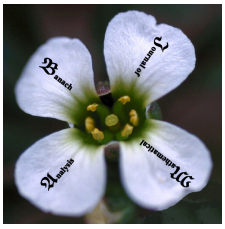

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A VERSION OF THE HERMITE–HADAMARD INEQUALITY IN A NONPOSITVE CURVATURE SPACE

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Abstract. We obtain some Hermite–Hadamard type inequalities for convex functions in a global non-positive curvature space.

1. Introduction

Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. For instance, a (strictly) convex function on an open set has no more than one minimum. Even in infinite-dimensional spaces, under suitable additional hypotheses, convex functions continue to satisfy such properties and, as a result, they are the most well-understood functionals in the calculus of variations. In particular, if f is a convex function defined on $I = [a, b]$ we have that

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

One of the most important inequalities, that has attracted many inequality experts in the last few decades, is the famous Hermite–Hadamard inequality which establishes a refinement of (1.1) and it involves the notions of convexity and geodesic (see Section 3).

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On the other hand, the study of non-positively curved spaces began with the work of Hadamard in the early years of the last century, and the work of Cartan about twenty years later. But the foundations of the theory of metric spaces with upper curvature bounds were laid in the 50's with the work of Alexandrov and Busemann [1, 2, 6], who actually coined the term "non-positively curved space".

Non-positive curvature in the sense of Alexandrov (NPC) states that sufficiently small geodesic triangles in the inner metric space (N, d) are at least as thin as corresponding Euclidean triangles, or equivalently, if the metric space (N, d) verifies the so-called CN-inequality of Bruhat and Tits [5]: for any $x \in X$ and any geodesic segment $\gamma \in X$,

$$\frac{1}{4}L(\gamma)^2 \leq \frac{1}{2}(d(x,\gamma_0)^2 + d(x,\gamma_1)^2) - d(x,\gamma_{1/2})^2,$$

provided γ is sufficiently close to x and $L(\gamma)$ denotes the length of γ.

The aim of this paper is to discuss an analogue of the Hermite–Hadamard inequality for convex functions defined on a space with curved geometry, more precisely on a metric space with global non-positive curvature. The NPC space theory offers a considerable insight into this matter, based on the fact that $(a + b)/2$ is the midpoint of the line segment which connects a with b, the mean value of f is the integral on the interval $[0, 1]$ of $f \circ \alpha_{a,b}$ (see Lemma 3.1) and in these spaces there is a notion of convex function.

2. Global NPC

We highlight here some of the results, terminology and definitions that we shall need in this paper.

Definition 2.1. Let (N, d) be a complete metric space, which is also a *geodesic length space* in the sense that the distance of N can be computed via the infimum of the length of the rectifiable arcs joining given endpoints in N (see [11, Section 2.2]). A geodesic length space is *globally non-positively curved in the sense of Busemann* if for given geodesic arcs α, β starting at $x \in X$, the distance map $t \mapsto d(\alpha(t), \beta(t))$ is a convex function.

We say that (N, d) is a *global NPC space* if for $x_1, x_2 \in N$ there exists a point $z \in N$ such that for each $x \in N$ we have

$$d(x,z)^2 \leq \frac{1}{2}d(x,x_1)^2 + \frac{1}{2}d(x,x_2)^2 - \frac{1}{4}d(x_1,x_2)^2.$$

Global NPC spaces are also called Hadamard spaces. If (N, d) is a global NPC space, then it is globally non-positively curved in the sense of Busemann. In a global NPC space each pair of points $x_1, x_2 \in N$ can be connected by a geodesic (that is, by a rectifiable curve $\gamma : [0, 1] \to N$ such that the length of $\gamma|_{[t_1,t_2]}$ is $d(\gamma(t_1), \gamma(t_2))$ for all $0 \leq t_1 \leq t_2 \leq 1$). Moreover, this geodesic is unique. Finally, note that the point z occuring in the preceding definition plays the role of a midpoint between x_1 and x_2.

The following classes of spaces are NPC: complete Riemannian manifolds with non-positive sectional curvature, Hilbert spaces, Bruhat Tits buildings, in particular metric trees.

