such that $\|f\| = \|g\| + \|h\|$. We use elementary methods to construct, via the state space of a C^*-algebra, a Banach space of C^* matrices that contains a closed subspace on which a version of Dixmier's theorem is proved. When the C^*-algebra is taken to be the complex numbers our approach gives elementary and transparent proofs of Dixmier's theorem and the trace formula tr(AB) = tr(BA), without using the operator theoretical machineries used in the known proofs.

1. INTRODUCTION AND NOTATION

Let f be a bounded linear functional on $\mathcal{B}(\ell^2)$ (the space of bounded linear operators on the Hilbert sequence space ℓ^2). Then f defines a bounded linear functional on $\mathcal{K}(\ell^2)$, the ideal of compact operators on ℓ^2. Thus there is a trace class operator (or matrix) A_f such that $f(B) = \mathrm{tr}(A_f B)$, where tr denotes the trace function, for all $B \in \mathcal{K}(\ell^2)$ [5, p. 46, Theorem 1]. Since the trace class operators form an ideal in $\mathcal{B}(\ell^2)$ [5, p. 42, Theorem 5], the function $g(B) = \mathrm{tr}(A_f B)$ for $B \in \mathcal{B}(\ell^2)$ defines a bounded linear functional on $\mathcal{B}(\ell^2)$. The functional $h = f - g$ vanishes on $\mathcal{K}(\ell^2)$ is also known as a *singular linear functional*. Dixmier's theorem

Date: Received: 4 October 2010; Revised: 7 March 2011; Accepted: 27 April 2011.
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2010 *Mathematics Subject Classification.* Primary 46E40; Secondary 46L30.
Key words and phrases. C^*-algebra, state space, weak* topology, dual space.

([2], [5, p. 50, Theorem 1]), which has also been attributed to Schatten, states that this decomposition is unique and satisfies the norm equality $\|f\| = \|g\| + \|h\|$.

As defined in the 1976 paper [1] of Alfsen and Effros, a closed subspace J of a Banach space X is an M-ideal if the annihilator $J^{\perp}$ is complemented as an ℓ^1 summand in the dual space $X^{\#}$ of X, i.e., $X^{\#} = J^{\perp} \oplus_1 E$ for some closed subspace E of $X^{\#}$. This theorem of Dixmier can now be restated as the compact operators form an M-ideal in $\mathcal{B}(\ell^2)$ (later it is also known as the only nontrivial one [6, 7]). See also [3]. Most spaces with known M-ideal structures are Banach algebras, mainly bounded operators on certain Banach spaces.

Since a C^*-algebra resemble the complex field in many ways, here we will use a fixed C^*-algebra $\mathcal{A}$ with identity 1 and state space $s(\mathcal{A})$, together with the pair $\mathcal{K}(\ell^2)$ and $\mathcal{B}(\ell^2)$, to build a Banach space of matrices over $\mathcal{A}$ with an M-ideal that corresponds to $\mathcal{K}(\ell^2)$. The resulting space is not a Banach algebra. When the C^*-algebra is taken to be $\mathbb{C}$, the space is exactly $\mathcal{B}(\ell^2)$. Since there is no parallel machinery available for our setting, this approach also gives elementary alternate proofs of Dixmier's theorem and the trace formula tr(AB) = tr(BA), without using the theory of trace class operators and other machineries.

Let $\mathcal{A}$ be a C^*-algebra with identity 1 and state space $s(\mathcal{A})$ (consisting of all states, i.e., bounded positive linear functionals of norm 1, on $\mathcal{A}$) with the weak* topology (as a subspace of the dual space $\mathcal{A}^{\#}$ of $\mathcal{A}$). For each matrix $B = \left[b_{jk}\right]$ with entries $b_{jk} \in \mathcal{A}$, and each $\psi \in s(\mathcal{A})$, denote by $\widetilde{\psi}(B)$ the complex matrix $\left[\psi(b_{jk})\right]$. Let $\mathcal{M}$ be the space of all matrices $A = \left[a_{jk}\right]$ over $\mathcal{A}$ such that (the scalar matrix)

$$\begin{aligned}
&\widetilde{\varphi}(A) := \left[\varphi(a_{jk})\right] \in \mathcal{B}(\ell^2) \quad \text{for all } \varphi \in s(\mathcal{A}) \qquad \text{and} \\
&\text{the map} \quad \varphi \mapsto \widetilde{\varphi}(A) = \left[\varphi\left(a_{jk}\right)\right] \quad \text{is continuous from} \\
&s(\mathcal{A}) \text{ with the weak}^* \text{ topology to } \mathcal{B}(\ell^2) \text{ with the norm topology.}
\end{aligned}$$

Thus each $A \in \mathcal{M}$ defines a continuous map, $\varphi \mapsto \widetilde{\varphi}(A)$, from $s(\mathcal{A})$ to $\mathcal{B}(\ell^2)$. Since $s(\mathcal{A})$ with the weak* topology is a compact Hausdorff space [4, p. 257], it is well known that $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$ is a Banach space with the norm

$$\|A\| = \sup_{\varphi \in s(\mathcal{A})} \|\widetilde{\varphi}(A)\|_{\mathcal{B}(\ell^2)}.$$

Each $A \in \mathcal{M}$ induces an element $\widetilde{A}$ in $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$:

$$\widetilde{A}(\varphi) = \widetilde{\varphi}(A), \qquad \varphi \in s(\mathcal{A}).$$

So $\mathcal{M}$ can be considered as a subspace of the Banach space $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$. The map $A \mapsto \widetilde{A}$ does not map $\mathcal{M}$ onto $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$, even when ℓ^2 is replaced by the one dimensional $\mathbb{C}$ and in the very simple case of $\mathcal{A} = C([0,1])$ (the algebra of continuous complex-valued functions on the interval $[0,1]$).

