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THE REFINED SOBOLEV SCALE, INTERPOLATION, AND ELLIPTIC PROBLEMS

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Abstract. The paper gives a detailed survey of recent results on elliptic problems in Hilbert spaces of generalized smoothness. The latter are the isotropic Hörmander spaces $H^{s,\varphi} := B_{2,\mu}$, with $\mu(\xi) = \langle\xi\rangle^s \varphi(\langle\xi\rangle)$ for $\xi \in \mathbb{R}^n$. They are parametrized by both the real number s and the positive function φ varying slowly at $+\infty$ in the Karamata sense. These spaces form the refined Sobolev scale, which is much finer than the Sobolev scale $\{H^s\} \equiv \{H^{s,1}\}$ and is closed with respect to the interpolation with a function parameter. The Fredholm property of elliptic operators and elliptic boundary-value problems is preserved for this new scale. Theorems of various type about a solvability of elliptic problems are given. A local refined smoothness is investigated for solutions to elliptic equations. New sufficient conditions for the solutions to have continuous derivatives are found. Some applications to the spectral theory of elliptic operators are given.

1. Introduction

In the theory of partial differential equations, the questions concerning the existence, uniqueness, and regularity of solutions are in the focus of investigations. Note that the regularity properties are usually formulated in terms of the belonging of solutions to some standard classes of function spaces. Thus, the finer a used scale of spaces is calibrated, the sharper and more informative results will be.

In contrast to the ordinary differential equations with smooth coefficients, the above questions are complicated enough. Indeed, some linear partial differential

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equations with smooth coefficients and right-hand sides are known to have no solutions in a neighbourhood of a given point, even in the class of distributions [58], [43, Sec. 6.0 and 7.3], [45, Sec. 13.3]. Next, certain homogeneous equations (specifically, of elliptic type) with smooth but not analytic coefficients have nontrivial solutions supported on a compact set [110], [45, Theorem 13.6.15]. Hence, the nontrivial null-space of this equation cannot be removed by any homogeneous boundary-value conditions; i.e., the operator of an arbitrary boundary-value problem is not injective. Finally, the question about the regularity of solutions is not simple either. For example, it is known [32, Ch. 4, Notes] that

$$\triangle u = f \in C(\Omega) \not\Rightarrow u \in C^2(\Omega),$$

with $\triangle$ being the Laplace operator, and Ω being an arbitrary Euclidean domain.

These questions have been investigated most completely for the elliptic equations, systems, and boundary-value problems. This was done in the 1950s and 1960s by S. Agmon, A. Douglis, L. Nirenberg, M.S. Agranovich, A.C. Dynin, Yu.M. Berezansky, S.G. Krein, Ya.A. Roitberg, F. Browder, L. Hermander, J.-L. Lions, E. Magenes, M. Schechter, L.N. Slobodetsky, V.A. Solonnikov, L.R. Volevich and some others (see, e.g., M.S. Agranovich's surveys [7, 8] and the references given therein). Note that the elliptic equations and problems have been investigated in the classical scales of Hölder spaces (of noninteger order) and Sobolev spaces (both of positive and negative orders).

The fundamental result of the theory of elliptic equations consists in that they generate bounded and Fredholm operators (i.e., the operators with finite index) between appropriate function spaces. For instance, let $Au = f$ be an elliptic linear differential equation of order m given a closed smooth manifold Γ. Then the operator

$$A : H^{s+m}(\Gamma) \to H^s(\Gamma), \quad s \in \mathbb{R},$$

is bounded and Fredholm. Moreover, the finite-dimensional spaces formed by solutions to homogeneous equations $Au = 0$ and $A^+v = 0$ both lie in $C^\infty(\Gamma)$. Here A^+ is the formally adjoint operator to A, whereas $H^{s+m}(\Gamma)$ and $H^s(\Gamma)$ are inner product Sobolev spaces over Γ and of the orders $s + m$ and s respectively. It follows from this that the solution u have an important regularity property on the Sobolev scale, namely

$$(f \in H^s(\Gamma) \text{ for some } s \in \mathbb{R}) \Rightarrow u \in H^{s+m}(\Gamma). \tag{1.1}$$

If the manifold has a boundary, then the Fredholm operator is generated by an elliptic boundary-value problem for the equation $Au = f$, specifically, by the Dirichlet problem.

Some of these results were extended by H. Triebel [133, 134] and the second author [92, 93] of the survey to finer scales of function spaces, namely the Nikolsky–Besov, Zygmund, and Lizorkin–Triebel scales.

The results mentioned above have various applications in the theory of differential equations, mathematical physics, the spectral theory of differential operators; see M.S. Agranovich's surveys [7, 8] and the references therein.

As for applications, especially to the spectral theory, the case of Hilbert spaces is of the most interest. Until recently, the Sobolev scale had been a unique scale of Hilbert spaces in which the properties of elliptic operators were investigated systematically. However, it turns out that this scale is not fine enough for a number of important problems.

We will give two representative examples. The first of them concerns with the smoothness properties of solutions to the elliptic equation $Au = f$ on the manifold Γ. According to Sobolev's Imbedding Theorem, we have

$$H^{\sigma}(\Gamma) \subset C^{r}(\Gamma) \quad \Leftrightarrow \quad \sigma > r + n/2, \tag{1.2}$$

where the integer $r \geq 0$ and $n := \dim \Gamma$. This result and property (1.1) allow us to investigate the classical regularity of the solution u. Indeed, if $f \in H^{s}(\Gamma)$ for some $s > r - m + n/2$, then $u \in H^{s+m}(\Gamma) \subset C^{r}(\Gamma)$. However, this is not true for $s = r - m + n/2$; i.e., the Sobolev scale cannot be used to express unimprovable sufficient conditions for belonging of the solution u to the class $C^{r}(\Gamma)$. An analogous situation occurs in the theory of elliptic boundary-value problems too.

The second demonstrative example is related to the spectral theory. Suppose that the differential operator A is of order $m > 0$, elliptic on Γ, and self-adjoint on the space $L_2(\Gamma)$. Given a function $f \in L_2(\Gamma)$, consider the spectral expansion

$$f = \sum_{j=1}^{\infty} c_j(f)\, h_j, \tag{1.3}$$

where $(h_j)_{j=1}^{\infty}$ is a complete orthonormal system of eigenfunctions of A, and $c_j(f)$ is the Fourier coefficient of f with respect to h_j. The eigenfunctions are enumerated so that the absolute values of the corresponding eigenvalues form a (non-strictly) increasing sequence. According to the Menshov–Rademacher theorem, which are valid for the general orthonormal series too, the expansion (1.3) converges almost everywhere on Γ provided that

$$\sum_{j=1}^{\infty} |c_j(f)|^2 \, \log^2(j+1) < \infty. \tag{1.4}$$

This hypotheses cannot be reformulated in equivalent manner in terms of the belonging of f to Sobolev spaces because

$$\|f\|^2_{H^{s}(\Gamma)} \asymp \sum_{j=1}^{\infty} |c_j(f)|^2 \, j^{2s}$$

for every $s > 0$. We may state only that the condition $f \in H^{s}(\Gamma)$ for some $s > 0$ implies convergence of the series (1.3) almost everywhere on Γ. This condition does not adequately express the hypotheses (1.4) of the Menshov–Rademacher theorem.

In 1963 L. Hörmander [43, Sec. 2.2] proposed a broad and informative generalization of the Sobolev spaces in the category of Hilbert spaces (also see [45, Sec. 10.1]). He introduced spaces that are parametrized by a general enough weight function, which serves as an analog of the differentiation order or smoothness index used for the Sobolev spaces. In particular, Hörmander considered the following Hilbert spaces

$$B_{2,\mu}(\mathbb{R}^n) := \{u : \mu\,\mathcal{F}u \in L_2(\mathbb{R}^n)\}, \qquad (1.5)$$
$$\|u\|_{B_{2,\mu}(\mathbb{R}^n)} := \|\mu\,\mathcal{F}u\|_{L_2(\mathbb{R}^n)}.$$

Here $\mathcal{F}u$ is the Fourier transform of a tempered distribution u given on $\mathbb{R}^n$, and μ is a weight function of n arguments.

In the case where

$$\mu(\xi) = \langle\xi\rangle^s, \quad \langle\xi\rangle := (1 + |\xi|^2)^{1/2}, \quad \xi \in \mathbb{R}^n, \quad s \in \mathbb{R},$$

we have the Sobolev space $B_{2,\mu}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ of differentiation order s.

The Hörmander spaces occupy a central position among the spaces of generalized smoothness, which is characterized by a function parameter, rather than a number. These spaces are under various and profound investigations; a good deal of the work was done in the last decades. We refer to G.A. Kalyabin and P.I. Lizorkin's survey [48], H. Triebel's monograph [135, Sec. 22], the recent papers by A.M. Caetano and H.-G. Leopold [17], W. Farkas, N. Jacob, and R.L. Schilling [27], W. Farkas and H.-G. Leopold [28], P. Gurka and B. Opic [36], D.D. Haroske and S.D. Moura [38, 39], S.D. Moura [91], B. Opic and W. Trebels [104], and references given therein. Various classes of spaces of generalized smoothness appear naturally in embedding theorems for function spaces, the theory of interpolation of function spaces, approximation theory, the theory of differential and pseudodifferential operators, theory of stochastic processes; see the monographs by D.D. Haroske [37], N. Jacob [47], V.G. Maz'ya and T.O. Shaposhnikova [66, Sec. 16], F. Nicola and L. Rodino [101], B.P. Paneah [106], A.I. Stepanets [130, Part I, Ch. 3, Sec. 7.1], and also the papers by F. Cobos and D.L. Fernandez [18], C. Merucci [69], M. Schechter [123] devoted to the interpolation of function spaces, and the papers by D.E. Edmunds and H. Triebel [23, 24], V.A. Mikhailets and V.M. Molyboga [72, 73, 74] on spectral theory of some elliptic operators appearing in mathematical physics.

Already in 1963 L. Hörmander applied the spaces (1.5) and more general Banach spaces $B_{p,\mu}(\mathbb{R}^n)$, with $1 \leq p \leq \infty$, to an investigation of regularity properties of solutions to the partial differential equations with constant coefficients and to some classes of equations with varying coefficients. However, as distinct from the Sobolev spaces, the Hörmander spaces have not got a broad application to the general elliptic equations on manifolds and to the elliptic boundary-value problems. This is due to the lack of a reasonable definition of the Hörmander spaces on smooth manifolds (the definition should be independent of a choice of local charts covering the manifold) an on the absence of analytic tools fit to use these spaces effectively.

Such a tool exists in the Sobolev spaces case; this is the interpolation of spaces. Namely, an arbitrary fractional order Sobolev space can be obtained by the interpolation of a certain couple of integer order Sobolev spaces. This fact essentially facilitates both the investigation of these spaces and proofs of various theorems of the theory of elliptic equations because the boundedness and the Fredholm property (if the defect is invariant) preserve for linear operators under the interpolation.

Therefore it seems reasonable to distinguish the Hörmander spaces that are obtained by the interpolation (with a function parameter) of couples of Sobolev spaces; we will consider only inner product spaces. For this purpose we introduce the following class of isotropic spaces

$$H^{s,\varphi}(\mathbb{R}^n) := B_{2,\mu}(\mathbb{R}^n) \quad \text{for} \quad \mu(\xi) = \langle\xi\rangle^s \varphi(\langle\xi\rangle). \tag{1.6}$$

Here the number parameter s is real, whereas the positive function parameter φ varies slowly at $+\infty$ in the Karamata sense [15, 126]. (We may assume that φ is constant outside of a neighbourhood of $+\infty$.) For example, φ is admitted to be the logarithmic function, its arbitrary iteration, their real power, and a product of these functions.

The class of spaces (1.6) contains the Sobolev Hilbert scale $\{H^s\} \equiv \{H^{s,1}\}$, is attached to it by the number parameter, but is calibrated much finer than the Sobolev scale. Indeed,

$$H^{s+\varepsilon}(\mathbb{R}^n) \subset H^{s,\varphi}(\mathbb{R}^n) \subset H^{s-\varepsilon}(\mathbb{R}^n) \quad \text{for every} \quad \varepsilon > 0.$$

Therefore the number parameter s defines the main (power) smoothness, whereas the function parameter φ determines an additional (subpower) smoothness on the class of spaces (1.6). Specifically, if $\varphi(t) \to \infty$ (or $\varphi(t) \to 0$) as $t \to \infty$, then φ determines an additional positive (or negative) smoothness. Thus, the parameter φ refines the main smoothness s. Therefore the class of spaces (1.6) is naturally called the refined Sobolev scale.

This scale possesses the following important property: every space $H^{s,\varphi}(\mathbb{R}^n)$ is a result of the interpolation, with an appropriate function parameter, of the couple of Sobolev spaces $H^{s-\varepsilon}(\mathbb{R}^n)$ and $H^{s+\delta}(\mathbb{R}^n)$, with $\varepsilon, \delta > 0$. The parameter of the interpolation is a function that varies regularly (in the Karamata sense) of index $\theta \in (0, 1)$ at $+\infty$; namely $\theta := \varepsilon/(\varepsilon + \delta)$. Moreover, the refined Sobolev scale proves to be closed with respect to this interpolation.