Definition 2.2. A subset $C \subseteq N$ is called convex if for each geodesic $\gamma : [0,1] \to N$ joining two arbitrary points in C holds that $\gamma([0,1]) \subseteq C$.

A function $f : C \to \mathbb{R}$ is called convex if the function $f \circ \gamma : [0,1] \to \mathbb{R}$ is convex whenever $\gamma : [0,1] \to C$ is geodesic, that is, for all $t \in [0,1]$

$$f(\gamma(t)) \leq (1-t)f(\gamma(0)) + tf(\gamma(1)).$$

Now, we state a lemma which is interesting on his own right and contains some basic properties of geodesics and convex functions in a NPC which we use in the sequel.

Lemma 2.3. *Let (N,d) be a global NPC space, $C \subseteq N$ a convex set and $\gamma : [0,1] \to C$ a geodesic connecting $\gamma(0)$ with $\gamma(1)$. Then*

(1) *For $t_1, t_2 \in [0,1]$ the curve $\gamma|_{[t_1,t_2]}(\lambda) = \gamma((1-\lambda)t_1 + \lambda t_2)$ is the unique geodesic connecting $\gamma(t_1)$ with $\gamma(t_2)$.*
(2) *For any $t_0 \in [0,1]$ the midpoint between $\gamma(t_0)$ and $\gamma(1-t_0)$ is given by $\gamma(1/2)$.*
(3) *If $f : C \to \mathbb{R}$ is convex, then $\displaystyle\int_0^1 f(\gamma(\nu))d\nu = \int_0^1 f(\gamma(1-\lambda))d\lambda$.*

Proof. (1) For $\lambda \in [0,1]$ we get

$$\begin{aligned} d(\gamma|_{[t_1,t_2]}(\lambda), \gamma(t_1)) &= d(\gamma((1-\lambda)t_1 + \lambda t_2), \gamma(t_1)) \\ &= |t_1 - (1-\lambda)t_1 - \lambda t_2| d(\gamma(0), \gamma(1)) \\ &= \lambda |t_2 - t_1| d(\gamma(0), \gamma(1)) = \lambda d(\gamma(t_1), \gamma(t_2)), \end{aligned}$$

and

$$\begin{aligned} d(\gamma|_{[t_1,t_2]}(\lambda), \gamma(t_2)) &= d(\gamma((1-\lambda)t_1 + \lambda t_2), \gamma(t_2)) \\ &= |t_2 - (1-\lambda)t_1 - \lambda t_2| d(\gamma(0), \gamma(1)) \\ &= (1-\lambda)|t_2 - t_1| d(\gamma(0), \gamma(1)) = (1-\lambda) d(\gamma(t_1), \gamma(t_2)). \end{aligned}$$

(2) Clear.
(3) With the change of variables $\nu = 1 - \lambda$ we obtain the equality desired. □

3. Hermite–Hadamard Inequality

For a convex function f on $I = [a,b]$, the double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{3.1}$$

is known as the Hermite–Hadarmad (H-H) inequality. This inequality was published by Hermite in 1883 and was independently proved by Hadamard in 1893 and it gives us an estimation of the mean value of a convex function f and it is important to note that (3.1) provides a refinement to the Jensen inequality. The interested reader is referred to [3, 9, 13, 14] and references therein for more information and other extensions of H-H inequality.

In order to prove that a H-H inequality in NPC, we shall need the following lemma which describes the mean value of f in the interval $[a,b]$ using the "geodesic" connecting a to b.

Lemma 3.1. *Let f be an integrable function on I. Then*

$$\frac{1}{b-a}\int_a^b f(x)dx \;=\; \int_0^1 f(\lambda a+(1-\lambda)b)d\lambda = \int_0^1 f(\lambda b+(1-\lambda)a)d\lambda.$$

Thus the H-H inequality can be written as follows

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f(\lambda a+(1-\lambda)b)d\lambda \leq \frac{f(a)+f(b)}{2}. \tag{3.2}$$