Example 1.1. *With $\mathcal{A} = C[0,1]$ there is a continuous map $\Psi : s(\mathcal{A}) \to \mathbb{C}$ such that there does not exist $a \in \mathcal{A}$ that satisfies $\Psi(\varphi) = \varphi(a)$ for all $\varphi \in s(\mathcal{A})$.*

Proof. Each $t \in [0,1]$ induces a state φ_t on $\mathcal{A}$: the evaluation functional $\varphi_t(a) = a(t)$ for all $a \in \mathcal{A}$. Let $a_1 \in \mathcal{A}$ be given by $a_1(t) = t$ for all $t \in [0,1]$. Let

$$\mathcal{V} = \left\{ \varphi \in s(\mathcal{A}) : \left| \varphi(a_1) - \varphi_{1/2}(a_1) \right| < \frac{1}{4} \right\},$$

a weak* neighborhood of $\varphi_{1/2}$. Since $s(\mathcal{A})$, with the relative weak* topology, being compact and Hausdorff [4, p. 257], is normal, there is a continuous map $\Psi : s(\mathcal{A}) \to \mathbb{C}$ such that $\Psi(\varphi_{1/2}) = 1$ and $\Psi(\varphi) = 0$ for all $\varphi \in s(\mathcal{A}) \setminus \mathcal{V}$. In particular $\Psi(\varphi_0) = 0$. Suppose there is an $a \in \mathcal{A}$ such that

$$\Psi(\varphi) = \varphi(a) \quad \text{ for all } \varphi \in s(\mathcal{A}).$$

Then $1 = \Psi(\varphi_{1/2}) = a(1/2)$ and $0 = \Psi(\varphi_0) = a(0)$. Let $\hat{\varphi} := \frac{1}{5}\varphi_{1/2} + \frac{4}{5}\varphi_0$. Then $\hat{\varphi} \in s(\mathcal{A})$ and

$$\left| \hat{\varphi}(a_1) - \varphi_{1/2}(a_1) \right| = \left| \frac{1}{5}\varphi_{1/2}(a_1) + \frac{4}{5}\varphi_0(a_1) - \varphi_{1/2}(a_1) \right| = \frac{2}{5} > \frac{1}{4}.$$

Thus $\hat{\varphi} \in s(\mathcal{A}) \setminus \mathcal{V}$, and hence,

$$0 = \Psi(\hat{\varphi}) = \hat{\varphi}(a) = \frac{1}{5} a\,(1/2) + \frac{4}{5} a\,(0) = \frac{1}{5},$$

which is a contradiction. □

It will be shown in Proposition 2.1 that the image of $\mathcal{M}$ under the map $A \mapsto \widetilde{A}$ is a closed subspace of $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$, and $\mathcal{M}$ is a Banach space with the norm

$$\|A\| = \sup_{\varphi \in s(\mathcal{A})} \|\widetilde{\varphi}(A)\|_{\mathcal{B}(\ell^2)}.$$

Let $A \in \mathcal{M}$. For each $n \in \mathbb{N}$, $A_{n_\lrcorner}$ denotes the $n-$th compression matrix of A; that is, the (j,k)-th entry of $A_{n_\lrcorner}$ is exactly the same as that of A for $1 \le j,k \le n$, and is zero otherwise. Denote by $A_{\underline{n}}$ [respectively, $A_{n|}$] the matrix whose first n rows [respectively, columns] coincide with that of A and all other rows [respectively, columns] are zero. Dually, $A_{\overline{n}}$ [respectively, $A_{|n}$] is the matrix whose first n rows [respectively, columns] are zero and all other rows [respectively, columns] coincide with that of A. Denote by $\mathcal{K}$ the space of all $A \in \mathcal{M}$ with the property that

$$\left\| A - A_{\underline{n}} \right\| = \|A_{\overline{n}}\| \to 0 \quad \text{ as } n \to \infty.$$

Note that this is equivalent to the compactness of A (i.e., $A \in \mathcal{K}(\ell^2)$) when $\mathcal{A}$ is the complex field $\mathbb{C}$.

We will show that the annihilator $\mathcal{K}^\perp$ of $\mathcal{K}$ behaves in the dual space $\mathcal{M}^\#$ of $\mathcal{M}$ just like $[\mathcal{K}(\ell^2)]^\perp$ in $[\mathcal{B}(\ell^2)]^\#$, as in Dixmier's theorem. That is $\mathcal{K}$ is an M-ideal in $\mathcal{M}$.

2. Preliminary results

We begin the section by showing that $\mathcal{M}$ is a Banach space.

Proposition 2.1. *$\mathcal{M}$ is a Banach space with the norm*

$$\|A\| = \sup_{\varphi \in s(\mathcal{A})} \|\widetilde{\varphi}(A)\|_{\mathcal{B}(\ell^2)}.$$

The *state norm* $\|\cdot\|_s$ on $\mathcal{A}$ is defined by

$$\|a\|_s = \sup_{\varphi \in s(\mathcal{A})} |\varphi(a)|, \qquad a \in \mathcal{A}.$$

The state norm is a norm and [8, Proposition 2.3]

$$\|a\|_s \le \|a\| \le 2\|a\|_s \qquad \text{for all } a \in \mathcal{A}.$$

The state norm and the C^*-norm on $\mathcal{A}$ are equivalent.

Proof. It suffices to show that the image of $\mathcal{M}$ under the map $A \mapsto \widetilde{A}$ is closed in $C(s(\mathcal{A}), \mathcal{B}(\ell^2))$. Let $\{A_n\}$ be a sequence in $\mathcal{M}$ such that $\widetilde{A}_n \to \Psi$ for some $\Psi \in C(s(\mathcal{A}), \mathcal{B}(\ell^2))$. Let $A_n = \left[a_{jk}^{(n)}\right]$. For each $j, k \in \mathbb{N}$,

$$\begin{aligned} \left\|a_{jk}^{(n)} - a_{jk}^{(m)}\right\|_s &= \sup_{\varphi \in s(\mathcal{A})} \left|\varphi(a_{jk}^{(n)}) - \varphi(a_{jk}^{(m)})\right| \\ &\le \sup_{\varphi \in s(\mathcal{A})} \|\widetilde{\varphi}(A_n) - \widetilde{\varphi}(A_m)\|_{\mathcal{B}(\ell^2)} \to 0 \end{aligned}$$

as $n, m \to \infty$. By the equivalence of the state norm and the norm on $\mathcal{A}$, the sequence $\left\{a_{jk}^{(n)}\right\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}$. Thus there is an $a_{jk} \in \mathcal{A}$ such that $\left\|a_{jk}^{(n)} - a_{jk}\right\| \to 0$. We also have

$$\left\|\widetilde{A}_n(\varphi) - \Psi(\varphi)\right\|_{\mathcal{B}(\ell^2)} \to 0 \qquad \text{for all } \varphi \in s(\mathcal{A}).$$

For each $\varphi \in s(\mathcal{A})$, let $\Psi(\varphi) = \left[\psi_{jk}(\varphi)\right]$. It follows that

$$\varphi(a_{jk}^{(n)}) \to \psi_{jk}(\varphi) \qquad \text{for all } \varphi \in s(\mathcal{A}).$$

But we also have

$$\varphi(a_{jk}^{(n)}) \to \varphi(a_{jk}) \qquad \text{for all } \varphi \in s(\mathcal{A}),$$

and hence

$$\varphi(a_{jk}) = \psi_{jk}(\varphi) \qquad \text{for all } \varphi \in s(\mathcal{A}).$$

Let $A = \left[a_{jk}\right]$. Then

$$\Psi(\varphi) = \left[\psi_{jk}(\varphi)\right] = \left[\varphi(a_{jk})\right] = \widetilde{\varphi}(A) = \widetilde{A}(\varphi) \qquad \text{for all } \varphi \in s(\mathcal{A}).$$

That is $\widetilde{A} = \Psi \in C(s(\mathcal{A}), \mathcal{B}(\ell^2))$, and $A \in \mathcal{M}$. ∎

Now we prove some properties of $\mathcal{K}$ that are parallel to well-known properties of compact operators.