Thus, every Hörmander space $H^{s,\varphi}(\mathbb{R}^n)$ possesses the interpolation property with respect to the Sobolev Hilbert scale. This means that each linear operator bounded on both the spaces $H^{s-\varepsilon}(\mathbb{R}^n)$ and $H^{s+\delta}(\mathbb{R}^n)$ is also bounded on $H^{s,\varphi}(\mathbb{R}^n)$. The interpolation property plays a decisive role here; namely, it permits us to establish some important properties of the refined Sobolev scale. They enable this scale to be applied in the theory of elliptic equations. Thus, we can prove with the help of the interpolation that each space $H^{s,\varphi}(\mathbb{R}^n)$, as the Sobolev spaces, is invariant with respect to diffeomorphic transformations of $\mathbb{R}^n$. This permits the space $H^{s,\varphi}(\Gamma)$ to be well defined over a smooth closed manifold Γ because the set of distributions and the topology in this space does not depend on a choice of local charts covering Γ. The spaces $H^{s,\varphi}(\mathbb{R}^n)$ and $H^{s,\varphi}(\Gamma)$ are useful in the theory of elliptic operators on manifolds and in the theory of elliptic boundary-value problems; these spaces are present implicitly in a number of problems appearing in calculus.

Let us dwell on some results that demonstrate advantages of the introduced scale as compared with the Sobolev scale. These results deal with the examples considered above. As before, let A be an elliptic differential operator given on Γ, with $m := \operatorname{ord} A$. Then A sets the bounded and Fredholm operators

$$A: \; H^{s+m,\varphi}(\Gamma) \to H^{s,\varphi}(\Gamma) \quad \text{for all} \quad s \in \mathbb{R}, \;\; \varphi \in \mathcal{M}.$$

Here $\mathcal{M}$ is the class of slowly varying function parameters φ used in (1.6). Note that the differential operator A leaves invariant the function parameter φ, which refines the main smoothness s. Besides, we have the following regularity property of a solution to the elliptic equation $Au = f$:

$$(f \in H^{s,\varphi}(\Gamma) \;\; \text{for some} \;\; s \in \mathbb{R}, \; \varphi \in \mathcal{M}) \; \Rightarrow \; u \in H^{s+m,\varphi}(\Gamma).$$

For the refined Sobolev scale, we have the following sharpening of Sobolev's Imbedding Theorem: given an integer $r \geq 0$ and function $\varphi \in \mathcal{M}$, the embedding $H^{r+n/2,\varphi}(\Gamma) \subset C^{r}(\Gamma)$ is equivalent to that

$$\int\limits_{1}^{\infty} \frac{dt}{t\,\varphi^{2}(t)} < \infty. \tag{1.7}$$

Therefore, if $f \in H^{r-m+n/2,\varphi}(\Gamma)$ for some parameter $\varphi \in \mathcal{M}$ satisfying (1.7), then the solution $u \in C^{r}(\Gamma)$.

Similar results are also valid for the elliptic systems and elliptic boundary-value problems.

Now let us pass to the analysis of the spectral expansion (1.3) convergence. We additionally suppose that the operator A is of order $m > 0$ and is unbounded and self-adjoint on the space $L_2(\Gamma)$. Condition (1.4) for the convergence of (1.3) almost everywhere on Γ is equivalent to the inclusion

$$f \in H^{0,\varphi}(\Gamma), \quad \text{with} \quad \varphi(t) := \max\{1, \log t\}.$$

The latter is much wider than the condition $f \in H^{s}(\Gamma)$ for some $s > 0$. We can also similarly represent conditions for unconditional convergence almost everywhere or convergence in the Hölder space $C^{r}(\Gamma)$, with integral $r \geq 0$.

The above and some other results show that the refined Sobolev scale is helpful and convenient. This scale can be used in different topics of the modern analysis as well; see, e.g., the articles by M. Hegland [40, 41], P. Mathé and U. Tautenhahn [65].

This paper is a detailed survey of our recent articles [75–87, 94–100], which are summed up in the monograph [85] published in Russian in 2010. In them, we have built a theory of general elliptic (both scalar and matrix) operators and elliptic boundary-value problems on the refined Sobolev scales of function spaces.

Let us describe the survey contents in greater detail. The paper consists of 13 sections.

Section 1 is Introduction, which we are presenting now.

Section 2 is preliminary and contains a necessary information about regularly varying functions and about the interpolation with a function parameter. Here we distinguish important Theorem 2.12, which gives a description of all interpolation parameters for the category of separable Hilbert spaces.

In Section 3, we consider the Hörmander spaces, give a definition of the refined Sobolev scale, and study its properties. Among them, we especially note the interpolation properties of this scale, formulated as Theorems 3.8 and 3.9. They are very important for applications.

Section 4 deals with uniformly elliptic pseudodifferential operators that are studied on the refined Sobolev scale over $\mathbb{R}^n$. We get an a priory estimate for a solution of the elliptic equation and investigate an interior smoothness of the solution. As an application, we obtain a sufficient condition for the existence of continuous bounded derivatives of the solution.

Next in Section 5, we define a class of Hörmander spaces, the refined Sobolev scale, over a smooth closed manifold. We give three equivalent definitions of these spaces: local (in terms of local properties of distributions), interpolational (by means of the interpolation of Sobolev spaces with an appropriate function parameter), and operational (via the completion of the set of infinitely smooth functions with respect to the norm generated by a certain function of the Beltrami–Laplace operator). These definitions are similar to those used for the Sobolev spaces. We study properties of the refined Sobolev scale over the closed manifold. Important applications of these results are given in Sections 6 and 7.

Section 6 deals with elliptic pseudodifferential operators on a closed manifold. We show that they are Fredholm (i.e. have a finite index) on appropriate couples of Hörmander spaces. As in Section 4, a priory estimates for solutions of the elliptic equations are obtained, and the solutions regularity is investigated. Using elliptic operators, we give equivalent norms on Hörmander spaces over the manifold.

In Section 7, we investigate a convergence of spectral expansions corresponding to elliptic normal operators given on the closed manifold. We find sufficient conditions for the following types of the convergence: almost everywhere, unconditionally almost everywhere, and in the space C^k, with integral $k \geq 0$. These conditions are formulated in constructive terms of the convergence on some function classes, which are Hörmander spaces.

Section 8 deals with the classes of Hörmander spaces that relate to the refined Sobolev scale and are given over Euclidean domains being open or closed. For these classes, we study interpolation properties, embeddings, traces, and riggings of the space of square integrable functions with Hörmander spaces. The results of this section are applied in next Sections 9–12, where a regular elliptic boundary-value problem is investigated in appropriate Hörmander spaces.

In Section 9, this problem is studied on the one-sided refined Sobolev scale. We show that the problem generates a Fredholm operator on this scale. We investigate some properties of the problem; namely, a priory estimates for solutions and local regularity are given. Moreover, a sufficient condition for the weak solution to be classical is found in terms of Hörmander spaces.

Section 10 deals with semihomogeneous elliptic boundary-value problems. They are considered on Hörmander spaces which form an appropriate two-sided refined Sobolev scale. We show that the operator corresponding to the problem is bounded and Fredholm on this scale.

In Sections 11–12, we give various theorems about a solvability of nonhomogeneous regular elliptic boundary-value problems in Hörmander spaces of an arbitrary real main smoothness. Developing the methods suggested by Ya.A. Roitberg [118] and J.-L. Lions, E. Magenes [61], we establish a certain generic theorem and a wide class of individual theorems on the solvability. The generic theorem is featured by that the domain of the elliptic operator does not depend on the coefficients of the elliptic equation and is common for all boundary-value problems of the same order. Conversely, the individual theorems are characterized by that the domain depends essentially on the coefficients, even of the lower order derivatives. In Section 11, we elaborate on Roitberg's approach in connection with Hörmander spaces and then deduce the generic theorem about the solvability of elliptic boundary-value problems on the two-sided refined Sobolev scale modified in the Roitberg sense.

Section 12 is devoted to J.-L. Lions and E. Magenes' approach, which we develop for various Hilbert scales consisting of Sobolev or Hörmander spaces. For the space of right-hand sides of an elliptic equation, we find a sufficiently general condition under which the operator of the problem is bounded and Fredholm (see key Theorems 12.6 and 12.16). As a consequence, we obtain new various individual theorems on the solvability of elliptic boundary-value problems considered in Sobolev or Hörmander spaces, both nonweighted and weighted.

In final Section 13, we indicate application of Hörmander spaces to other important classes of elliptic problems. They are nonregular boundary-value problems, parameter-elliptic problems, certain mixed elliptic problems, elliptic systems and corresponding boundary-value problems.

It is necessary to note that some results given in the survey are new even for the Sobolev spaces. These results are Theorem 10.1 in the case of half-integer s and individual Theorems 12.6, 12.10, and 12.14.

In addition, note that we have also investigated a certain class of Hörmander spaces, which is wider than the refined Sobolev scale. Interpolation properties of this class are studied and then applied to elliptic operators [85, 87, 99, 140]. It is remarkable that this class consists of all the Hilbert spaces which possess the interpolation property with respect to the Sobolev Hilbert scale. These results fall beyond the limits of our survey.

2. Preliminaries

In this section we recall some important results concerning the regularly varying functions and the interpolation with a function parameter of couples of Hilbert spaces. These results will be necessary for us in the sequel.

2.1. **Regularly varying functions.** We recall the following notion.

Definition 2.1. A positive function ψ defined on a semiaxis $[b,+\infty)$ is said to be regularly varying of index $\theta\in\mathbb{R}$ at $+\infty$ if ψ is Borel measurable on $[b_0,+\infty)$ for some number $b_0\geq b$ and

$$\lim_{t\to+\infty}\frac{\psi(\lambda\,t)}{\psi(t)}=\lambda^{\theta}\quad\text{for each}\quad\lambda>0.$$

A function regularly varying of the index $\theta = 0$ at $+\infty$ is called slowly varying at $+\infty$.

The theory of regularly varying functions was founded by Jovan Karamata [49, 50] in the 1930s. These functions are closely related to the power functions and have numerous applications, mainly due to their special role in Tauberian-type theorems (see the monographs [15, 64, 113, 126] and references therein).

Example 2.2. The well-known standard case of functions regularly varying of the index θ at $+\infty$ is

$$\psi(t) := t^{\theta} (\log t)^{r_1} (\log\log t)^{r_2} \ldots (\log\ldots\log t)^{r_k} \quad \text{for} \quad t \gg 1 \tag{2.1}$$

with arbitrary parameters $k \in \mathbb{Z}_+$ and $r_1, r_2, \ldots, r_k \in \mathbb{R}$. In the case where $\theta = 0$ these functions form the logarithmic multiscale, which has a number of applications in the theory of function spaces.

We denote by SV the set of all functions slowly varying at $+\infty$. It is evident that ψ is a function regularly varying at $+\infty$ of index θ if and only if $\psi(t) = t^{\theta}\varphi(t)$, $t \gg 1$, for some function $\varphi \in \mathrm{SV}$. Thus, the investigation of regularly varying functions is reduced to the case of slowly varying functions.

The study and application of regularly varying functions are based on two fundamental theorems: the Uniform Convergence Theorem and Representation Theorem. They were proved by Karamata [49] in the case of continuous functions and, in general, by a number of authors later (see the monographs cited above).

Theorem 2.3 (Uniform Convergence Theorem)**.** *Suppose that $\varphi \in \mathrm{SV}$; then $\varphi(\lambda t)/\varphi(t) \to 1$ as $t \to +\infty$ uniformly on each compact λ-set in $(0, \infty)$.*

Theorem 2.4 (Representation Theorem)**.** *A function φ belongs to SV if and only if it can be written in the form*

$$\varphi(t) = \exp\left(\beta(t) + \int_b^t \frac{\alpha(\tau)}{\tau}\, d\tau\right), \quad t \geq b, \tag{2.2}$$

for some number $b > 0$, continuous function $\alpha : [b, \infty) \to \mathbb{R}$ approaching zero at ∞, and Borel measurable bounded function $\beta : [b, \infty) \to \mathbb{R}$ that has the finite limit at ∞.

The Representation Theorem implies the following sufficient condition for a function to be slowly varying at infinity [126, Sec. 1.2].

Theorem 2.5. *Suppose that a function $\varphi : (b, \infty) \to (0, \infty)$ has a continuous derivative and satisfies the condition $t\varphi'(t)/\varphi(t) \to 0$ as $t \to \infty$. Then $\varphi \in \mathrm{SV}$.*

Using Theorem 2.5 one can give many interesting examples of slowly varying functions. Among them we mention the following.

Example 2.6. Let $\varphi(t) := \exp\psi(t)$, with ψ being defined according to (2.1), where $\theta = 0$ and $r_1 < 1$. Then $\varphi \in \mathrm{SV}$.