Let X be a vector space, $a,b\in X; a\neq b$. We denote by

$$\alpha_{a,b}=\{(1-\lambda)a+\lambda b, \lambda\in[0,1]\}$$

the line segment connecting a to b and $\alpha_{a,b}(\lambda_0)=(1-\lambda_0)a+\lambda_0 b$. We consider function $f:\alpha_{a,b}\to\mathbb{R}$ and the associated function

$$g_{a,b}:[0,1]\to\mathbb{R} \qquad g_{a,b}(\lambda)=f(\alpha_{a,b}(\lambda)),$$

then f is convex on $\alpha_{a,b}$ if and only if $g_{a,b}$ is convex on $[0,1]$. Then for any convex function defined on a segment $\alpha_{a,b}\subseteq X$, we have the Hermite–Hadamard integral inequality (see [8])

$$f\left(\alpha_{a,b}(1/2)\right)\leq\int_0^1 f(\alpha_{a,b}(\lambda))d\lambda\leq\alpha_{f(a),f(b)}(1/2).$$

Remark 3.2. Let $(X,\|.\|)$ be a normed space, $x,y\in X$ and f a convex function defined on the segment $\alpha_{x,y}\subset X$. Therefore, we obtain as a consequence of (3.2) the following norm inequality for $1\leq p<\infty$

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-\lambda)x+\lambda y\|^p d\lambda \leq \frac{\|x\|^p+\|y\|^p}{2}, \tag{3.3}$$

by considering $f(x)=\|x\|^p$. This example has been obtained by Dragomir in [8]. For $p=1$, we improve a refinement of the triangle inequality. The right hand side of (3.3) resembles the p-norm of the pair $(x,y)\in X^2$, see [12].

From the previous statement, we conclude that the necessary notions to obtain an analogue of the H-H inequality in a global NPC space N are: the existence of a unique geodesic connecting two points given, the notion of convex function associated with this privileged curve and convexity. Since these concepts exist in the context of global NPC spaces, we obtain a generalization of the H-H inequality to such spaces.

Theorem 3.3. *Let (N,d) be a global NPC space, $C\subseteq N$ a convex set and $f:C\to\mathbb{R}$ a convex function. Then*

$$f\left(\gamma(1/2)\right)\leq\int_0^1 f(\gamma(\lambda))\,d\lambda\leq\alpha_{f(\gamma(0)),f(\gamma(1))}(1/2) \tag{3.4}$$

for all geodesic $\gamma:[0,1]\to C$.

Proof. Since f is convex, we have

$$f\left(\gamma(1/2)\right) \leq \frac{1}{2}f(\gamma(\lambda)) + \frac{1}{2}f(\gamma(1-\lambda)) \leq \frac{1}{2}f(\gamma(0)) + \frac{1}{2}f(\gamma(1)) \tag{3.5}$$

for all $\lambda \in [0,1]$. Integrating (3.5) over $[0,1]$ and using Lemma 2.3 we get (3.4). □

Remark 3.4. (1) Note that the first inequality, in (3.4), is stronger than the second, i.e.

$$0 \leq \int_0^1 f(\gamma(\lambda))\, d\lambda - f\left(\gamma(1/2)\right) \leq \alpha_{f(\gamma(0)),f(\gamma(1))}(1/2) - \int_0^1 f(\gamma(\lambda))\, d\lambda.$$

Indeed, if we suppose that γ is the unique geodesic connecting a to b and $m = m(a,b)$ denotes the unique midpoint between a and b, then

$$\begin{aligned} 2\int_0^1 f(\gamma_{a,b}(\lambda))\, d\lambda &= 2\int_0^{1/2} f(\gamma_{a,b}(\lambda))\, d\lambda + 2\int_{1/2}^1 f(\gamma_{a,b}(\lambda))\, d\lambda \\ &= \int_0^1 f(\gamma_{a,m}(\lambda))\, d\lambda + \int_0^1 f(\gamma_{m,b}(\lambda))\, d\lambda \\ &\leq \frac{f(\gamma_{a,m}(0)) + f(\gamma_{a,m}(1))}{2} + \frac{f(\gamma_{m,b}(0)) + f(\gamma_{m,b}(1))}{2} \\ &= f(\gamma_{a,b}(1/2)) + \frac{f(\gamma_{a,b}(0)) + f(\gamma_{a,b}(1))}{2}. \end{aligned}$$