Proposition 2.2. *$\mathcal{K}$ is a closed proper subspace of $\mathcal{M}$.*

Proof. Let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{K}$ such that $\|A_k - A\| \to 0$ for some $A \in \mathcal{M}$. Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that

$$\|A_k - A\| < \frac{\epsilon}{4} \quad \text{for all } k \geq N.$$

Since $A_N \in \mathcal{K}$, there is an $n_0 \in \mathbb{N}$ such that

$$\left\|(A_N)_{\underline{n}} - A_N\right\| < \frac{\epsilon}{4} \quad \text{for all } n \geq n_0.$$

Let $n \geq n_0$.

$$\begin{aligned}\left\|A_{\underline{n}} - A\right\| &\leq \left\|A_{\underline{n}} - (A_N)_{\underline{n}}\right\| + \left\|(A_N)_{\underline{n}} - A_N\right\| + \|A_N - A\| \\ &< \left\|(A - A_N)_{\underline{n}}\right\| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \|A_N - A\| + \frac{\epsilon}{2} < \epsilon\end{aligned}$$

That is $\left\|A - A_{\underline{n}}\right\| \to 0$ as $n \to \infty$, and hence $A \in \mathcal{K}$.

By definition, we have $\mathcal{K} \subseteq \mathcal{M}$. To see that the inclusion is proper, we note that the matrix A with 1 (the identity of $\mathcal{A}$) on the diagonal and 0 elsewhere (i.e., $A(j,k) = \delta_{jk}1$) is in $\mathcal{M}$ but not in $\mathcal{K}$. Weak* to norm continuity of the map $\varphi \mapsto \widetilde{\varphi}(A)$ follows immediately from the fact that $\widetilde{\varphi}(A)$ is the identity matrix in $\mathcal{B}(\ell^2)$ for each $\varphi \in s(\mathcal{A})$. Thus $A \in \mathcal{M}$. But $\left\|\widetilde{\varphi}(A - A_{\underline{n}})\right\| = 1$ for all $\varphi \in s(\mathcal{A})$ and all $n \in \mathbb{N}$, which implies that $A \notin \mathcal{K}$. □

Proposition 2.3. *Let $A \in \mathcal{M}$ satisfy $A = A_{N\rceil}$ (respectively, $A = A_{\underline{N}}$) for some fixed $N \in \mathbb{N}$. Then $A \in \mathcal{K}$, and $\left\|A - A_{\nu\lrcorner}\right\| \to 0$ as $\nu \to \infty$.*

Proof. Suppose $A = A_{\underline{N}} \in \mathcal{M}$. For $n \geq N$, we have $A_{\underline{n}} = A_{\underline{N}} = A$. Thus $\left\|A - A_{\underline{n}}\right\| = 0$ for all $n \geq N$, and hence $A \in \mathcal{K}$.

If $A = A_{N\rceil} \in \mathcal{M}$, then the transpose of A,

$$B = A^T \qquad \left(B_{jk} = (A^T)_{jk} = A_{kj} \ \forall \ j,\ k \in \mathbb{N}\right),$$

satisfies

$$B = A^T = [A_{N\rceil}]^T = B_{\underline{N}},$$

and hence,

$$\left\|B - B_{\underline{n}}\right\| = 0 \quad \text{for all } n \geq N.$$

For each $n \geq N$ we have

$$\begin{aligned}\left\|A - A_{n\rceil}\right\| &= \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}\left(A - A_{n\rceil}\right)\right\|_{\mathcal{B}(\ell^2)} = \sup_{\varphi \in s(\mathcal{A})} \left\|\left(\widetilde{\varphi}\left(A - A_{n\rceil}\right)\right)^T\right\|_{\mathcal{B}(\ell^2)} \\ &= \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}\left(A^T - \left(A_{n\rceil}\right)^T\right)\right\|_{\mathcal{B}(\ell^2)} = \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}\left(B - B_{\underline{n}}\right)\right\|_{\mathcal{B}(\ell^2)} \\ &= \left\|B - B_{\underline{n}}\right\| = 0.\end{aligned}$$

Since A is assumed to be in $\mathcal{M}$, this shows that $A = A_{N\rceil} \in \mathcal{K}$, and hence

$$\left\|A - A_{\nu\lrcorner}\right\| = \left\|A - A_{\underline{\nu}}\right\| \to 0 \quad \text{as } \nu \to \infty.$$

For the case $A = A_{\underline{N}}$, we see as above that $C = A^T$ satisfies $C = C_{N|} \in \mathcal{K}$, and hence

$$\|A - A_{\nu_\lrcorner}\| = \left\|\left(A - A_{\nu_\lrcorner}\right)^T\right\| = \|C - C_{\underline{\nu}}\| \to 0 \quad \text{as} \;\; \nu \to \infty.$$

□

For each $A = \left[a_{jk}\right] \in \mathcal{M}$, A^* is defined by

$$(A^*)_{jk} = a^*_{kj} \qquad \text{for all } j,\; k \in \mathbb{N}.$$

It is easy to see that $A^* \in \mathcal{M}$ whenever $A \in \mathcal{M}$.

Proposition 2.4. *Let $A = \left[a_{jk}\right]$ be a matrix over $\mathcal{A}$.*

(1) *$A \in \mathcal{K}$ iff the map $\varphi \mapsto \widetilde{\varphi}(A)$ is continuous form $s(\mathcal{A})$ with the weak* topology to $\mathcal{K}(\ell^2)$ with the operator norm topology.*

(2) *$A \in \mathcal{K}$ iff $A^* = \left[a^*_{jk}\right]^T \in \mathcal{K}$.*

(3) *If $A \in \mathcal{M}$, then $A \in \mathcal{K}$ iff $\left\|A - A_{n|}\right\| = \left\|A_{|n}\right\| \to 0$ as $n \to \infty$.*

Proof. (1) [⇒] Suppose $A \in \mathcal{K}$. Then $A \in \mathcal{M}$. Thus $\varphi \mapsto \widetilde{\varphi}(A)$ is continuous from $s(\mathcal{A})$ with weak* topology to $\mathcal{B}(\ell^2)$ with norm topology. It suffices to show that $\widetilde{\varphi}(A) \in \mathcal{K}(\ell^2)$ for all $\varphi \in s(\mathcal{A})$. Let $\varphi \in s(\mathcal{A})$. We have

$$\left\|\widetilde{\varphi}(A) - [\widetilde{\varphi}(A)]_{\underline{n}}\right\|_{\mathcal{B}(\ell^2)} = \left\|\widetilde{\varphi}(A - A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \le \left\|A - A_{\underline{n}}\right\| \to 0 \quad \text{as} \;\; n \to \infty$$

and hence $\widetilde{\varphi}(A) \in \mathcal{K}(\ell^2)$.