Example 2.7. Let $\alpha, \beta, \gamma \in \mathbb{R}$, $\beta \neq 0$, and $0 < \gamma < 1$. We set $\omega(t) := \alpha + \beta \sin \log^{\gamma} t$ and $\varphi(t) := (\log t)^{\omega(t)}$ for $t > 1$. Then $\varphi \in \mathrm{SV}$.

Example 2.8. Let $\alpha,\beta,\gamma\in\mathbb{R}$, $\alpha\neq 0$, $0<\gamma<\beta<1$, and
$$\varphi(t):=\exp(\alpha(\log t)^{1-\beta}\,\sin\log^{\gamma}t)\quad\text{for}\quad t>1.$$
Then $\varphi\in\mathrm{SV}$.

The last two examples show that a function φ varying slowly at $+\infty$ may exhibit infinite oscillation, that is
$$\liminf_{t\to+\infty}\varphi(t)=0\quad\text{and}\quad\limsup_{t\to+\infty}\varphi(t)=+\infty.$$

We will use regularly varying functions as parameters when we define certain Hilbert spaces. If the function parameters are equivalent in a neighbourhood of $+\infty$, we get the same space up to equivalence of norms. Therefore it is useful to introduced the following notion [81, p. 90].

Definition 2.9. We say that a positive function ψ defined on a semiaxis $[b,+\infty)$ is quasiregularly varying of index $\theta\in\mathbb{R}$ at $+\infty$ if there exist a number $b_1\geq b$ and a function $\psi_1:[b_1,+\infty)\to(0,+\infty)$ regularly varying of the same index $\theta\in\mathbb{R}$ at $+\infty$ such that $\psi\asymp\psi_1$ on $[b_1,+\infty)$. A function quasiregularly varying of the index $\theta=0$ at $+\infty$ is called quasislowly varying at $+\infty$.

As usual, the notation $\psi\asymp\psi_1$ on $[b_1,+\infty)$ means that the functions ψ and ψ_1 are equivalent there, that is both the functions ψ/ψ_1 and ψ_1/ψ are bounded on $[b_1,+\infty)$.

We denote by QSV the set of all functions varying quasislowly at $+\infty$. It is evident that ψ is quasiregularly varying of the index θ at $+\infty$ if and only if $\psi(t)=t^{\theta}\varphi(t)$, $t\gg 1$, for some function $\varphi\in\mathrm{QSV}$.

We note the following properties of the class QSV.

Theorem 2.10. *Let $\varphi,\chi\in\mathrm{QSV}$. The next assertions are true:*

i) *There is a function $\varphi_1\in C^{\infty}((0;+\infty))\cap\mathrm{SV}$ such that $\varphi\asymp\varphi_1$ in a neighbourhood of $+\infty$.*
ii) *If $\theta>0$, then both $t^{-\theta}\varphi(t)\to 0$ and $t^{\theta}\varphi(t)\to+\infty$ as $t\to+\infty$.*
iii) *All the functions $\varphi+\chi$, $\varphi\,\chi$, φ/χ and φ^{σ}, with $\sigma\in\mathbb{R}$, belong to* QSV.
iv) *Let $\theta\geq 0$, and in the case where $\theta=0$ suppose that $\varphi(t)\to+\infty$ as $t\to+\infty$. Then the composite function $\chi(t^{\theta}\varphi(t))$ of t belongs to* QSV.

Theorem 2.10 are known for slowly varying functions, even with the strong equivalence $\varphi(t)\sim\varphi_1(t)$ as $t\to+\infty$ being in assertion i); see, e.g., [15, Sec. 1.3] and [126, Sec. 1.5]. This implies the case when $\varphi,\chi\in\mathrm{QSV}$ [81, p. 91].

2.2. **The interpolation with a function parameter of Hilbert spaces.** It is a natural generalization of the classical interpolation method by J.-L. Lions and S.G. Krein (see, e.g., [30, Ch. IV, § 9] and [61, Ch. 1, Sec. 2 and 5]) to the case when a general enough function is used as an interpolation parameter instead of a number parameter. The generalization appeared in the paper by C. Foiaş and J.-L. Lions [29, p. 278] and then was studied by W.F. Donoghue [21], E.I. Pustyl'nik [111], V.I. Ovchinnikov [105, Sec. 11.4], and the authors [81].

We recall the definition of this interpolation. For our purposes, it is sufficient to restrict ourselves to the case of separable Hilbert spaces.

Let an ordered couple $X := [X_0, X_1]$ of complex Hilbert spaces X_0 and X_1 be such that these spaces are separable and that the continuous dense embedding $X_1 \hookrightarrow X_0$ holds true. We call this couple admissible. For the couple X there exists an isometric isomorphism $J : X_1 \leftrightarrow X_0$ such that J is a self-adjoint positive operator on the space X_0 with the domain X_1 (see [61, Ch. 1, Sec. 2.1] and [30, Ch. IV, Sec. 9.1]). The operator J is said to be generating for the couple X and is uniquely determined by X.

We denote by $\mathcal{B}$ the set of all functions $\psi : (0, \infty) \to (0, \infty)$ such that:

a) ψ is Borel measurable on the semiaxis $(0, +\infty)$;
b) ψ is bounded on each compact interval $[a, b]$ with $0 < a < b < +\infty$;
c) $1/\psi$ is bounded on each set $[r, +\infty)$ with $r > 0$.

Let $\psi \in \mathcal{B}$. Generally, the unbounded operator $\psi(J)$ is defined in the space X_0 as a function of J. We denote by $[X_0, X_1]_\psi$ or simply by X_ψ the domain of the operator $\psi(J)$ endowed with the inner product $(u, v)_{X_\psi} := (\psi(J)u, \psi(J)v)_{X_0}$ and the corresponding norm $\|u\|_{X_\psi} := (u, u)^{1/2}_{X_\psi}$. The space X_ψ is Hilbert and separable.

Definition 2.11. We say that a function $\psi \in \mathcal{B}$ is an interpolation parameter if the following property is fulfilled for all admissible couples $X = [X_0, X_1]$, $Y = [Y_0, Y_1]$ of Hilbert spaces and an arbitrary linear mapping T given on X_0. If the restriction of the mapping T to the space X_j is a bounded operator $T : X_j \to Y_j$ for each $j = 0$, 1, then the restriction of the mapping T to the space X_ψ is also a bounded operator $T : X_\psi \to Y_\psi$.

Otherwise speaking, ψ is an interpolation parameter if and only if the mapping $X \mapsto X_\psi$ is an interpolation functor given on the category of all admissible couples X of Hilbert spaces. (For the notion of interpolation functor, see, e.g., [14, Sec. 2.4] and [133, Sec. 1.2.2]) In the case where ψ is an interpolation parameter, we say that the space X_ψ is obtained by the interpolation with the function parameter ψ of the admissible couple X. Then the continuous dense embeddings $X_1 \hookrightarrow X_\psi \hookrightarrow X_0$ are fulfilled.

The classical result by J.-L. Lions and S.G. Krein consists in the fact that the power function $\psi(t) := t^\theta$ is an interpolation parameter whenever $0 < \theta < 1$; see [30, Ch. IV, § 9, Sec. 3] and [61, Ch. 1, Sec. 5.1].

We have the following criterion for a function to be an interpolation parameter.

Theorem 2.12. *A function $\psi \in \mathcal{B}$ is an interpolation parameter if and only if ψ is pseudoconcave in a neighbourhood of $+\infty$, i.e. $\psi \asymp \psi_1$ for some concave positive function ψ_1*

This theorem follows from Peetre's results [108] on interpolations functions (see also the monograph [14, Sec. 5.4]). The corresponding proof is given in [81, Sec. 2.7].

For us, it is important the next consequence of Theorem 2.12.

Corollary 2.13. *Suppose a function $\psi \in \mathcal{B}$ to be quasiregularly varying of index θ at $+\infty$, with $0 < \theta < 1$. Then ψ is an interpolation parameter.*

The direct proof of this assertion is given in [76, Sec. 2].

3. Hörmander spaces

In 1963 Lars Hörmander [43, Sec. 2.2] introduced the spaces $B_{p,\mu}(\mathbb{R}^n)$, which consist of distributions in $\mathbb{R}^n$ and are parametrized by a number $p \in [1,\infty]$ and a general enough weight function μ of argument $\xi \in \mathbb{R}^n$; see also [45, Sec. 10.1]. The number parameter p characterizes integrability properties of the distributions, whereas the function parameter μ describes their smoothness properties. In this section, we recall the definition of the spaces $B_{p,\mu}(\mathbb{R}^n)$, some their properties, and an application to constant-coefficient partial differential equations. Further we consider the important case where the Hörmander space $B_{p,\mu}(\mathbb{R}^n)$ is Hilbert, i.e. $p = 2$, and μ is a quasiregularly varying function of $(1 + |\xi|^2)^{1/2}$ at infinity.

3.1. **The spaces** $B_{p,\mu}(\mathbb{R}^n)$**.** Let an integer $n \geq 1$ and a parameter $p \in [1,\infty]$. We use the following conventional designations, where Ω is an nonempty open set in $\mathbb{R}^n$, in particular $\Omega = \mathbb{R}^n$:

a) $L_p(\Omega) := L_p(\Omega, d\xi)$ is the Banach space of complex-valued functions $f(\xi)$ of $\xi \in \Omega$ such that $|f|^p$ is integrable over Ω (if $p = \infty$, then f is essentially bounded in Ω);
b) $C^k_{\mathrm{b}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{C}$ having continuous and bounded derivatives of order $\leq k$ on Ω;
c) $C^\infty_0(\Omega)$ is the linear topological space of infinitely differentiable functions $u : \mathbb{R}^n \to \mathbb{C}$ such that their supports are compact and belong to Ω; we will identify functions from $C^\infty_0(\Omega)$ with their restrictions to Ω;
d) $\mathcal{D}'(\Omega)$ is the linear topological space of all distributions given in Ω; we always suppose that distributions are antilinear complex-valued functionals;
e) $\mathcal{S}'(\mathbb{R}^n)$ is the linear topological Schwartz space of tempered distributions given in $\mathbb{R}^n$;
f) $\widehat{u} := \mathcal{F}u$ is the Fourier transform of a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$; $\mathcal{F}^{-1}f$ is the inverse Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$;
g) $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$ is a smoothed modulus of $\xi \in \mathbb{R}^n$.

Suppose a continuous function $\mu : \mathbb{R}^n \to (0,\infty)$ to be such that, for some numbers $c \geq 1$ and $l > 0$, we have

$$\frac{\mu(\xi)}{\mu(\eta)} \leq c\,(1 + |\xi - \eta|)^l \quad \text{for all} \quad \xi, \eta \in \mathbb{R}^n. \tag{3.1}$$

The function μ is called a weight function.

Definition 3.1. The Hörmander space $B_{p,\mu}(\mathbb{R}^n)$ is a linear space of the distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that the Fourier transform $\widehat{u}$ is locally Lebesgue integrable on $\mathbb{R}^n$ and, moreover, $\mu\,\widehat{u} \in L_p(\mathbb{R}^n)$. The space $B_{p,\mu}(\mathbb{R}^n)$ is endowed with the norm $\|u\|_{B_{p,\mu}(\mathbb{R}^n)} := \|\mu\,\widehat{u}\|_{L_p(\mathbb{R}^n)}$.

The space $B_{p,\mu}(\mathbb{R}^n)$ is complete and continuously embedded in $\mathcal{S}'(\mathbb{R}^n)$. If $1 \leq p < \infty$, then this space is separable, and the set $C^\infty_0(\mathbb{R}^n)$ is complete in it [43, Sec. 2.2]. Of special interest is the $p = 2$ case, when $B_{p,\mu}(\mathbb{R}^n)$ becomes a Hilbert space.

Remark 3.2. Hörmander assumes initially that μ satisfies a stronger condition than (3.1); namely, there exist some positive numbers c and l such that

$$\frac{\mu(\xi)}{\mu(\eta)} \leq (1 + c\,|\xi - \eta|)^l \quad \text{for all} \quad \xi, \eta \in \mathbb{R}^n. \tag{3.2}$$

But he notices that two sets of functions satisfying either (3.1) or (3.2) lead to the same class of spaces $B_{p,\mu}(\mathbb{R}^n)$ [43, the remark at the end of Sec. 2.1].

The term 'Hörmander space' was suggested by H. Triebel in [133, Sec. 4.11.4].

The following Hörmander's theorem establishes an important relation between the spaces $B_{p,\mu}(\mathbb{R}^n)$ and $C^k_{\mathrm{b}}(\mathbb{R}^n)$ [43, Sec. 2.2, Theorem 2.2.7].

Theorem 3.3 (Hörmander's Embedding Theorem). *Let $p, q \in [1, \infty]$, $1/p + 1/q = 1$, and an integer $k \geq 0$. Then the condition*

$$\langle\xi\rangle^k\,\mu^{-1}(\xi) \in L_q(\mathbb{R}^n, d\xi) \tag{3.3}$$

entails the continuous embedding $B_{p,\mu}(\mathbb{R}^n) \hookrightarrow C^k_{\mathrm{b}}(\mathbb{R}^n)$. Conversely, if

$$\{u \in B_{p,\mu}(\mathbb{R}^n) : \operatorname{supp} u \subset V\} \subset C^k(\mathbb{R}^n)$$

for some nonempty open set $V \subseteq \mathbb{R}^n$, then (3.3) is valid.