(2) If (N,d) is a global NPC space and $z \in N$, the distance from a point z, $d_z(x) = d(x,z)$, provides a basic example of a convex function. Moreover, its square is strictly convex and as a consequence, the balls in a global NPC space are convex sets. For more details, see [4]. Then, given $z, x_0, x_1 \in N$ and $p \geq 1$, we get that $f_z(t) = d^p(z, \gamma_{x_0,x_1}(t))$, with $t \in [0,1]$, is a convex function. Hence

$$d^p(z, \gamma_{x_0,x_1}(1/2)) \leq \int_0^1 d^p(z, \gamma_{x_0,x_1}(\lambda))\, d\lambda \leq \frac{d^p(z,x_0) + d^p(z,x_1)}{2}. \tag{3.6}$$

This inequality is analogous to (3.3) in the context of global NPC spaces.

On the other hand, using the convexity of the function $g : [0,1] \to \mathbb{R}$, $g(t) = d^2(\gamma(t), \eta(t))$ with γ, η geodesics we obtain the following refinement of (3.6) for $p = 2$.

Corollary 3.5. *Let (N,d) be a global NPC space and $\gamma_{x_0,x_1}, \eta_{y_0,y_1}$ two geodesics in N. Then*

$$\begin{aligned} d^2(\eta_{y_0,y_1}(1/2), \gamma_{x_0,x_1}(1/2)) &\leq \int_0^1 d^2(\eta_{y_0,y_1}(\lambda), \gamma_{x_0,x_1}(\lambda))\, d\lambda \\ &\leq \frac{d^2(y_0,x_0) + d^2(y_1,x_1)}{2} - \frac{1}{6}[d(y_0,y_1) - d(x_0,x_1)]^2 \\ &\leq \frac{d^2(y_0,x_0) + d^2(y_1,x_1)}{2}. \end{aligned} \tag{3.7}$$

In particular, if $\eta_{y_0,y_1}(t) = z$ *for all* $t \in [0,1]$ *we get*

$$\begin{aligned} d^2(z, \gamma_{x_0,x_1}(1/2)) &\leq \int_0^1 d^2(z, \gamma_{x_0,x_1}(\lambda))\, d\lambda \\ &\leq \frac{d^2(z,x_0) + d^2(z,x_1)}{2} - \frac{1}{6} d^2(x_0,x_1) \\ &\leq \frac{d^2(z,x_0) + d^2(z,x_1)}{2}. \end{aligned}$$

Proof. We recall the called *geodesic comparison* (see [15]) which establishes that in any global NPC and $\lambda \in [0,1]$ holds

$$\begin{aligned} d^2(\eta_{y_0,y_1}(\lambda), \gamma_{x_0,x_1}(\lambda)) &\leq (1-\lambda) d^2(y_0,x_0) + \lambda d^2(y_1,x_1) \\ &- \lambda(1-\lambda)[d(y_0,y_1) - d(x_0,x_1)]^2. \end{aligned} \tag{3.8}$$

Then the inequality (3.7) is consequence of the H-H inequality for global NPC spaces and (3.8).

□

Remark 3.6. It is well-known that in a Hilbert space H one has

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

with $x, y \in H$ and $\lambda \in [0,1]$. In a p-uniformly convex Banach space E (with $p > 1$) or equivalently in a Banach space where the following inequality holds (see [16])

$$\|(1-\lambda)x + \lambda y\|^p \leq (1-\lambda)\|x\|^p + \lambda\|y\|^p - W_p(\lambda) c \|x-y\|^p,$$

for all $x, y \in E$, $\lambda \in [0,1]$, $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$ and c a positive constant, we obtain the following refinement of (3.3) for all $p > 1$,

$$\begin{aligned} \left\| \frac{x+y}{2} \right\|^p &\leq \int_0^1 \|(1-\lambda)x + \lambda y\|^p d\lambda \leq \frac{\|x\|^p + \|y\|^p}{2} - \frac{2c}{(p+1)(p+2)} \|x-y\|^p \\ &\leq \frac{\|x\|^p + \|y\|^p}{2}. \end{aligned}$$

From now on, we obtain different refinements of (3.4).