(1) [⇐] Let $\epsilon > 0$. By continuity, for each $\varphi \in s(\mathcal{A})$, there is a weak* open set $V_\varphi \subseteq s(\mathcal{A})$ such that

$$\varphi \in V_\varphi \quad \text{and} \quad \left\|\widetilde{\varphi}(A) - \widetilde{\psi}(A)\right\|_{\mathcal{K}(\ell^2)} = \left\|\widetilde{\varphi}(A) - \widetilde{\psi}(A)\right\|_{\mathcal{B}(\ell^2)} < \frac{\epsilon}{4} \;\; \forall \;\; \psi \in V_\varphi.$$

Since $s(\mathcal{A})$ with the weak* topology is a compact Hausdorff space [4, p. 257], and

$$s(\mathcal{A}) \subseteq \bigcup_{\varphi \in s(\mathcal{A})} V_\varphi,$$

there are $\varphi_1, \dots \varphi_k \in s(\mathcal{A})$ such that

$$s(\mathcal{A}) \subseteq \bigcup_{j=1}^{k} V_{\varphi_j}.$$

For each $j = 1, \dots, k$, since $\widetilde{\varphi}_j(A) \in \mathcal{K}(\ell^2)$, there is an $N_j \in \mathbb{N}$ such that

$$\left\|\widetilde{\varphi}_j(A) - [\widetilde{\varphi}_j(A)]_{\underline{n}}\right\|_{\mathcal{B}(\ell^2)} = \left\|\widetilde{\varphi}_j(A) - [\widetilde{\varphi}_j(A_{\underline{n}})]\right\|_{\mathcal{B}(\ell^2)} < \frac{\epsilon}{4} \qquad \text{for all } n \ge N_j.$$

Put $N = \max\{N_j : j = 1, \dots, k\}$. Then for $n \geq N$ and $\varphi \in s(\mathcal{A})$, we have $\varphi \in V_{\varphi_j}$ for some $j = 1, \dots, k$, and thus

$$\begin{aligned}
&\left\|\widetilde{\varphi}(A) - \widetilde{\varphi}(A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \\
\leq &\left\|\widetilde{\varphi}(A) - \widetilde{\varphi}_j(A)\right\|_{\mathcal{B}(\ell^2)} + \left\|\widetilde{\varphi}_j(A) - \widetilde{\varphi}_j(A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} + \left\|\widetilde{\varphi}_j(A_{\underline{n}}) - \widetilde{\varphi}(A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \\
< &\frac{\epsilon}{4} + \frac{\epsilon}{4} + \left\|\left[\widetilde{\varphi}_j(A) - \widetilde{\varphi}(A)\right]_{\underline{n}}\right\|_{\mathcal{B}(\ell^2)} < \frac{\epsilon}{2} + \left\|\widetilde{\varphi}_j(A) - \widetilde{\varphi}(A)\right\|_{\mathcal{B}(\ell^2)} < \frac{3\epsilon}{4}
\end{aligned}$$

Since $\varphi \in s(\mathcal{A})$ is arbitrary,

$$\begin{aligned}
\left\|A - A_{\underline{n}}\right\| &= \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}(A - A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} = \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}(A) - \widetilde{\varphi}(A_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \\
&\leq \frac{3\epsilon}{4} < \epsilon \qquad \text{for all } \; n \geq N.
\end{aligned}$$

(2) [$\Rightarrow$] Suppose that $A \in \mathcal{K}$. Then $\varphi \mapsto \widetilde{\varphi}(A)$ is weak* to norm continuous from $s(\mathcal{A})$ to $\mathcal{K}(\ell^2)$. Let $\epsilon > 0$. For each $\varphi \in s(\mathcal{A})$, there is a weak* neighborhood U_φ of φ such that

$$\text{for all } \psi \in U_\varphi, \; \widetilde{\psi}(A) \in \mathcal{K} \quad \text{and} \quad \left\|\widetilde{\varphi}(A) - \widetilde{\psi}(A)\right\|_{\mathcal{B}(\ell^2)} < \epsilon.$$

Since ψ is a positive linear functional, $\psi(a^*) = \overline{\psi(a)}$ for all $a \in \mathcal{A}$ [4, p. 255]. From $\widetilde{\psi}(A) \in \mathcal{K}(\ell^2)$, we have $\widetilde{\psi}(A^*) = [\widetilde{\psi}(A)]^* \in \mathcal{K}(\ell^2)$, and

$$\left\|\widetilde{\varphi}(A^*) - \widetilde{\psi}(A^*)\right\|_{\mathcal{B}(\ell^2)} = \left\|[\widetilde{\varphi}(A)]^* - [\widetilde{\psi}(A)]^*\right\|_{\mathcal{B}(\ell^2)} = \left\|\widetilde{\varphi}(A) - \widetilde{\psi}(A)\right\|_{\mathcal{B}(\ell^2)} < \epsilon.$$

Thus the map $\varphi \mapsto \widetilde{\varphi}(A^*)$ is continuous from $s(\mathcal{A})$ with weak* topology to $\mathcal{K}(\ell^2)$ with norm topology. Hence $A^* \in \mathcal{K}$ by part (1).

(2) [$\Leftarrow$] Suppose that $A^* \in \mathcal{K}$. Then $A = (A^*)^* \in \mathcal{K}$.

(3) [$\Rightarrow$] Suppose $A \in \mathcal{K}$. Then $A^* \in \mathcal{K}$ and hence

$$\left\|A^* - (A^*)_{\underline{n}}\right\| \to 0 \quad \text{as } n \to \infty.$$

Thus

$$\left\|A - A_{n|}\right\| = \left\|(A - A_{n|})^*\right\| = \left\|A^* - (A^*)_{\underline{n}}\right\| \to 0 \quad \text{as } n \to \infty.$$

(3) [$\Leftarrow$] Suppose $\left\|A - A_{n|}\right\| \to 0$. Since each $A_{n|} \in \mathcal{K}$ by Proposition 2.3, and since $\mathcal{K}$ is closed under the operator norm, $A \in \mathcal{K}$. □

3. The dual of $\mathcal{K}$

In this section we will obtain a functional matrix representation of the dual $\mathcal{K}^\#$ of $\mathcal{K}$. First note that for $A = [a_{jk}] \in \mathcal{M}$, and each $j, k \in \mathbb{N}$, we have

$$\left\|a_{jk}\right\| \leq 2\left\|a_{jk}\right\|_s = 2 \sup_{\varphi \in s(\mathcal{A})} \left|\varphi(a_{jk})\right| \leq 2 \sup_{\varphi \in s(\mathcal{A})} \left\|\widetilde{\varphi}(A)\right\|_{\mathcal{B}(\ell^2)} = 2\left\|A\right\|.$$