The spaces $B_{p,\mu}(\mathbb{R}^n)$ were applied by Hörmander to investigation of regularity properties of solutions to some partial differential equations (see [43, Ch. IV, VII] and [45, Ch. 11, 13]). We state one of his results relating to elliptic equations [43, Sec 7.4].

Let Ω be a nonempty open set in $\mathbb{R}^n$. In Ω, consider a partial differential equation $P(x, D)u = f$ of an order r with coefficients belonging to $C^\infty(\Omega)$. Introduce the local Hörmander space over Ω:

$$B^{\mathrm{loc}}_{p,\mu}(\Omega) := \{f \in \mathcal{D}'(\Omega) : \; \chi f \in B_{p,\mu}(\Omega) \;\; \forall \;\; \chi \in C^\infty_0(\Omega)\}.$$

Here $B_{p,\mu}(\Omega)$ is the space of restrictions of all the distributions $u \in B_{p,\mu}(\mathbb{R}^n)$ to Ω.

Theorem 3.4 (Hörmander's Regularity Theorem). *Let the operator $P(x, D)$ be elliptic in Ω, and $u \in \mathcal{D}'(\Omega)$. If $P(x, D)u \in B^{\mathrm{loc}}_{p,\mu}(\Omega)$ for some $p \in [1, \infty]$ and weight function μ, then $u \in B^{\mathrm{loc}}_{p,\mu_r}(\Omega)$ with $\mu_r(\xi) := \langle\xi\rangle^r \mu(\xi)$.*

For applications of the spaces $B_{p,\mu}(\mathbb{R}^n)$, the Hilbert case of $p = 2$ is the most interesting. This case was investigated by B. Malgrange [63] and L.R. Volevich, B.P. Paneah [138] (see also Paneah's monograph [106, Sec. 1.4]). Specifically, if $\mu(\xi) = \langle\xi\rangle^s$ for all $\xi \in \mathbb{R}^n$ with some $s \in \mathbb{R}$, then $B_{2,\mu}(\mathbb{R}^n)$ becomes the Sobolev inner product space $H^s(\mathbb{R}^n)$ of order s.

In what follows we consider the isotropic Hörmander inner product spaces $B_{2,\mu}(\mathbb{R}^n)$, with $\mu(\xi)$ being a radial function, i.e. depending only on $\langle\xi\rangle$.

3.2. **The refined Sobolev scale.** It useful to have a class of the Hörmander inner product spaces $B_{2,\mu}(\mathbb{R}^n)$ that are close to the Sobolev spaces $H^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$. For this purpose we choose $\mu(\xi) := \langle\xi\rangle^s \varphi(\langle\xi\rangle)$ for some functions $\varphi \in \mathrm{QSV}$; then μ is a quasiregularly varying function of $\langle\xi\rangle$ at infinity of index s. In this case it is naturally to rename the Hörmander space $B_{2,\mu}(\mathbb{R}^n)$ by $H^{s,\varphi}(\mathbb{R}^n)$. Let us formulate the corresponding definitions. First we introduce the following set $\mathcal{M} \subset \mathrm{QSV}$ of function parameters φ.

By $\mathcal{M}$ we denote the set of all functions $\varphi : [1;+\infty) \to (0;+\infty)$ such that:

a) φ is Borel measurable on $[1;+\infty)$;

b) φ and $1/\varphi$ are bounded on every compact interval $[1;b]$, where $1 < b < +\infty$;

c) $\varphi \in \mathrm{QSV}$.

It follows from Theorem 2.4 that $\varphi \in \mathcal{M}$ if and only if φ can be written in the form (2.2) with $b = 1$ for some continuous function $\alpha : [1,\infty) \to \mathbb{R}$ approaching zero at $+\infty$ and Borel measurable bounded function $\beta : [1,\infty) \to \mathbb{R}$.

Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$.

Definition 3.5. The space $H^{s,\varphi}(\mathbb{R}^n)$ is the Hörmander inner product space $B_{2,\mu}(\mathbb{R}^n)$ with $\mu(\xi) := \langle\xi\rangle^s \varphi(\xi)$ for $\xi \in \mathbb{R}^n$.

Thus $H^{s,\varphi}(\mathbb{R}^n)$ consists of the distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that the Fourier transform $\widehat{u}$ is a function locally Lebesgue integrable on $\mathbb{R}^n$ and

$$\int\limits_{\mathbb{R}^n} \langle\xi\rangle^{2s} \varphi^2(\langle\xi\rangle)\, |\widehat{u}(\xi)|^2 \, d\xi < \infty.$$

The inner product in the space $H^{s,\varphi}(\mathbb{R}^n)$ is defined by the formula

$$(u_1, u_2)_{H^{s,\varphi}(\mathbb{R}^n)} := \int\limits_{\mathbb{R}^n} \langle\xi\rangle^{2s} \varphi^2(\langle\xi\rangle)\, \widehat{u_1}(\xi)\, \overline{\widehat{u_2}(\xi)} \, d\xi$$

and induces the norm in the usual way, $H^{s,\varphi}(\mathbb{R}^n)$ being a Hilbert space.

The function μ used in Definition 3.5 is a weight function that follows from the integral representation of the set $\mathcal{M}$ given above. We consider the Borel measurable weight functions μ, rather than continuous as Hörmander does. By Theorem 2.10 i) we do not obtain the spaces different from those considered by Hörmander.

In the simplest case where $\varphi(\cdot) \equiv 1$, the space $H^{s,\varphi}(\mathbb{R}^n) = H^{s,1}(\mathbb{R}^n)$ coincides with the Sobolev space $H^s(\mathbb{R}^n)$.

By Theorem 2.10 (ii), for each $\varepsilon > 0$ there exist a number $c_\varepsilon \geq 1$ such that

$$c_\varepsilon^{-1} t^{-\varepsilon} \leq \varphi(t) \leq c_\varepsilon t^\varepsilon \quad \text{for all} \quad t \geq 1.$$

This implies the inclusions

$$\bigcup_{\varepsilon>0} H^{s+\varepsilon}(\mathbb{R}^n) =: H^{s+}(\mathbb{R}^n) \subset H^{s,\varphi}(\mathbb{R}^n) \subset H^{s-}(\mathbb{R}^n) := \bigcap_{\varepsilon>0} H^{s-\varepsilon}(\mathbb{R}^n). \tag{3.4}$$

They show that in the class of spaces

$$\left\{ H^{s,\varphi}(\mathbb{R}^n) :\ s \in \mathbb{R},\, \varphi \in \mathcal{M} \right\} \tag{3.5}$$

the functional parameter φ defines a supplementary (subpower) smoothness to the basic (power) s-smoothness. If $\varphi(t) \to \infty$ [$\varphi(t) \to 0$] as $t \to \infty$, then φ defines a positive [negative] supplementary smoothness. Otherwise speaking, φ *refines* the power smoothness s. Therefore, it is naturally to give

Definition 3.6. The class of spaces (3.5) is called the refined Sobolev scale over $\mathbb{R}^n$.

Obviously, the scale (3.5) is much finer than the Hilbert scale of Sobolev spaces. The scale (3.5) was considered by the authors in [75, 77, 81]. Let us formulate some important properties of it.

Theorem 3.7. *Let $s \in \mathbb{R}$ and $\varphi, \varphi_1 \in \mathcal{M}$. The following assertions are true:*

i) *The dense continuous embedding $H^{s+\varepsilon,\varphi_1}(\mathbb{R}^n) \hookrightarrow H^{s,\varphi}(\mathbb{R}^n)$ is valid for each $\varepsilon > 0$.*

ii) *The function φ/φ_1 is bounded in a neighbourhood of $+\infty$ if and only if $H^{s,\varphi_1}(\mathbb{R}^n) \hookrightarrow H^{s,\varphi}(\mathbb{R}^n)$. This embedding is continuous and dense.*

iii) *Let an integer $k \geq 0$ be given. The inequality*

$$\int_1^\infty \frac{dt}{t\,\varphi^2(t)} < \infty \tag{3.6}$$

is equivalent to the embedding

$$H^{k+n/2,\varphi}(\mathbb{R}^n) \hookrightarrow C^k_{\mathrm{b}}(\mathbb{R}^n). \tag{3.7}$$

The embedding is continuous.

iv) *The spaces $H^{s,\varphi}(\mathbb{R}^n)$ and $H^{-s,1/\varphi}(\mathbb{R}^n)$ are mutually dual with respect to the inner product in $L_2(\mathbb{R}^n)$.*

Assertion i) of this theorem follows from (3.4), whereas assertions ii) – iv) are inherited from the Hörmander spaces properties [43, Sec. 2.2], in particular, iii) from Theorem 3.3. Note that $\varphi \in \mathcal{M} \Leftrightarrow 1/\varphi \in \mathcal{M}$, so the space $H^{-s,1/\varphi}(\mathbb{R}^n)$ in assertion iv) is defined as an element of the refined Sobolev scale.

The refined Sobolev scale possesses the interpolation property with respect to the Sobolev scale because every space $H^{s,\varphi}(\mathbb{R}^n)$ is obtained by the interpolation, with an appropriate function parameter, of a couple of inner product Sobolev spaces.

Theorem 3.8. *Let a function $\varphi \in \mathcal{M}$ and positive numbers ε, δ be given. We set*

$$\psi(t) := \begin{cases} t^{\varepsilon/(\varepsilon+\delta)}\,\varphi(t^{1/(\varepsilon+\delta)}) & \text{for} \quad t \geq 1, \\ \varphi(1) & \text{for} \quad 0 < t < 1. \end{cases} \tag{3.8}$$

Then the following assertions are true:

i) *The function ψ belongs to the set $\mathcal{B}$ and is an interpolation parameter.*

ii) *For an arbitrary $s \in \mathbb{R}$, we have*

$$[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)]_\psi = H^{s,\varphi}(\mathbb{R}^n) \tag{3.9}$$

with equality of norms in the spaces.

Assertion i) holds true by Corollary 2.13 because the function (3.8) is quasiregularly varying of index $\theta := \varepsilon/(\varepsilon+\delta) \in (0,1)$ at $+\infty$. Assertion ii) is directly verified if we note that the operator $J : u \mapsto \mathcal{F}^{-1}(\langle\xi\rangle^{\varepsilon+\delta}\,\widehat{u}(\xi))$ is generating for the couple on the left of (3.9). Then the operator $\psi(J) : u \mapsto \mathcal{F}^{-1}(\langle\xi\rangle^{\varepsilon}\varphi(\langle\xi\rangle)\,\widehat{u}(\xi))$

maps $H^{s,\varphi}(\mathbb{R}^n)$ onto $H^{s-\varepsilon}(\mathbb{R}^n)$ that means (3.9); for details, see [77, Sec. 3] or [81, Sec. 3.2].

The refined Sobolev scale is closed with respect to the interpolation with the functions parameters that are quasiregularly varying at $+\infty$.

Theorem 3.9. *Let $s_0, s_1 \in \mathbb{R}$, $s_0 \leq s_1$, and $\varphi_0, \varphi_1 \in \mathcal{M}$. In the case where $s_0 = s_1$ we suppose that the function φ_0/φ_1 is bounded in a neighbourhood of ∞. Let $\psi \in \mathcal{B}$ be a quasiregularly varying function of an index $\theta \in (0, 1)$ at ∞. We represent $\psi(t) = t^\theta \chi(t)$ with $\chi \in \mathrm{QSV}$ and set $s := (1-\theta)s_0 + \theta s_1$,*

$$\varphi(t) := \varphi_0^{1-\theta}(t)\, \varphi_1^{\theta}(t)\, \chi\Big(t^{s_1-s_0}\, \frac{\varphi_1(t)}{\varphi_0(t)}\Big) \quad \textit{for} \quad t \geq 1.$$

Then $\varphi \in \mathcal{M}$, and

$$[H^{s_0,\varphi_0}(\mathbb{R}^n), H^{s_1,\varphi_1}(\mathbb{R}^n)]_\psi = H^{s,\varphi}(\mathbb{R}^n) \tag{3.10}$$

with equality of norms in the spaces.

This theorem can be proved by means of the repeated application of Theorem 3.8 if we employ the reiteration formula $[X_f, X_g]_\psi = X_\omega$, where X is an admissible couple of Hilbert spaces, $f, g, \psi \in \mathcal{B}$, f/g is bounded in a neighbourhood of ∞, and $\omega(t) := f(t)\, \psi(g(t)/f(t))$ for $t > 0$; see [81, Sec. 2.3]. Besides, it is possible to give the direct proof, which is similar to that used for Theorem 3.8.

Remark 3.10. The interpolation of the Hörmander spaces $B_{p,\mu}(\mathbb{R}^n)$, with $1 \leq p \leq \infty$, was studied by M. Schechter [123] with the help of the complex method of interpolation. C. Merucci [69] and F. Cobos, D.L. Fernandez [18] considered the interpolation of various Banach spaces of generalized smoothness by means of the real method involving a function parameter.