Proposition 3.7. *Let* (N,d) *be a global NPC space,* $C \subseteq N$ *a convex set and* $f : C \to \mathbb{R}$ *a convex function, then*

$$\begin{aligned} f(\gamma(1/2)) &\leq \frac{1}{2}[f(\gamma(2/3)) + f(\gamma(1/3))] \leq \frac{1}{2}[f(\gamma(1/4)) + f(\gamma(3/4))] \\ &\leq \int_0^1 f(\gamma(t))\, dt \\ &\leq \frac{1}{2}\left[f(\gamma(1/2)) + \frac{f(\gamma(0)) + f(\gamma(1))}{2} \right] \leq \alpha_{f(\gamma(0)),f(\gamma(1))}(1/2), \end{aligned}$$

for all geodesic $\gamma : [0,1] \to C$.

Proof. To prove the first and the second inequality, we use the convexity of f and the following inequalities: $\frac{1}{2} = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{3}\right)$, $\frac{1}{3} = \frac{5}{6}\frac{1}{4} + \frac{1}{6}\frac{3}{4}$, and $\frac{2}{3} = \frac{1}{6}\frac{1}{4} + \frac{5}{6}\frac{3}{4}$.

On the other hand, utilizing the Hermite–Hadamard inequality (3.4) we can write

$$f\left(\gamma(1/4)\right) \leq \int_0^1 f(\gamma|_{[0,1/2]}(\lambda))d\lambda \leq \alpha_{f(\gamma(0)),f(\gamma(1/2))}(1/2),$$

and

$$f\left(\gamma(3/4)\right) \leq \int_0^1 f(\gamma|_{[1/2,1]}(\lambda))d\lambda \leq \alpha_{f(\gamma(1/2)),f(\gamma(1))}(1/2),$$

which by a change of variable in the integrals, sumation and division by two we get the third and fourth desired inequalities. Finally, the last inequality is clear by (3.4). □

We generalize the previous result as follows:

Theorem 3.8. *Let (N,d) a global NPC space, $C \subseteq N$ a convex set, $f : C \to \mathbb{R}$ a convex function and k, p are positive integers, then*

$$\begin{aligned} f\left(\gamma(1/2)\right) &\leq \frac{1}{k^p}\sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) \leq \int_0^1 f(\gamma(t))\,dt \\ &\leq \frac{1}{2k^p}\sum_{i=0}^{k^p-1}\left[f\left(\gamma\left(\frac{i+1}{k^p}\right)\right) + f\left(\gamma\left(\frac{i}{k^p}\right)\right)\right] \\ &\leq \alpha_{f(\gamma(0)),f(\gamma(1))}(1/2), \end{aligned} \tag{3.9}$$

for all geodesic $\gamma : [0,1] \to C$.

Proof. Utilizing the Hermite–Hadamard inequality, we get

$$f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) \leq \int_0^1 f(\gamma|_{[\frac{i}{k^p},\frac{i+1}{k^p}]}(t))\,dt \leq \frac{f\left(\gamma(\frac{i}{k^p})\right) + f\left(\gamma(\frac{i+1}{k^p})\right)}{2}.$$

Summation of the above inequalities over $i = 0, 1, \cdots, k^p - 1$ yields

$$\sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) \leq k^p \int_0^1 f\left(\gamma(t)\right)\,dt \leq \sum_{i=0}^{k^p-1} \frac{f\left(\gamma(\frac{i}{k^p})\right) + f\left(\gamma(\frac{i+1}{k^p})\right)}{2}.$$

Hence

$$\frac{1}{k^p}\sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) \leq \int_0^1 f(\gamma(t))\,dt \leq \frac{1}{k^p}\sum_{i=0}^{k^p-1} \frac{f\left(\gamma(\frac{i}{k^p})\right) + f\left(\gamma(\frac{i+1}{k^p})\right)}{2}. \tag{3.10}$$