We will need the following lemma in the proofs of Propositions 3.2 and 3.3

Lemma 3.1. *Let* $\{f_n\}$ *be a sequence in the dual space* $X^{\#}$ *of a Banach space* X *such that* $f(x) = \sum_{k=1}^{\infty} f_k(x)$ *converges for all* $x \in X$*. Then* $f \in X^{\#}$*.*

Proof. A routine argument shows that f is linear. For the boundedness of f, let $g_n = \sum_{k=1}^{n} f_k$ for each $n \in \mathbb{N}$. Then $g_n \in X^{\#}$. For each $x \in X$, since $\sum_{k=1}^{\infty} f_k(x)$ converges, there is an $\alpha_x \geq 0$ such that $|g_n(x)| \leq \alpha_x$ for all $n \in \mathbb{N}$. So $\{g_n\}$ is a sequence in $X^{\#}$ that is pointwise bounded. The uniform boundedness principle implies that $\{g_n\}$ is uniformly bounded; i.e., there is a β such that $\|g_n\| \leq \beta$ for all $n \in \mathbb{N}$. For each $x \in X$, we have

$$|f(x)| = \lim_{n\to\infty} \left| \sum_{k=1}^{n} f_k(x) \right| = \lim_{n\to\infty} |g_n(x)| \leq \limsup_{n\to\infty} \|g_n\| \, \|x\| \leq \beta \, \|x\| .$$

Thus $f \in X^{\#}$ with $\|f\| \leq \beta$. □

Proposition 3.2. *For each* $f \in \mathcal{K}^{\#}$*, there exists a unique matrix* $[f_{jk}]$*, with* $f_{jk} \in \mathcal{A}^{\#}$*, such that*

$$f(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{for all } \; A = [a_{jk}] \in \mathcal{K}.$$

Conversely, each matrix $[g_{jk}]$ *over* $\mathcal{A}^{\#}$ *with the property that*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \quad \text{converges for every } \; A = [a_{jk}] \in \mathcal{K},$$

defines a bounded linear functional

$$g(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \quad \left(A = [a_{jk}] \in \mathcal{K}\right) \quad \text{on } \mathcal{K}.$$

Moreover, in this case,

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} g_{jk}(a_{jk}) \quad \text{converges,} \quad \text{and,}$$

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} g_{jk}(a_{jk}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \qquad \text{for all } \; [a_{jk}] \in \mathcal{K}.$$

Thus $\mathcal{K}^{\#}$ is identified with the space of all such matrices. The norm of such a matrix is defined to be the norm of the bounded linear functional it represents, i.e., $\|[f_{jk}]\| = \|f\|$ if $[f_{jk}]$ represents $f \in \mathcal{K}^{\#}$.

Proof. Let $f \in \mathcal{K}^{\#}$. For each $(j,k) \in \mathbb{N} \times \mathbb{N}$ and each $a \in \mathcal{A}$, since the matrix $E_{jk}(a)$ with (j,k) entry a and all others 0 is easily seen from Proposition 2.3 to be in $\mathcal{K}$ with

$$\|E_{jk}(a)\| = \|a\|_s \leq \|a\| ,$$

we define f_{jk} by

$$f_{jk}(a) = f(E_{jk}(a)) \qquad \text{for all } a \in \mathcal{A}.$$

It is readily seen that f_{jk} is linear, and

$$\left|f_{jk}(a)\right| = \left|f(E_{jk}(a))\right| \le \|f\| \left\|E_{jk}(a)\right\| \le \|f\| \, \|a\| \, .$$

Hence $f_{jk} \in \mathcal{A}^{\#}$ with $\left\|f_{jk}\right\| \le \|f\|$. Let $A = \left[a_{jk}\right] \in \mathcal{K}$. For each $n \in \mathbb{N}$, $A_{\underline{n}} \in \mathcal{K}$, and, by Proposition 2.3,

$$\left\|A_{\underline{n}} - [A_{\underline{n}}]_{\nu_\lrcorner}\right\| \to 0 \quad \text{ as } \nu \to \infty.$$

Thus, by linearity,

$$\sum_{j=1}^{n} \sum_{k=1}^{\nu} f_{jk}(a_{jk}) = f([A_{\underline{n}}]_{\nu_\lrcorner}) \to f(A_{\underline{n}}) \quad \text{ as } \nu \to \infty.$$

That is

$$f(A_{\underline{n}}) = \sum_{j=1}^{n} \sum_{k=1}^{\infty} f_{jk}(a_{jk}).$$

Since $\left\|A - A_{\underline{n}}\right\| \to 0$ as $n \to \infty$, $f(A_{\underline{n}}) \to f(A)$, and hence

$$f(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}).$$

Now suppose $\left[g_{jk}\right]$ is a matrix over $\mathcal{A}^{\#}$ such that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) \;\text{ converges for every }\; A = \left[a_{jk}\right] \in \mathcal{K}.$$

For each fixed $m, n \in \mathbb{N}$, define $\hat{g}_{mn} : \mathcal{K} \to \mathbb{C}$ by

$$\hat{g}_{mn}(A) = g_{mn}(a_{mn}) \quad \text{ for each }\; A = \left[a_{jk}\right] \in \mathcal{K}.$$

Then

$$|\hat{g}_{mn}(A)| \le \|g_{mn}\| \, \|a_{mn}\| \le 2 \, \|A\| \, \|g_{mn}\|$$

i.e., $\hat{g}_{mn} \in \mathcal{K}^{\#}$. Since by assumption

$$g_m(A) := \sum_{k=1}^{\infty} \hat{g}_{mk}(A) = \sum_{k=1}^{\infty} g_{mk}(a_{mk}) \;\text{ converges for every } A = \left[a_{jk}\right] \in \mathcal{K},$$

by Lemma 3.1, $g_m \in \mathcal{K}^{\#}$. Since we also assume that

$$g(A) := \sum_{m=1}^{\infty} g_m(A) = \sum_{m=1}^{\infty} \; \sum_{k=1}^{\infty} g_{mk}(a_{mk}) \quad \text{converges for every } A = \left[a_{jk}\right] \in \mathcal{K},$$

by Lemma 3.1 again, the functional g is bounded, i.e., $g \in \mathcal{K}^{\#}$.