4. Elliptic operators in $\mathbb{R}^n$

In this section we consider an arbitrary uniformly elliptic classical pseudodifferential operator (PsDO) A on the scale (3.5). We establish an a priory estimate for a solution to the equation $Au = f$ and investigate the solution smoothness in this scale. Our results refine the classical theorems on elliptic operators on the Sobolev scale; see, e.g., [7, Sec. 1.8] or [46, Sec. 18.1].

Following [7, Sec. 1.1], we denote by $\Psi^r(\mathbb{R}^n)$ with $r \in \mathbb{R}$ the class of all the PsDOs A in $\mathbb{R}^n$ (generally, not classical) such that their symbols $a(x, \xi)$ are complex-valued infinitely smooth functions satisfying the following condition. For arbitrary multi-indexes α and β, there exist a number $c_{\alpha,\beta} > 0$ such that

$$|\, \partial_x^\alpha\, \partial_\xi^\beta\, a(x, \xi)\, | \leq c_{\alpha,\beta}\, \langle \xi \rangle^{r-|\beta|} \quad \text{for every} \quad x, \xi \in \mathbb{R}^n.$$

Lemma 4.1. *Let $A \in \Psi^r(\mathbb{R}^n)$ with $r \in \mathbb{R}$. Then the restriction of the mapping $u \mapsto Au$, $u \in \mathcal{S}'(\mathbb{R}^n)$, to the space $H^{s,\varphi}(\mathbb{R}^n)$ is a bounded linear operator*

$$A : H^{s,\varphi}(\mathbb{R}^n) \to H^{s-r,\,\varphi}(\mathbb{R}^n)$$

for each $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$.

This lemma follows from the Sobolev $\varphi\equiv 1$ case [7, Sec. 1.1, Theorem 1.1.2] by the interpolation formula (3.9).

By $\Psi^r_{\mathrm{ph}}(\mathbb{R}^n)$ we denote the subset in $\Psi^r(\mathbb{R}^n)$ that consists of all the classical (polyhomogeneous) PsDOs of the order r; see [7, Sec. 1.5]. An important example of PsDO from $\Psi^r_{\mathrm{ph}}(\mathbb{R}^n)$ is given by a partial differential operator of order r with coefficients belonging to $C^\infty_{\mathrm{b}}(\mathbb{R}^n)$.

Definition 4.2. A PsDO $A\in\Psi^r_{\mathrm{ph}}(\mathbb{R}^n)$ is called uniformly elliptic in $\mathbb{R}^n$ if there exists a number $c>0$ such that $|a_0(x,\xi)|\geq c$ for each $x,\xi\in\mathbb{R}^n$ with $|\xi|=1$. Here $a_0(x,\xi)$ is the principal symbol of A.

Let $r\in\mathbb{R}$. Suppose a PsDO $A\in\Psi^r_{\mathrm{ph}}(\mathbb{R}^n)$ to be uniformly elliptic in $\mathbb{R}^n$.

Theorem 4.3. *Let $s\in\mathbb{R}$, $\varphi\in\mathcal{M}$, and $\sigma<s$. The following a priori estimate holds true:*

$$\|u\|_{H^{s,\varphi}(\mathbb{R}^n)}\leq c\,\bigl(\,\|Au\|_{H^{s-r,\varphi}(\mathbb{R}^n)}+\|u\|_{H^{\sigma,\varphi}(\mathbb{R}^n)}\,\bigr)\quad\textit{for all}\quad u\in H^{s,\varphi}(\mathbb{R}^n). \tag{4.1}$$

Here $c=c(s,\varphi,\sigma)$ is a positive number not depending on u.

We prove this theorem with the help of the left parametrix of A if we apply Lemma 4.1. As knows [7, Sec. 1.8, Theorem 1.8.3] there exists a PsDO $B\in\Psi^{-r}_{\mathrm{ph}}(\mathbb{R}^n)$ such that $BA=I+T$, where I is identical operator and $T\in\Psi^{-\infty}:=\bigcap_{m\in\mathbb{R}}\Psi^m(\mathbb{R}^n)$. The operator B is called the left parametrix of A. Writing $u=BAu-Tu$, we easily get (4.1) by Lemma 4.1.

Let Ω be an arbitrary nonempty open subset in $\mathbb{R}^n$. We study an interior smoothness of a solution to the equation $Au=f$ in Ω.

Let us introduce some relevant spaces. By $H^{-\infty}(\mathbb{R}^n)$ we denote the union of all the spaces $H^{s,\varphi}(\mathbb{R}^n)$ with $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$. The linear space $H^{-\infty}(\mathbb{R}^n)$ is endowed with the inductive limit topology. We set

$$\begin{aligned} H^{s,\varphi}_{\mathrm{int}}(\Omega):=\bigl\{f\in H^{-\infty}(\mathbb{R}^n):\;\chi f\in H^{s,\varphi}(\mathbb{R}^n)\\ \text{for all }\;\chi\in C^\infty_{\mathrm{b}}(\mathbb{R}^n),\;\operatorname{supp}\chi\subset\Omega,\;\operatorname{dist}(\operatorname{supp}\chi,\partial\Omega)>0\bigr\}.\end{aligned} \tag{4.2}$$

A topology in $H^{s,\varphi}_{\mathrm{int}}(\Omega)$ is defined by the seminorms $f\mapsto\|\chi f\|_{H^{s,\varphi}(\mathbb{R}^n)}$ with χ being the same as in (4.2).

Theorem 4.4. *Let $u\in H^{-\infty}(\mathbb{R}^n)$ be a solution to the equation $Au=f$ in Ω with $f\in H^{s,\varphi}_{\mathrm{int}}(\Omega)$ for some $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$. Then $u\in H^{s+r,\varphi}_{\mathrm{int}}(\Omega)$.*

The special case when $\Omega=\mathbb{R}^n$ (global smoothness) follows at once from the equality $u=Bf-Tu$, with B being the left parametrix, and Lemma 4.1. In general, we deduce Theorem 4.4 from this case if we rearrange A and the operator of multiplication by a function χ satisfying (4.2). Then we write

$$A\chi u=A\chi\eta u=\chi\,A\eta u+A'\eta u=\chi f+\chi\,A(\eta-1)u+A'\eta u, \tag{4.3}$$

where $A'\in\Psi^{r-1}(\mathbb{R}^n)$, and the function η has the same properties as χ and is equal to 1 in a neighbourhood of $\operatorname{supp}\chi$. Now, if $u\in H^{s+r-k,\varphi}_{\mathrm{int}}(\Omega)$ for some integer $k\geq 1$, then the right-hand side of (4.3) belongs to $H^{s-k+1,\varphi}(\mathbb{R}^n)$ that implies $\chi u\in H^{s+r-k+1,\varphi}(\mathbb{R}^n)$, i.e. $u\in H^{s+r-k+1,\varphi}_{\mathrm{int}}(\Omega)$. By induction in k we have $u\in H^{s+r,\varphi}_{\mathrm{int}}(\Omega)$.

It is useful to compare Theorem 4.4 with Hörmander's Regularity Theorem. If A is a partial *differential* operator, and Ω is bounded, then Theorem 4.4 is a consequence of the Hörmander theorem.

Applying Theorems 4.4 and 3.7 iii) we get the following sufficient condition for the solution u to have continuous and bounded derivatives of the prescribed order.

Theorem 4.5. *Let $u \in H^{-\infty}(\mathbb{R}^n)$ be a solution to the equation $Au = f$ in Ω, with $f \in H^{k-r+n/2,\varphi}_{\mathrm{int}}(\Omega)$ for some integer $k \geq 0$ and function parameter $\varphi \in \mathcal{M}$. Suppose that φ satisfies (3.6). Then u has the continuous partial derivatives on Ω up to the order k, and they are bounded on every set $\Omega_0 \subset \Omega$ with $\operatorname{dist}(\Omega_0, \partial\Omega) > 0$. In particular, if $\Omega = \mathbb{R}^n$, then $u \in C^k_{\mathrm{b}}(\mathbb{R}^n)$.*

This theorem shows an advantage of the refined Sobolev scale over the Sobolev scale when a classical smoothness of a solution is under investigation. Indeed, if we restrict ourselves to the Sobolev case of $\varphi \equiv 1$, then we have to replace the condition $f \in H^{k-r+n/2,\varphi}_{\mathrm{int}}(\Omega)$ with the condition $f \in H^{k-r+\varepsilon+n/2,1}_{\mathrm{int}}(\Omega)$ for some $\varepsilon > 0$. The last condition is far stronger than previous one.

Note that the condition (3.6) not only is sufficient in Theorem 3.3 but also is necessary on the class of all the considered solutions u. Namely, (3.6) is equivalent to the implication

$$\big(u \in H^{-\infty}(\mathbb{R}^n), \quad f := Au \in H^{k-r+n/2,\varphi}_{\mathrm{int}}(\Omega) \big) \;\; \Rightarrow \;\; u \in C^k(\Omega). \tag{4.4}$$

Indeed, if $u \in H^{k+n/2,\varphi}_{\mathrm{int}}(\Omega)$, then $f = Au \in H^{k-r+n/2,\varphi}_{\mathrm{int}}(\Omega)$, whence $u \in C^k(\Omega)$ if (4.4) holds. Thus (4.4) entails (3.6) in view of Hörmander's Theorem 3.3.

The analogs of Theorems 4.3–4.5 were proved in [98] for uniformly elliptic matrix PsDOs.

5. Hörmander spaces over a closed manifold

In this section we introduce a certain class of Hörmander spaces over a closed (compact) smooth manifold. Namely, using the spaces $H^{s,\varphi}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$ we construct their analogs for the manifold. We give three equivalent definitions of the analogs; these definitions are similar to those used for the Sobolev spaces (see, e.g., [132, Ch. 1, Sec. 5]).

5.1. **The equivalent definitions.** In what follows except Subsection 7.1, Γ is a closed (i.e. compact and without a boundary) infinitely smooth oriented manifold of an arbitrary dimension $n \geq 1$. We suppose that a certain C^∞-density dx is defined on Γ. As usual, $\mathcal{D}'(\Gamma)$ denotes the linear topological space of all distributions on Γ. The space $\mathcal{D}'(\Gamma)$ is antidual to the space $C^\infty(\Gamma)$ with respect to the natural extension of the scalar product in $L_2(\Gamma) := L_2(\Gamma, dx)$ by continuity. This extension is denoted by $(f, w)_\Gamma$ for $f \in \mathcal{D}'(\Gamma)$ and $w \in C^\infty(\Gamma)$.

Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. We give the following three equivalent definitions of the Hörmander space $H^{s,\varphi}(\Gamma)$.

The first definition exhibits the local properties of those distributions $f \in \mathcal{D}'(\Gamma)$ that form $H^{s,\varphi}(\Gamma)$. From the C^∞-structure on Γ, we arbitrarily choose a finite collection of the local charts $\alpha_j : \mathbb{R}^n \leftrightarrow \Gamma_j$, $j = 1, \ldots, \varkappa$, such that the open sets

Γ_j form the finite covering of Γ. Let functions $\chi_j \in C^\infty(\Gamma)$, $j = 1, \ldots, \varkappa$, form a partition of unity on Γ satisfying the condition $\operatorname{supp} \chi_j \subset \Gamma_j$.

Definition 5.1. The linear space $H^{s,\varphi}(\Gamma)$ is defined by the formula

$$H^{s,\varphi}(\Gamma) := \bigl\{ f \in \mathcal{D}'(\Gamma) : \ (\chi_j f) \circ \alpha_j \in H^{s,\varphi}(\mathbb{R}^n) \ \ \forall\, j = 1, \ldots \varkappa \bigr\}.$$

Here $(\chi_j f) \circ \alpha_j$ is the representation of the distribution $\chi_j f$ in the local chart α_j. The inner product in the space $H^{s,\varphi}(\Gamma)$ is introduced by the formula

$$(f_1, f_2)_{H^{s,\varphi}(\Gamma)} := \sum_{j=1}^{\varkappa} ((\chi_j f_1) \circ \alpha_j, (\chi_j\, f_2) \circ \alpha_j)_{H^{s,\varphi}(\mathbb{R}^n)}$$

and induces the norm in the usual way.

In the special case where $\varphi \equiv 1$ the space $H^{s,\varphi}(\Gamma)$ coincides with the inner product Sobolev space $H^s(\Gamma)$ of order s. The Sobolev spaces on Γ are known to be complete and independent (up to equivalence of norms) of the choice of the local charts and the partition of unity.

The second definition connects the space $H^{s,\varphi}(\Gamma)$ with the Sobolev scale by means of the interpolation.

Definition 5.2. Let two integers k_0 and k_1 be such that $k_0 < s < k_1$. We define

$$H^{s,\varphi}(\Gamma) := [H^{k_0}(\Gamma), H^{k_1}(\Gamma)]_\psi, \tag{5.1}$$

where the interpolation parameter ψ is given by the formula (3.8) with $\varepsilon := s - k_0$ and $\delta := k_1 - s$.