By the convexity of $f \circ \gamma$ we have

$$\frac{1}{k^p}\sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) \geq f\left[\gamma\left(\frac{1}{k^p}\sum_{i=0}^{k^p-1}\left(\frac{2i+1}{2k^p}\right)\right)\right] = f\left(\gamma(1/2)\right) \tag{3.11}$$

and

$$\frac{1}{k^p}\sum_{i=0}^{k^p-1}\frac{f\left(\gamma(\frac{i}{k^p})\right)+f\left(\gamma(\frac{i+1}{k^p})\right)}{2}=\frac{1}{2k^p}\sum_{i=0}^{k^p-1}\left[f\left(\gamma\left(\frac{i+1}{k^p}\right)\right)+f\left(\gamma\left(\frac{i}{k^p}\right)\right)\right]$$
$$\leq\frac{f(\gamma(0))+f(\gamma(1))}{2}. \tag{3.12}$$

Now (3.10), (3.11) and (3.12) yield the whole inequalities (3.9) as desired. □

We remark, that for $k^p=2$ in the previous Theorem we obtain the Proposition 3.7. The following statement was motivated by [10].

Theorem 3.9. *Let (N,d) a global NPC space, $C\subseteq N$ a convex set and $f:C\to\mathbb{R}$ a convex function, then*

$$f\left(\gamma(1/2)\right)\leq l(\lambda)\leq\int_0^1 f(\gamma(t))dt\leq L(\lambda)\leq\alpha_{f(\gamma(0)),f(\gamma(1))}(1/2),$$

for all geodesic $\gamma:[0,1]\to C$ and $\lambda\in[0,1]$, where

$$l(\lambda)=\lambda f\left(\gamma\left(\frac{\lambda}{2}\right)\right)+(1-\lambda)f\left(\gamma\left(\frac{1+\lambda}{2}\right)\right),$$

and

$$L(\lambda)=\frac{1}{2}[f(\gamma(\lambda))+\lambda f(\gamma(0))+(1-\lambda)f(\gamma(1))].$$

Proof. Let f a convex function on C, applying (3.4) to the geodesic which connects $\gamma(0)$ to $\gamma(\lambda)$, with $\lambda\neq 0$, we get

$$f\left(\gamma\left(\frac{\lambda}{2}\right)\right)\leq\int_0^1 f(\gamma|_{[0,\lambda]}(t))dt\leq\frac{f(\gamma(0))+f(\gamma(\lambda))}{2}, \tag{3.13}$$

and

$$f\left(\gamma\left(\frac{1+\lambda}{2}\right)\right)\leq\int_0^1 f(\gamma|_{[\lambda,1]}(t))dt\leq\frac{f(\gamma(\lambda))+f(\gamma(1))}{2}. \tag{3.14}$$

Multiplying (3.13) by λ, (3.14) by $(1-\lambda)$ and adding the resulting inequalities we obtain

$$l(\lambda)\leq\lambda\int_0^1 f(\gamma|_{[0,\lambda]}(t))dt+(1-\lambda)\int_0^1 f(\gamma|_{[\lambda,1]}(t))dt=\int_0^1 f(\gamma(t))dt\leq L(\lambda).$$

On the other hand, using the fact that f is convex we get

$$\begin{aligned}f(\gamma(1/2)) &= f\left(\gamma\left(\lambda\frac{\lambda}{2}+(1-\lambda)\frac{(1+\lambda)}{2}\right)\right)\leq l(\lambda)\leq\int_0^1 f(\gamma(t))dt\leq L(\lambda)\\ &\leq \alpha_{f(\gamma(0)),f(\gamma(1))}(1/2).\end{aligned}$$

□

From the last result, we can conclude under the same hypothesis that

$$f\left(\gamma(1/2)\right)\leq\sup_{\lambda\in[0,1]}l(\lambda)\leq\int_0^1 f(\gamma(t))dt\leq\inf_{\lambda\in[0,1]}L(\lambda)\leq\alpha_{f(\gamma(0)),f(\gamma(1))}(1/2).$$

Remark 3.10. The different results of this paper remain valid in the context of *Alexandrov p-space*, i.e. if (N, d) is a geodesic length space that verifies the following geodesic curvature condition: given $x_1, x_2 \in N$ there exists a point $z \in N$ and a constant $K > 0$ such that for each $x \in N$ we have

$$d(x,z)^p \leq \frac{1}{2}(d(x,x_1)^p + d(x,x_2)^p) - \frac{1}{(2K)^p} d(x_1,x_2)^p.$$

A reference for this subject is [7].

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