For each $A = \left[a_{jk}\right] \in \mathcal{K}$, since the matrix $A_{|k|} = A_{k|} - A_{(k-1)|}$, with the k-th column the same as that of A and all others 0, is in $\mathcal{K}$,

$$\sum_{j=1}^{\infty} g_{jk}(a_{jk}) = g(A_{|k|}) \;\text{ converges, for all } k \in \mathbb{N}.$$

Since $\left\| A - A_{m|} \right\| \to 0$ as $m \to \infty$,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g_{jk}(a_{jk}) = g(A) = \lim_{m\to\infty} g(A_{m|}) = \lim_{m\to\infty} \left[\sum_{k=1}^{m} g(A_{|k|}) \right]$$

$$= \lim_{m\to\infty} \left[\sum_{k=1}^{m} \sum_{j=1}^{\infty} g_{jk}(a_{jk}) \right] = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} g_{jk}(a_{jk}).$$

□

Next we show that if $[f_{jk}] \in \mathcal{K}^{\#}$, then the two double sums both converge and are equal for each $A = [a_{jk}] \in \mathcal{M}$, not just for elements in $\mathcal{K}$.

Proposition 3.3. *For each $f = [f_{jk}] \in \mathcal{K}^{\#}$ and each $A = [a_{jk}] \in \mathcal{M}$, both*

$$\hat{f}(A) := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \qquad \text{and} \qquad g(A) := \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f_{jk}(a_{jk})$$

converge, and they have the same sum. Furthermore $\hat{f}$ is a bounded linear functional on $\mathcal{M}$ with norm $\left\| \hat{f} \right\|_{\mathcal{M}^{\#}} = \| f \|_{\mathcal{K}^{\#}}$.

Proof. Let $A = [a_{jk}] \in \mathcal{M}$. Then for each $j \in \mathbb{N}$, the row j matrix $A_{\underline{j}} = A_{\underline{j}} - A_{\underline{j-1}} \in \mathcal{K}$. Thus

$$\sum_{k=1}^{\infty} f_{jk}(a_{jk}) \text{ converges for every } j \in \mathbb{N}.$$

Suppose

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \text{ does not converge.}$$

Then there are an $\epsilon > 0$ and two sequences $\{j_\nu\}$, $\{l_\nu\}$ in $\mathbb{N}$ such that

$$1 \le j_1 < l_1 < j_2 < l_2 < \ldots < j_\nu < l_\nu < \ldots, \quad \text{and}$$

$$\left| \sum_{j=j_\nu}^{l_\nu} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \right| > \epsilon \quad \text{for all } \nu \in \mathbb{N}.$$

Let $A_\nu = A_{\underline{l_\nu}} - A_{\underline{j_\nu - 1}}$, the matrix whose rows from j_ν-th through l_ν-th coincide with that of A and all others are 0; let

$$\alpha_\nu = \frac{1}{\nu} \operatorname{sgn} \left[\sum_{j=j_\nu}^{l_\nu} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \right]; \quad \text{and} \quad B = \sum_{\nu=1}^{\infty} \alpha_\nu A_\nu.$$

We show that $B \in \mathcal{K}$ but the sum for $f(B)$ diverges. Let $\eta > 0$. There is a $\nu_0 \in \mathbb{N}$ such that

$$\sum_{\nu=\nu_0}^{\infty} \frac{\|A\|^2}{\nu^2} < \frac{\eta^2}{4}.$$

For $n \geq j_{\nu_0}$, $\varphi \in s(\mathcal{A})$, and $x = \{x_k\} \in \ell^2$, let ν_1 be the largest ν such that $j_\nu \leq n$. Thus $\nu_1 \geq \nu_0$, and hence,

$$\left\|\widetilde{\varphi}\left(B - B_{\underline{n}}\right) x\right\|_{\ell^2}^2 = \left\|\left[\widetilde{\varphi}(B) - \widetilde{\varphi}(B_{\underline{n}})\right] x\right\|_{\ell^2}^2$$

$$= \sum_{j=n+1}^{l_{\nu_1}} \left|\alpha_{\nu_1} \sum_{k=1}^{\infty} \varphi(a_{jk}) x_k\right|^2 + \sum_{\nu=\nu_1+1}^{\infty} \sum_{j=j_\nu}^{l_\nu} \left|\alpha_\nu \sum_{k=1}^{\infty} \varphi(a_{jk}) x_k\right|^2$$

$$= \left|\alpha_{\nu_1}\right|^2 \sum_{j=n+1}^{l_{\nu_1}} \left|\sum_{k=1}^{\infty} \varphi(a_{jk}) x_k\right|^2 + \sum_{\nu=\nu_1+1}^{\infty} |\alpha_\nu|^2 \sum_{j=j_\nu}^{l_\nu} \left|\sum_{k=1}^{\infty} \varphi(a_{jk}) x_k\right|^2$$

$$\leq \frac{1}{\nu_1^2} \sum_{j=1}^{\infty} \left|\sum_{k=1}^{\infty} \varphi(a_{jk}) x_k\right|^2 + \sum_{\nu=\nu_1+1}^{\infty} \frac{1}{\nu^2} \sum_{j=1}^{\infty} \left|\sum_{k=1}^{\infty} \varphi(a_{jk}) x_k\right|^2$$

$$\leq \frac{\|A\|^2}{\nu_1^2} \|x\|_{\ell^2}^2 + \sum_{\nu=\nu_2}^{\infty} \frac{\|A\|^2}{\nu^2} \|x\|_{\ell^2}^2 < \frac{\eta^2}{4} \|x\|_{\ell^2}^2 .$$

Since this is true for all $x \in \ell^2$, we see that

$$\left\|\widetilde{\varphi}(B - B_{\underline{n}})\right\|_{\mathcal{B}(\ell^2)} \leq \frac{\eta}{2}.$$

But $\varphi \in s(\mathcal{A})$ is also arbitrary,

$$\left\|B - B_{\underline{n}}\right\| \leq \frac{\eta}{2} < \eta.$$

Since this is true for all $n \geq j_{\nu_0}$, we conclude that $B \in \mathcal{K}$.
On the other hand we also have

$$f(B) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{\nu=1}^{\infty} \alpha_\nu \sum_{j=j_\nu}^{l_\nu} \sum_{k=1}^{\infty} f_{jk}(a_{jk})$$

$$= \sum_{\nu=1}^{\infty} \frac{1}{\nu} \left|\sum_{j=j_\nu}^{l_\nu} \sum_{k=1}^{\infty} f_{jk}(a_{jk})\right| \geq \sum_{\nu=1}^{\infty} \frac{\epsilon}{\nu} = \infty,$$

contradicting $B \in \mathcal{K}$ and $f \in \mathcal{K}^\#$. Therefore

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \text{ converges.}$$

A similar argument shows that the sum in the other order for g also converges.

Uniform boundedness arguments similar to that used in the proof of Proposition 3.2 show that $\hat{f}$ and g are both bounded linear functionals on $\mathcal{M}$.