It is useful in the spectral theory to have the third definition of $H^{s,\varphi}(\Gamma)$ that connects the norm in $H^{s,\varphi}(\Gamma)$ with a certain function of $1 - \Delta_\Gamma$. As usual, Δ_Γ is the Beltrami-Laplace operator on the manifold Γ endowed with the Riemannian metric that induces the density dx; see, e.g., [127, Sec. 22.1].

Definition 5.3. The space $H^{s,\varphi}(\Gamma)$ is defined to be the completion of $C^\infty(\Gamma)$ with respect to the Hilbert norm

$$f \ \mapsto \ \|(1 - \Delta_\Gamma)^{s/2} \varphi((1 - \Delta_\Gamma)^{1/2})\, f\|_{L_2(\Gamma)}, \quad f \in C^\infty(\Gamma). \tag{5.2}$$

Theorem 5.4. *Definitions 5.1, 5.2, and 5.3 are mutually equivalent, that is they define the same Hilbert space $H^{s,\varphi}(\Gamma)$ up to equivalence of norms.*

Let us explain how to prove this fundamental theorem.

The equivalence of Definitions 5.1 and 5.2. We use Definition 5.1 as a starting point and show that the equality (5.1) holds true up to equivalence of norms. We apply the $\mathbb{R}^n$-analog of (5.1), due to Theorem 3.8, and pass to local coordinates on Γ. Namely, let the mapping T take each $f \in \mathcal{D}'(\Gamma)$ to the vector with components $(\chi_j f) \circ \alpha_j$, $j = 1, \ldots, \varkappa$. We get the bounded linear operator

$$T : \ H^{s,\varphi}(\Gamma) \to (H^{s,\varphi}(\mathbb{R}^n))^\varkappa. \tag{5.3}$$

It has the right inverse bounded linear operator

$$K : \ (H^{s,\varphi}(\mathbb{R}^n))^\varkappa \to H^{s,\varphi}(\Gamma), \tag{5.4}$$

where the mapping K can be constructed with the help of the local charts and is independent of parameters s and φ. If we consider these operators in the Sobolev case of $\varphi\equiv 1$ and use the $\mathbb{R}^n$-analog of (5.1), then we get the bounded operators

$$T:\,[H^{k_0}(\Gamma),H^{k_1}(\Gamma)]_\psi\rightarrow(H^{s,\varphi}(\mathbb{R}^n))^\varkappa, \tag{5.5}$$

$$K:\,(H^{s,\varphi}(\mathbb{R}^n))^\varkappa\rightarrow[H^{k_0}(\Gamma),H^{k_1}(\Gamma)]_\psi. \tag{5.6}$$

Now it follows from (5.3) and (5.6) that the identity mapping KT establishes the continuous embedding of the space $H^{s,\varphi}(\Gamma)$ into the interpolation space $[H^{k_0}(\Gamma),H^{k_1}(\Gamma)]_\psi$, whereas the inverse continuous embedding is valid by (5.4) and (5.5). Thus the equality (5.1) holds true up to equivalence of norms (for details, see [77, Sec. 3] or [81, Sec. 3.3]).

By equivalence of Definitions 5.1 and 5.2, the space $H^{s,\varphi}(\Gamma)$ is complete and independent (up to equivalence of norms) of the choice of the local charts and the partition of unity on Γ. Moreover, the set $C^\infty(\Gamma)$ is dense in $H^{s,\varphi}(\Gamma)$.

The equivalence of Definitions 5.2 and 5.3. Let us use Definition 5.2 as a starting point. If $s>0$, then we choose $k_0:=0$ and $k_1:=2k>s$ for some integer k in (5.1). By $\Lambda_k(\Gamma)$ we denote the Sobolev space $H^{2k}(\Gamma)$ endowed with the equivalent norm $\|(1-\Delta_\Gamma)^k f\|_{L_2(\Gamma)}$ of f. We have

$$\begin{aligned}\|f\|_{H^{s,\varphi}(\Gamma)} &= \|f\|_{[H^0(\Gamma),H^{2k}(\Gamma)]_\psi}\asymp\|f\|_{[L_2(\Gamma),\Lambda_k(\Gamma)]_\psi}=\|\psi((1-\Delta_\Gamma)^k)f\|_{L_2(\Gamma)}\\ &= \|(1-\Delta_\Gamma)^{s/2}\varphi((1-\Delta_\Gamma)^{1/2})\,f\|_{L_2(\Gamma)},\end{aligned}$$

with $f\in C^\infty(\Gamma)$, because $(1-\Delta_\Gamma)^k$ is the generating operator for the couple $[L_2(\Gamma),\Lambda_k(\Gamma)]$. Thus the norm (5.2) is equivalent to the norm in $H^{s,\varphi}(\Gamma)$ on the dense set $C^\infty(\Gamma)$ if $s>0$. The case of $s\leq 0$ can be reduced to the previous one with the help of the homeomorphism

$$(1-\Delta_\Gamma)^k:\,H^{s+2k,\varphi}(\Gamma)\leftrightarrow H^{s,\varphi}(\Gamma),$$

with $s+2k>0$ for some integer $k\geq 1$. This homeomorphism follows from the Sobolev case of $\varphi\equiv 1$ by the interpolation formula (5.1) (for details, see [81, Sec. 3.4]).

Now we can give

Definition 5.5. The class of Hilbert spaces

$$\bigl\{H^{s,\varphi}(\Gamma):\,s\in\mathbb{R},\,\varphi\in\mathcal{M}\,\bigr\} \tag{5.7}$$

is called the refined Sobolev scale over the manifold Γ.

5.2. **The properties.** We consider some important properties of the scale (5.7). They are inherited from the refined Sobolev scale over $\mathbb{R}^n$.

Theorem 5.6. *Let $s\in\mathbb{R}$ and $\varphi,\varphi_1\in\mathcal{M}$. The following assertions are true:*

- i) *The dense compact embedding $H^{s+\varepsilon,\varphi_1}(\Gamma)\hookrightarrow H^{s,\varphi}(\Gamma)$ is valid for each $\varepsilon>0$.*
- ii) *The function φ/φ_1 is bounded in a neighbourhood of $+\infty$ if and only if $H^{s,\varphi_1}(\Gamma)\hookrightarrow H^{s,\varphi}(\Gamma)$. This embedding is continuous and dense. It is compact if and only if $\varphi(t)/\varphi_1(t)\rightarrow 0$ as $t\rightarrow+\infty$.*

iii) *Let integer $k \geq 0$ be given. Inequality (3.6) is equivalent to the embedding $H^{k+n/2,\varphi}(\Gamma) \hookrightarrow C^k(\Gamma)$. The embedding is compact.*

iv) *The spaces $H^{s,\varphi}(\Gamma)$ and $H^{-s,1/\varphi}(\Gamma)$ are mutually dual (up to equivalence of norms) with respect to the inner product in $L_2(\mathbb{R}^n)$.*

This theorem except the statements on compactness follows from Theorem 3.7 in view of Definition 5.1. The compactness of the embeddings is a consequence of the compactness of Γ. Namely, the statements in assertions i) and ii) follow from the next proposition. For each number $r > 0$, the embedding

$$\{u \in H^{s,\varphi_1}(\mathbb{R}^n) : \mathrm{dist}(0, \mathrm{supp}\, u) \leq r\} \hookrightarrow H^{s,\varphi}(\mathbb{R}^n)$$

is compact if and only if $\varphi(t)/\varphi_1(t) \to 0$ as $t \to \infty$. This proposition is a special case of Hörmander's result [43, Sec. 2, Theorem 2.2.3]. Now we get the compactness of the embedding in assertion iii) if we write

$$H^{k+n/2,\varphi}(\Gamma) \hookrightarrow H^{k+n/2,\varphi_1}(\Gamma) \hookrightarrow C^k(\Gamma)$$

for a function φ_1 such that the first embedding is compact.

Theorem 5.7. *Theorems 3.8 and 3.9 remain true if we replace the designation $\mathbb{R}^n$ with Γ, and the phrase 'equality of norms' with 'equivalence of norms'.*

Theorem 5.7 can be proved by means of a repeated application of the interpolation formula (5.1). We can also deduce this theorem from its $\mathbb{R}^n$-analogs with the help of operators T and K used above.

6. Elliptic operators on a closed manifold

Recall that Γ is a closed infinitely smooth oriented manifold. In this section we study an arbitrary elliptic classical PsDO A on the refined Sobolev scale over Γ. We prove that A is a bounded and Fredholm operator on the respective pairs of Hörmander spaces, and investigate the smoothness of a solution to the equation $Au = f$. Our results [96, Sec. 4 and 5] refine the classical theorems on elliptic operators on the Sobolev scale over a closed smooth manifold (see, e.g., [7, Sec. 2], [46, Sec. 19], and [127, § 8]). We also use some elliptic PsDOs to get an important class of equivalent norms in $H^{s,\varphi}(\Gamma)$ [81, Sec. 3.4].

6.1. The main properties.

By $\Psi^r(\Gamma)$ with $r \in \mathbb{R}$ we denote the class of all the PsDOs A on Γ (generally, not classical) such that the image of A in each local chart on Γ belongs to $\Psi^r(\mathbb{R}^n)$; see [7, Sec. 2.1].

Lemma 6.1. *Let $A \in \Psi^r(\Gamma)$, with $r \in \mathbb{R}$. Then the restriction of the mapping $u \mapsto Au$, $u \in \mathcal{D}'(\Gamma)$, to the space $H^{s,\varphi}(\Gamma)$ is the bounded linear operator*

$$A : H^{s,\varphi}(\Gamma) \to H^{s-r,\varphi}(\Gamma) \tag{6.1}$$

for each $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$.

This lemma follows from the Sobolev $\varphi \equiv 1$ case [7, Sec. 2.1, Theorem 2.1.2] by the interpolation in view of Theorem 5.7.

By $\Psi^r_{\mathrm{ph}}(\Gamma)$ we denote the subset in $\Psi^r(\Gamma)$ that consists of all classical PsDOs of the order r; see [7, Sec. 2.1]. The image of PsDO $A \in \Psi^r_{\mathrm{ph}}(\Gamma)$ in every local chart on Γ belongs to $\Psi^r_{\mathrm{ph}}(\mathbb{R}^n)$.

Definition 6.2. A PsDO $A \in \Psi^r_{\mathrm{ph}}(\mathbb{R}^n)$ is called elliptic on Γ if $a_0(x,\xi) \neq 0$ for each point $x \in \Gamma$ and covector $\xi \in T^*_x\Gamma \setminus \{0\}$. Here $a_0(x,\xi)$ is the principal symbol of A, and $T^*_x\Gamma$ is the cotangent space to Γ at x.

Let $r \in \mathbb{R}$. Suppose a PsDO $A \in \Psi^r_{\mathrm{ph}}(\Gamma)$ to be elliptic on Γ.

By A^+ we denote the PsDO formally adjoint to A with respect to the sesquilinear form $(\cdot,\cdot)_\Gamma$. Since both A and A^+ are elliptic on Γ, both the spaces

$$N := \{\, u \in C^\infty(\Gamma) : \; Au = 0 \;\text{ on }\; \Gamma \,\},$$
$$N^+ := \{\, v \in C^\infty(\Gamma) : \; A^+v = 0 \;\text{ on }\; \Gamma \,\}$$

are finite-dimensional [127, Sec. 8.2, Theorem 8.1].

Recall the following definition.

Definition 6.3. Let X and Y be Banach spaces. The bounded linear operator $T : X \to Y$ is said to be Fredholm if its kernel $\ker T$ and co-kernel $\operatorname{coker} T := Y/T(X)$ are finite-dimensional. The number $\operatorname{ind} T := \dim\ker T - \dim\operatorname{coker} T$ is called the index of the Fredholm operator T.

If the operator $T : X \to Y$ is Fredholm, then its range $T(X)$ is closed in Y; see, e.g., [46, Sec. 19.1, Lemma 19.1.1].

Theorem 6.4. *Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. For the elliptic PsDO A, the operator (6.1) is Fredholm, has the kernel N and the range*

$$A(H^{s,\varphi}(\Gamma)) = \bigl\{ f \in H^{s-r,\varphi}(\Gamma) : \; (f,v)_\Gamma = 0 \;\; \textit{for all} \;\; v \in N^+ \bigr\}. \tag{6.2}$$

The index of the operator (6.1) is equal to $\dim N - \dim N^+$ *and independent of s and φ.*

Theorem 6.4 is well known in the Sobolev case of $\varphi \equiv 1$; see, e.g., [46, Sec. 19.2, Theorem 19.2.1]. For an arbitrary $\varphi \in \mathcal{M}$, we deduce this theorem from the Sobolev case with the help of the interpolation. Indeed, consider the bounded Fredholm operators $A : H^{s\mp1}(\Gamma) \to H^{s\mp1-r}(\Gamma)$. By applying Theorem 5.7, we have the bounded operator

$$A : \; H^{s,\varphi}(\Gamma) = [H^{s-1}(\Gamma), H^{s+1}(\Gamma)]_\psi \to [H^{s-1-r}(\Gamma), H^{s+1-r}(\Gamma)]_\psi = H^{s-r,\varphi}(\Gamma).$$

Here ψ is the interpolation parameter defined by the formula (3.8) with $\varepsilon = \delta = 1$. This operator has the properties stated in Theorem 6.4 because of the next proposition.

Proposition 6.5. *Let $X = [X_0, X_1]$ and $Y = [Y_0, Y_1]$ be admissible couples of Hilbert spaces, and a linear mapping T be given on X_0. Suppose we have the Fredholm bounded operators $T : X_j \to Y_j$, with $j = 0, 1$, that possess the common kernel and the common index. Then, for an arbitrary interpolation parameter $\psi \in \mathcal{B}$, the bounded operator $T : X_\psi \to Y_\psi$ is Fredholm, has the same kernel and the same index, moreover its range $T(X_\psi) = Y_\psi \cap T(X_0)$.*

This proposition was proved by G. Geymonat [31, p. 280, Proposition 5.2] for arbitrary interpolation functors given on the category of couples of Banach spaces. The proof for Hilbert spaces is analogous.

If both the spaces N and N^+ are trivial, then the operator (6.1) is a homeomorphism. Generally, the index of (6.1) is equal to zero provided that $\dim\Gamma \geq 2$ (see [10] and [7, Sec. 2.3 f]). In the case where $\dim\Gamma = 1$, the index can be nonzero. If A is a differential operator, then the index is always zero.

The Fredholm property of A implies the following a priory estimate.

Theorem 6.6. *Let $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$, and $\sigma < s$. Then*

$$\|u\|_{H^{s,\varphi}(\Gamma)} \leq c\left(\|Au\|_{H^{s-r,\varphi}(\Gamma)} + \|u\|_{H^{\sigma,\varphi}(\Gamma)}\right) \quad \textit{for all} \quad u \in H^{s,\varphi}(\Gamma);$$

here the number $c > 0$ is independent of u.

We get the above implication if we use the compactness of the embedding $H^{s,\varphi}(\Gamma) \hookrightarrow H^{\sigma,\varphi}(\Gamma)$ for $\sigma < s$ and apply the following proposition [7, Sec. 2.3, Theorem 2.3.4].

Proposition 6.7. *Let X, Y, and Z be Banach spaces. Suppose that the compact embedding $X \hookrightarrow Z$ is valid, and a bounded linear operator $T : X \to Y$ is given. Then $\ker T$ is finite-dimensional and $T(X)$ is closed in Y if and only if there exists a number $c > 0$ such that*

$$\|u\|_X \leq c\,(\|Tu\|_Y + \|u\|_Z) \quad \textit{for all} \quad u \in X.$$

Now we study a local smoothness of a solution to the elliptic equation $Au = f$. Let Γ_0 be an nonempty open set on the manifold Γ, and define

$$H^{s,\varphi}_{\mathrm{loc}}(\Gamma_0) := \left\{ f \in \mathcal{D}'(\Gamma) : \chi f \in H^{s,\varphi}(\Gamma),\ \ \forall\, \chi \in C^\infty(\Gamma),\ \operatorname{supp}\chi \subseteq \Gamma_0 \right\}. \tag{6.3}$$

Theorem 6.8. *Let $u \in \mathcal{D}'(\Gamma)$ be a solution to the equation $Au = f$ on Γ_0 with $f \in H^{s,\varphi}_{\mathrm{loc}}(\Gamma_0)$ for some $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. Then $u \in H^{s+r,\varphi}_{\mathrm{loc}}(\Gamma_0)$.*

The special case when $\Gamma_0 = \Gamma$ (global smoothness) follows from Theorem 6.4. Indeed, using (6.2) we can write $f = Av$ for some $v \in H^{s+r,\varphi}(\Gamma)$, whence $u-v \in N$ and $u \in H^{s+r,\varphi}(\Gamma)$. In general, we deduce Theorem 6.8 from this case reasoning similar to the proof of Theorem 4.4.

If we bring Theorems 6.8 and 5.6 iii) together, then we get the following sufficient condition for the solution u to have continuous derivatives of the prescribed order on Γ_0. Recall that $n := \dim\Gamma$.

Theorem 6.9. *Let $u \in \mathcal{D}'(\Gamma)$ be a solution to the equation $Au = f$ on Γ_0, with $f \in H^{k-r+n/2,\varphi}_{\mathrm{loc}}(\Gamma_0)$ for some integer $k \geq 0$ and function parameter $\varphi \in \mathcal{M}$. If φ satisfies (3.6), then $u \in C^k(\Gamma_0)$.*

Here it is important that the condition (3.6) not only is sufficient for u to belong to $C^k(\Gamma_0)$ but also is necessary on the class of all the considered solutions u.

6.2. **The equivalent norms induced by elliptic operators.** Let $r > 0$, and a PsDO $A \in \Psi^r_{\mathrm{ph}}(\Gamma)$ be elliptic on Γ. We may consider A as a closed unbounded operator on $L_2(\Gamma)$ with the domain $H^r(\Gamma)$; see, e.g., [7, Sec. 2.3, Theorem 2.3.5]. Suppose the operator A to be positive in $L_2(\Gamma)$. Then A is self-adjoint in $L_2(\Gamma)$ [7, Sec. 2.3, Theorem 2.3.7].

For $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$, we set

$$\varphi_{s,r}(t):=t^{s/r}\varphi(t^{1/r})\ \text{ for }\ t\geq 1,\quad\text{ and }\quad \varphi_{s,r}(t):=\varphi(1)\ \text{ for }\ 0<t<1.$$

The operator $\varphi_{s,r}(A)$ is regarded as the Borel function $\varphi_{s,r}$ of the positive self-adjoint operator A in $L_2(\Gamma)$. Consider the norm

$$f\ \mapsto\ \|\varphi_{s,r}(A)f\|_{L_2(\Gamma)},\quad f\in C^\infty(\Gamma). \tag{6.4}$$

Theorem 6.10. *Let $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$. The norm in the space $H^{s,\varphi}(\Gamma)$ is equivalent to the norm (6.4) on the set $C^\infty(\Gamma)$. Thus $H^{s,\varphi}(\Gamma)$ is the completion of $C^\infty(\Gamma)$ with respect to the norm (6.4).*

The proof of this theorem is quite similar to the reasoning we did to demonstrate the equivalence of Definitions 5.2 and 5.3.

If $H^{s,\varphi}(\Gamma)\hookrightarrow L_2(\Gamma)$, then Theorem 6.10 entails the following.

Corollary 6.11. *Let $s\geq 0$ and $\varphi\in\mathcal{M}$. In the case where $s=0$ we suppose that the function $1/\varphi$ is bounded in a neighbourhood of ∞. Then the space $H^{s,\varphi}(\Gamma)$ coincides with the domain of the operator $\varphi_{s,r}(A)$, and the norm in the space $H^{s,\varphi}(\Gamma)$ is equivalent to the graphics norm of $\varphi_{s,r}(A)$.*

It is useful to have an analog of Theorem 6.10 formulated in terms of sequences. For this purpose, we recall some spectral properties of the operator A; see, e.g., [7, Sec. 6.1] or [127, Sec. 8.3 and 15.2].

There is an orthonormal basis $(h_j)_{j=1}^{\infty}$ of $L_2(\Gamma)$ formed by eigenfunctions $h_j\in C^\infty(\Gamma)$ of the operator A. Let $\lambda_j>0$ is the eigenvalue corresponding to h_j; the enumeration is such that $\lambda_j\leq\lambda_{j+1}$. Then the spectrum of A coincides with the set $\{\lambda_1,\lambda_2,\lambda_3,\ldots\}$ of all eigenvalues of A, and the asymptotics formula holds: $\lambda_j\sim c\,j^{\,r/n}$ as $j\to\infty$. Each distribution $f\in\mathcal{D}'(\Gamma)$ is expanded into the Fourier series

$$f=\sum_{j=1}^{\infty}\,c_j(f)\,h_j \tag{6.5}$$

convergent in $\mathcal{D}'(\Gamma)$; here $c_j(f):=(f,h_j)_\Gamma$ are the Fourier coefficients of f.

Theorem 6.12. *Let $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$. Then the next equality of spaces with equivalence of norms in them holds:*

$$H^{s,\varphi}(\Gamma)=\Bigl\{f\in\mathcal{D}'(\Gamma):\,\sum_{j=1}^{\infty}\,j^{\,2s/n}\,\varphi^2(j^{\,1/n})\,|c_j(f)|^2<\infty\Bigr\}, \tag{6.6}$$

$$\|f\|_{H^{s,\varphi}(\Gamma)}\asymp\Bigl(\,\sum_{j=1}^{\infty}\,j^{\,2s/n}\,\varphi^2(j^{\,1/n})\,|c_j(f)|^2\,\Bigr)^{1/2}. \tag{6.7}$$

This theorem follows from Theorem 6.10 since

$$\varphi_{s,r}(A)\,f=\sum_{j=1}^{\infty}\,\varphi_{s,r}(\lambda_j)\,c_j(f)\,h_j\quad(\text{convergence in }L_2(\Gamma))$$

for each function f from the domain of the operator $\varphi_{s,r}(A)$. Here $\varphi_{s,r}(\lambda_j)\asymp j^{\,s/n}\varphi(j^{\,1/n})$ with integers $j\geq 1$ in view of the asymptotics formula mentioned above. By applying Parseval's equality we get (6.7) and then can deduce (6.6).

Corollary 6.13. *Suppose $f \in H^{s,\varphi}(\Gamma)$, then the series (6.5) converges to f in the space $H^{s,\varphi}(\Gamma)$.*

This is a simple consequence of (6.7).

Example 6.14. Let Γ be a unit circle and $A := 1 - d^2/dt^2$, where t sets the natural parametrization on Γ. The eigenfunctions $h_j(t) := (2\pi)^{-1}e^{ijt}$, $j \in \mathbb{Z}$, of A form an orthonormal basis in $L_2(\Gamma)$, with $\lambda_j := 1 + j^2$ being the corresponding eigenvalues. Therefore the equivalence (6.7) becomes

$$\|f\|_{H^{s,\varphi}(\Gamma)} \asymp \|\varphi_{s,2}(A)f\|_{L_2(\Gamma)} = \Bigl(\sum_{j=-\infty}^{\infty} (1+j^2)^s \, \varphi^2((1+j^2)^{1/2}) \, |c_j(f)|^2 \Bigr)^{1/2}.$$

Note that we can chose the basis formed by the real-valued eigenfunctions $h_0(t) := (2\pi)^{-1}$, $h_j(t) := \pi^{-1}\cos jt$, and $h_{-j}(t) := \pi^{-1}\sin jt$, with integral $j \geq 1$. Then

$$\|f\|^2_{H^{s,\varphi}(\Gamma)} \asymp |a_0(f)|^2 + \sum_{j=1}^{\infty} j^{2s} \, \varphi^2(j) \, \bigl(|a_j(f)|^2 + |b_j(f)|^2\bigr),$$

where $a_0(f)$, $a_j(f)$, and $b_j(f)$ are the Fourier coefficients of f with respect to these eigenfunctions. In the considered case, $H^{s,\varphi}(\Gamma)$ is closely related to the spaces of periodic functions considered by A.I. Stepanets [130, Part I, Ch. 3, Sec. 7.1].

7. Applications to spectral expansions

In this section, we investigate the convergence of expansions in eigenfunctions of normal (in particular, self-adjoint) elliptic PsDOs given on a compact smooth manifold. Using the refined Sobolev scale, we find new sufficient conditions for the convergence almost everywhere; they are expressed in constructive terms of regularity of functions. We also give a criterion for convergence in the metrics of C^k on the classes being Hörmander spaces. Beforehand let us recall some classical results concerning the convergence almost everywhere of arbitrary orthogonal series. The results will be used below.

7.1. The classical results.

In this subsection, Γ is an arbitrary set with a finite measure μ. Suppose that $(h_j)_{j=1}^{\infty}$ is an orthonormal system of functions in $L_2(\Gamma) := L_2(\Gamma, d\mu)$, generally, complex-valued. The following proposition is a general version of the well-known Menshov-Rademacher convergence theorem.

Theorem 7.1. *Let a sequence of complex numbers $(a_j)_{j=1}^{\infty}$ be such that*

$$\sum_{j=2}^{\infty} |a_j|^2 \log^2 j < \infty. \tag{7.1}$$

Then the series

$$\sum_{j=1}^{\infty} a_j \, h_j(x) \tag{7.2}$$

converges almost everywhere on Γ.

This theorem was proved independently by D.E. Menshov [68] and H. Rademacher [112] in the classical case where Γ is a bounded interval on $\mathbb{R}$, μ is the Lebesgue measure, and the functions h_j are real-valued. The proof of the Menshov–Rademacher theorem given in [51, Ch. 8, § 1] remains valid in the general situation that we consider (apparently, the most general case is treated in [89]).

It is important that the Menshov–Rademacher theorem is precise. Menshov [68] gave an example of an orthonormal system $(h_j)_{j=1}^{\infty}$ in $L_2((0,1))$ for which Theorem 7.1 will not be true if one replaces, in (7.1), the sequence $(\log^2 j)_{j=1}^{\infty}$ by any increasing sequence of positive numbers $\omega_j = o(\log^2 j)$ with $j \to \infty$. This result is set forth, e.g., in the monograph [51, Ch. 8, § 1, Theorem 2].