For $A \in \mathcal{M}$, since $A_{n|} \in \mathcal{K}$, for each $n \in \mathbb{N}$, by last part of the preceding proposition,

$$|g(A)| = \lim_{n\to\infty} \left|g(A_{n|})\right| = \lim_{n\to\infty} \left|f(A_{n|})\right| \leq \limsup_{n\to\infty} \|f\| \left\|A_{n|}\right\| \leq \|f\| \|A\|,$$

thus $\|g\| \leq \|f\|$. Also $g|_{\mathcal{K}} = f$, we see that $\|g\| \geq \|f\|$, and thus $\|f\| = \|g\|$. Similarly $\left\|\hat{f}\right\| = \|f\|$.

To see that the two sums are equal, we first show that the sequence $\{g_n\}$ defined by

$$g_n(A) := \sum_{k=1}^{n} \sum_{j=1}^{\infty} f_{jk}(a_{jk}) \qquad \left(A = [a_{jk}] \in \mathcal{K} \right)$$

is a Cauchy sequence in $\mathcal{K}^\#$. Suppose $\{g_n\}$ is not a Cauchy sequence in $\mathcal{K}^\#$. Then there exist an $\epsilon > 0$ and sequences $\{k_\nu\}_{\nu\in\mathbb{N}}$, $\{l_\nu\}_{\nu\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$l_{\nu-1} + 1 \leq k_\nu < l_\nu \ \ (\text{where } l_0 = 0), \ \text{ and } \qquad \left\|g_{l_\nu} - g_{k_\nu}\right\| > 2\epsilon \quad \text{ for all } \nu \in \mathbb{N}.$$

Thus there are elements $A_\nu \in \mathcal{K}$ such that

$$\|A_\nu\| = 1 \ \text{ and } \ \left|g_{l_\nu}(A_\nu) - g_{k_\nu}(A_\nu)\right| > 2\epsilon.$$

Let

$$\alpha_\nu = \frac{1}{\nu} \operatorname{sgn}\left[g_{l_\nu}(A_\nu) - g_{k_\nu}(A_\nu)\right] \quad \text{ and } \quad B = \sum_{\nu=1}^{\infty} \alpha_\nu A_\nu.$$

Then an argument similar to that used above shows that

$$B \in \mathcal{K} \qquad \text{but} \qquad g(B) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f_{jk}(B(j,k)) \ \text{ diverges,}$$

which is a contradiction. Therefore $\{g_n\}$ is a Cauchy sequence in $\mathcal{K}^\#$. Thus there is an $h \in \mathcal{K}^\#$ such that

$$\|g_n - h\|_{\mathcal{K}^\#} \to 0.$$

But since each $A \in \mathcal{K}$ has $\left\|A - A_{n|}\right\| \to 0$, also $g \in \mathcal{K}^\#$ and $g_n(A) = g(A_{n|})$, we have

$$g_n(A) \to g(A) \ \text{ for each } A \in \mathcal{K}.$$

Thus $g = h$ and hence

$$\|g_n - g\|_{\mathcal{K}^\#} \to 0.$$

For each $A = [a_{jk}] \in \mathcal{M}$, since

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{ and } \quad \sum_{j=1}^{\infty} \sum_{k=1}^{n} f_{jk}(a_{jk}) \ \text{ converge for all } \ n \in \mathbb{N},$$

$$\begin{aligned}(\hat{f} - g_n)(A) =& \hat{f}(A) - g_n(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) - \sum_{j=1}^{\infty} \sum_{k=1}^{n} f_{jk}(a_{jk}) \\ =& \sum_{j=1}^{\infty} \sum_{k=n+1}^{\infty} f_{jk}(a_{jk}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \tilde{f}_{jk}(a_{jk})\end{aligned}$$

where

$$\tilde{f}_{jk} = \begin{cases} f_{jk} & \text{for } k > n \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\widehat{(f - g_n)} = \hat{f} - g_n$, and, by Proposition 3.2, that $f = g$ on $\mathcal{K}$. Thus, from the first part, we have

$$\begin{aligned} \lim_{n\to\infty} \left\| \hat{f} - g_n \right\|_{\mathcal{M}^\#} &= \lim_{n\to\infty} \left\| \widehat{(f - g_n)} \right\|_{\mathcal{M}^\#} = \lim_{n\to\infty} \| f - g_n \|_{\mathcal{K}^\#} \\ &= \lim_{n\to\infty} \| g - g_n \|_{\mathcal{K}^\#} = 0. \end{aligned}$$

Therefore

$$\hat{f}(A) = \lim_{n\to\infty} g_n(A) \qquad \text{for all} \quad A \in \mathcal{M},$$

and hence, for each $A = [a_{jk}] \in \mathcal{M}$,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) = \hat{f}(A) = \lim_{n\to\infty} g_n(A) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f_{jk}(a_{jk}).$$

□

Note that this proposition corresponds to the fact that the trace functional satisfies $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for a trace class A and bounded B on a Hilbert space. The proof of this proposition can easily be adapted to a proof of the trace identity. Since each $[f_{jk}] \in \mathcal{K}^\#$ defines a bounded linear functional

$$\hat{f}(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \qquad (A = [a_{jk}] \in \mathcal{M})$$

on $\mathcal{M}$ with the same norm $\left\| \hat{f} \right\|_{\mathcal{M}^\#} = \| [f_{jk}] \|_{\mathcal{K}^\#}$. The space of all such linear functionals $\hat{f}$ will be denoted by $\widehat{\mathcal{K}^\#}$

4. The main theorem

Now we are ready for the main Dixmier's theorem. Denote by $\mathcal{K}^\perp$ the subspace of $\mathcal{M}^\#$ consisting of bounded linear functionals on $\mathcal{M}$ that vanish on $\mathcal{K}$.

Theorem 4.1. *For each $f \in \mathcal{M}^\#$, there is a unique pair $g \in \widehat{\mathcal{K}^\#}$ and $h \in \mathcal{K}^\perp$ such that*

$$f = g + h \qquad \text{and} \qquad \|f\| = \|g\| + \|h\|.$$

Proof. For each $(j, k) \in \mathbb{N}\times\mathbb{N}$, define f_{jk} by $f_{jk}(a) = f(E_{jk}(a))$ for all $a \in \mathcal{A}$. Then $f_{jk} \in \mathcal{A}^\#$ with $\| f_{jk} \| \le \|f\|$. Then as in the proof of Proposition 3.2 the matrix $[f_{jk}]$ represents a bounded linear functional $\tilde{f} = f|_{\mathcal{K}}$ on $\mathcal{K}$. By Proposition 3.3, $[f_{jk}]$ defines a bounded linear functional $g = \widehat{(\tilde{f})} = \widehat{(f|_{\mathcal{K}})}$ on $\mathcal{M}$, where

$$g(A) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \quad \text{for all} \quad A = [a_{jk}] \in \mathcal{M},$$

and

$$\|g\|_{\mathcal{M}^\#} = \left\|\widetilde{f}\right\|_{\mathcal{K}^\#}.$$

Let $h = f - g$. It is clear that $h \in \mathcal{K}^\perp$. The uniqueness of the decomposition follows from the fact that $\widehat{\mathcal{K}^\#} \oplus \mathcal{K}^\perp = \mathcal{M}^\#$ is a direct sum.

Since $\|f\| \leq \|g\| + \|h\|$, it suffices to prove that $\|f\| \geq \|g\| + \|h\|$. Let $\epsilon > 0$. Since $\|g\|_{\mathcal{M}^\#} = \left\|g|_{\mathcal{K}}\right\|$, there is an $A = \left[a_{jk}\right] \in \mathcal{K}$ such that

$$\|A\| = 1 \quad \text{and} \quad g(A) > \|g\| - \frac{\epsilon}{8}.$$

There is also a $B = \left[b_{jk}\right] \in \mathcal{M}$ such that

$$\|B\| = 1 \quad \text{and} \quad h(B) > \|h\| - \frac{\epsilon}{8}.$$

Form the convergence of the double sum, there is a j_0 such that

$$\left|\sum_{j=n}^{\infty}\sum_{k=1}^{\infty} f_{jk}(a_{jk})\right| < \frac{\epsilon}{8} \quad \forall \;\; n > j_0.$$

There is also a k_0 such that

$$\left|\sum_{j=1}^{j_0}\sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk})\right| < \frac{\epsilon}{8}.$$

By Proposition 3.3,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f_{jk}(b_{jk}),$$

thus there is a $j_1 \geq j_0$ such that

$$\left|\sum_{j=j_1+1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(b_{jk})\right| < \frac{\epsilon}{8}.$$

Put

$$\hat{f}_{jk} = \begin{cases} 0 & \text{if } 1 \leq j \leq j_1 \\ f_{jk} & \text{if } j_1 < j \end{cases}$$

Then $\left[\hat{f}_{jk}\right] \in \mathcal{K}^\#$. Thus

$$\sum_{j=j_1+1}^{\infty}\sum_{k=1}^{\infty} f_{jk}(b_{jk}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \hat{f}_{jk}(b_{jk}) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \hat{f}_{jk}(b_{jk})$$

$$= \sum_{k=1}^{\infty}\sum_{j=j_1+1}^{\infty} f_{jk}(b_{jk})$$

converges, and hence there is a $k_1 \geq k_0$ such that

$$\left| \sum_{k=k_1+1}^{\infty} \sum_{j=j_1+1}^{\infty} f_{jk}(b_{jk}) \right| < \frac{\epsilon}{8}.$$

Let

$$A_0(j,k) = \begin{cases} a_{jk} & \text{if } 1 \leq j \leq j_0, \text{ and } 1 \leq k \leq k_0 \\ 0 & \text{otherwise,} \end{cases}$$

$$B_0(j,k) = \begin{cases} b_{jk} & \text{if } j_1 < j, \text{ and } k_1 < k \\ 0 & \text{otherwise;} \end{cases}$$

and let $C = \left[c_{jk}\right] = A_0 + B_0$. Then $\|C\| = \max\{\|A_0\|, \|B_0\|\} \leq 1$. Since $h \in \mathcal{K}^{\perp}$, and $A_0, B - B_0 \in \mathcal{K}$, we have $h(A_0) = 0$, and hence $h(B) = h(B_0)$. Therefore

$$\begin{aligned} \|f\| &\geq |f(C)| = |g(A_0) + g(B_0) + h(A_0) + h(B_0)| \\ &\geq |g(A_0) + h(B_0)| - |g(B_0)| > \mathrm{Re}[g(A_0)] + \mathrm{Re}[h(B_0)] - \frac{\epsilon}{8} \\ &= \mathrm{Re}\left[g(A) - \sum_{j=j_0+1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) - \sum_{j=1}^{j_0} \sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk}) \right] + h(B) - \frac{\epsilon}{8} \\ &> \|g\| - \frac{\epsilon}{8} - \left| \sum_{j=j_0+1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(a_{jk}) \right| - \left| \sum_{j=1}^{j_0} \sum_{k=k_0+1}^{\infty} f_{jk}(a_{jk}) \right| + \|h\| - \frac{\epsilon}{4} \\ &> \|g\| + \|h\| - \frac{5\epsilon}{8} > \|g\| + \|h\| - \epsilon. \end{aligned}$$

Since the preceding argument holds for every $\epsilon > 0$, we conclude that

$$\|f\| \geq \|g\| + \|h\|.$$

□

We note that when $\mathcal{A}$ is the complex field $\mathbb{C}$, then $s(\mathcal{A})$ consists of the identity map alone. So a matrix A over $\mathbb{C}$ is in $\mathcal{M}$ iff A is in $\mathcal{B}(\ell^2)$ and A is in $\mathcal{K}$ iff A is in $\mathcal{K}(\ell^2)$. A matrix defines a bounded linear functional on $\mathcal{K}(\ell^2)$ iff it is represented by a trace class matrix and hence it is a trace class matrix itself. Thus Dixmier's Theorem is an immediate consequence of this result.

Acknowledgments. We would like to thank the referee for his/her suggestions for improvements. T. Wootijirattikal was supported in part by a research grant from Ubon Ratchathani University. This work was done while S.-C Ong was on a sabbatical leave from Central Michigan University visiting Silpakorn University and Ubon Ratchathani University. He wishes to thank Jitti Rakbud and Titarii Wootijirattikal and their departments for their warm hospitality during his visit.

References

[1] E.M. Alfsen and E.G. Effros, *Structure in real Banach spaces*, Ann. of Math. **96** (1972), 98–173.

[2] J. Dixmier, *Les fonctionnelles lin'earies sur l'ensemble des op'erateurs born'es dans espace de Hilbert,*, Ann. of Math. **51** (1950), 387–408.

[3] P. Harmand, D. Werner and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Mathematics, vol. 1547, Springer-Verlag, 1993.

[4] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, vol. I, Academic Press, New York, 1983.

[5] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Erg. d. Math. N. F., vol. 27, Springer-Verlag, Berlin, 1960.

[6] R.R. Smith and J.D. Ward, *M-ideal structure in Banach algebras*, J. Funct. Anal. **27** (1978), 337–349.

[7] ———, *Application of convexity and M-ideal theory to quotient Banach algebras*, Quart. J. Math. Oxford **30** (1979), 365–384.

[8] O. Wootijiruttikal, S.C. Ong, P. Chaisuriya and J. Rakbud, *Banach spaces of fucntions taking values in a C^*-algebra*, Oper. Matrices **3** (2009), 373–396.

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