Note that, for series (7.2) with coefficients subject to (7.1), the convergence almost everywhere need not be unconditional; see, e.g., [51, Ch. 8, § 2]. Recall that a series of functions is said to be unconditionally convergent almost everywhere on a set if it remains convergent almost everywhere on the set after arbitrary permutation of its terms (the null measure set of divergence may vary.) The following proposition is a general version of the Orlicz theorem on unconditional convergence of orthogonal series of functions.

Theorem 7.2. *Let a sequence of complex numbers $(a_j)_{j=1}^{\infty}$ and increasing sequence of positive numbers $(\omega_j)_{j=1}^{\infty}$ satisfy the conditions*

$$\sum_{j=2}^{\infty} |a_j|^2 \, (\log^2 j) \, \omega_j < \infty, \quad \sum_{j=2}^{\infty} \frac{1}{j \, (\log j) \, \omega_j} < \infty.$$

Then the series (7.2) converges unconditionally almost everywhere on Γ.

This equivalent statement of W. Orlicz' theorem [103] was given by P.L. Ulyanov [136, § 4]; they considered the classical case mentioned above. In our (more general) case, Theorem 6.2 follows from K. Tandori's theorem [131], which remains valid for arbitrary measure space [88, 89]. As Tandori proved [131], the Orlicz theorem is the best possible in the sense that its condition on the sequence $(\omega_j)_{j=1}^{\infty}$ cannot be weaken.

7.2. **The convergence of spectral expansions.** Further, Γ is a closed infinitely smooth oriented manifold, and $n = \dim \Gamma$. A C^{∞}-density dx is given on Γ and defines the finite measure there. Let a PsDO $A \in \Psi^{r}_{\mathrm{ph}}(\Gamma)$, with $r > 0$, be elliptic on Γ. Suppose that A is a normal (unbounded) operator on $L_2(\Gamma) = L_2(\Gamma, dx)$. Let $(h_j)_{j=1}^{\infty}$ be a complete orthonormal system of eigenfunctions of this operator. They are enumerated so that $|\lambda_j| \leq |\lambda_{j+1}|$ for $j = 1, 2, 3, \ldots$, where $Ah_j = \lambda_j h_j$. For an arbitrary function $f \in L_2(\Gamma)$, we consider its expansion into the Fourier series (6.5) with respect to the system $(h_j)_{j=1}^{\infty}$.

We say that the series (6.5) converges on a function class $X(\Gamma)$ in the indicated sense if, for every function $f \in X(\Gamma)$, the series converges to f in the indicated manner.

We investigate the convergence almost everywhere of the spectral expansion (6.5) with the help of Theorems 6.12, 7.1, and 7.2. Let $\log^* t := \max\{1, \log t\}$ for $t \geq 1$.

Theorem 7.3. *The series* (6.5) *converges almost everywhere on* Γ *on the class* $H^{0,\log^*}(\Gamma)$.

Indeed, if A is a positive operator on $L_2(\Gamma)$, then by Theorem 6.12 we have

$$|c_1(f)|^2 + \sum_{j=2}^{\infty} |c_j(f)|^2 \log^2 j \asymp \|f\|^2_{H^{0,\log^*}(\Gamma)} < \infty \quad \text{for} \quad f \in H^{0,\log^*}(\Gamma).$$

This and Theorem 7.1 yields Theorem 7.3. In general, if A is a normal operator, we should exchange A for the positive elliptic PsDO $B := 1 + A^*A$ in our consideration and use the fact that $(h_j)_{j=1}^{\infty}$ is a complete system of eigenfunctions of B.

Similarly, if we bring together Theorems 6.12 and 7.2, we will get the following result.

Theorem 7.4. *Let an increasing function* $\varphi \in \mathcal{M}$ *be such that*

$$\int_2^{\infty} \frac{dt}{t\,(\log t)\,\varphi^2(t)} < \infty.$$

Then the series (6.5) *converges unconditionally almost everywhere on* Γ *on the class* $H^{0,\varphi\log^*}(\Gamma)$.

Note that the applying of Hörmander spaces permits us to use the conditions of Theorems 7.1 and 7.2 in an exhaustive manner. If we remain in the framework of the Sobolev scale, we deduce that the series (6.5) converges (unconditionally) almost everywhere on Γ on the class $H^{0+}(\Gamma) := \bigcup_{\varepsilon>0} H^{\varepsilon}(\Gamma)$. The result is far rougher than those formulated in Theorems 7.3 and 7.4. In the special case of $A = \Delta_{\Gamma}$, this result was proved by C. Meaney [67]. (As Meaney noted, it has "all the qualities of a folk theorem".)

To compete this section we give a criterion for the convergence of the spectral expansions in the metrics of $C^k(\Gamma)$ on the classes $H^{s,\varphi}(\Gamma)$.

Theorem 7.5. *Let an integer* $k \geq 0$ *and function* $\varphi \in \mathcal{M}$ *be given. The series* (6.5) *converges in* $C^k(\Gamma)$ *on the class* $H^{k+n/2,\varphi}(\Gamma)$ *if and only if* φ *satisfies* (3.6).

This theorem results from Corollary 6.13 and Theorem 5.6 iii).

Note that the convergence conditions in Theorems 7.3, 7.4, and 7.5 are given in constructive terms of regularity of functions. The regularity properties can be determined locally on Γ according to Definition 5.1.

8. Hörmander spaces over Euclidean domains

In the next sections, we will consider some applications of Hörmander spaces to elliptic boundary problems in a bounded domain $\Omega \subset \mathbb{R}^n$. For this purpose we need the Hörmander spaces that consists of distributions given in Ω. The spaces of distributions supported on the closure $\overline{\Omega}$ of the domain Ω is also of use. These spaces are constructed from the Hörmander spaces over $\mathbb{R}^n$ in the standard way [138, Ch. 1, § 3]. We are interested in the Hörmander spaces that form the refined Sobolev scales over Ω and $\overline{\Omega}$. In this section, we give the definitions of these spaces and consider their properties, among them the interpolation properties being of great importance for us. We also study a connection between the refined Sobolev scales over Ω and its boundary (the trace theorems) and introduce riggings of $L_2(\Omega)$ with some Hörmander spaces.

8.1. **The definitions.** Let $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$.

Definition 8.1. Suppose that Q is a nonempty closed set in $\mathbb{R}^n$. The linear space $H^{s,\varphi}_Q(\mathbb{R}^n)$ is defined to consist of the distributions $u\in H^{s,\varphi}(\mathbb{R}^n)$ such that $\operatorname{supp} u\subseteq Q$. The space $H^{s,\varphi}_Q(\mathbb{R}^n)$ is endowed with the inner product and norm from $H^{s,\varphi}(\mathbb{R}^n)$.

The space $H^{s,\varphi}_Q(\mathbb{R}^n)$ is complete (i.e., Hilbert) because of the continuous embedding of the Hilbert space $H^{s,\varphi}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$.

Definition 8.2. Suppose that Ω is a nonempty open set in $\mathbb{R}^n$. The linear space $H^{s,\varphi}(\Omega)$ is defined to consist of the restrictions $v=u\restriction\Omega$ of all the distributions $u\in H^{s,\varphi}(\mathbb{R}^n)$ to Ω. The space $H^{s,\varphi}(\Omega)$ is endowed with the norm

$$\|v\|_{H^{s,\varphi}(\Omega)}:=\inf\bigl\{\,\|u\|_{H^{s,\varphi}(\mathbb{R}^n)}:\,u\in H^{s,\varphi}(\mathbb{R}^n),\;\;v=u\;\text{ in }\;\Omega\,\bigr\}. \tag{8.1}$$

By Definition 8.2, $H^{s,\varphi}(\Omega)$ is a factor space $H^{s,\varphi}(\mathbb{R}^n)/H^{s,\varphi}_{\widehat{\Omega}}(\mathbb{R}^n)$, where $\widehat{\Omega}:=\mathbb{R}^n\setminus\Omega$. Hence, the space $H^{s,\varphi}(\Omega)$ is Hilbert; the norm 8.1 is induced by the inner product

$$\bigl(v_1,v_2\bigr)_{H^{s,\varphi}(\Omega)}:=\bigl(u_1-\Pi u_1,u_2-\Pi u_2\bigr)_{H^{s,\varphi}(\mathbb{R}^n)}.$$

Here $u_j\in H^{s,\varphi}(\mathbb{R}^n)$, $u_j=v_j$ in Ω for $j=1,\,2$, and Π is the orthogonal projector of the space $H^{s,\varphi}(\mathbb{R}^n)$ onto the subspace $H^{s,\varphi}_{\widehat{\Omega}}(\mathbb{R}^n)$.

Both the spaces $H^{s,\varphi}_Q(\mathbb{R}^n)$ and $H^{s,\varphi}(\Omega)$ are separable. In the Sobolev case of $\varphi\equiv1$ we will omit the index φ in the designations of these and other $H^{s,\varphi}$-type spaces.

In what follows, we suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $n\geq2$, and its boundary $\partial\Omega$ is an infinitely smooth closed manifold of the dimension $n-1$. (Note that domains are defined to be an open and connected sets.) Consider the classes of Hörmander spaces

$$\bigl\{H^{s,\varphi}(\Omega):\,s\in\mathbb{R},\,\varphi\in\mathcal{M}\,\bigr\}\quad\text{and}\quad\bigl\{H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n):\,s\in\mathbb{R},\,\varphi\in\mathcal{M}\,\bigr\}. \tag{8.2}$$

The space $H^{s,\varphi}(\Omega)$ consists of distributions given in Ω, whereas the space $H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)$ consists of distributions supported on $\overline{\Omega}$.

Definition 8.3. The classes appearing in (8.2) are called the refined Sobolev scales over Ω and $\overline{\Omega}$ respectively.

8.2. **The interpolation properties.** The scales (8.2) have the interpolation properties analogous to those the refined Sobolev scale over $\mathbb{R}^n$ possesses.

Theorem 8.4. *Let a function $\varphi\in\mathcal{M}$ and positive numbers ε,δ be given, and let the interpolation parameter $\psi\in\mathcal{B}$ be defined by (3.8). Then, for each $s\in\mathbb{R}$, the* *following equalities of spaces with equivalence of norms in them hold:*

$$\big[H^{s-\varepsilon}(\Omega), H^{s+\delta}(\Omega)\big]_{\psi} = H^{s,\varphi}(\Omega) \tag{8.3}$$

$$\big[H^{s-\varepsilon}_{\overline{\Omega}}(\mathbb{R}^n), H^{s+\delta}_{\overline{\Omega}}(\mathbb{R}^n)\big]_{\psi} = H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n). \tag{8.4}$$

We will deduce this theorem from Theorem 3.8 with the help of the following result concerning the interpolation of subspaces and factor spaces.

Proposition 8.5. *Let $X = [X_0, X_1]$ be an admissible couple of Hilbert spaces, and Y_0 be a subspace in X_0. Then $Y_1 := X_1 \cap Y_0$ is a subspace in X_1. Suppose that there exists a linear mapping P which is a projector of X_j onto Y_j for $j = 0, 1$. Then the couples $[Y_0, Y_1]$ and $[X_0/Y_0, X_1/Y_1]$ are admissible, and, for each interpolation parameter $\psi \in \mathcal{B}$, the following equalities of spaces up to equivalence of norms in them hold:*

$$[Y_0, Y_1]_\psi = X_\psi \cap Y_0, \quad [X_0/Y_0, X_1/Y_1]_\psi = X_\psi/(X_\psi \cap Y_0).$$

Here $X_\psi \cap Y_0$ is a subspace in X_ψ.

Recall that, by definition, subspaces of a Hilbert space are closed, and projectors on subspaces are, generally, nonorthogonal. Proposition 8.5 was proved in H. Triebel's monograph [133, Sec. 1.17] for arbitrary interpolation functors given on the category of couples of Banach spaces. The proof for Hilbert spaces is quite similar.

Let us explain how to prove Theorem 8.4. It is known [133, Sec. 2.10.4, Theorem 2] that, for each integer $k > 0$, there exists a linear mapping $P_{k,Q}$ which is a projector of every space $H^\sigma(\mathbb{R}^n)$, with $|\sigma| < k$, onto the subspace $H^\sigma_Q(\mathbb{R}^n)$, where Q is a closed half-space in $\mathbb{R}^n$. Using the local coordinates methods and $P_{k,Q}$, we can construct a linear mapping that projects $H^\sigma(\mathbb{R}^n)$ onto $H^\sigma_{\overline{\Omega}}(\mathbb{R}^n)$ (or onto $H^\sigma_{\overline{\Omega}}(\mathbb{R}^n)$) with $|\sigma| < k$. Now Theorem 8.4 follows from Theorem 3.8 and Proposition 8.5.

Reasoning as above we can also deduce analogs of Theorem 3.9 (on interpolation) for the refined Sobolev scale given over Ω or $\overline{\Omega}$. We will not formulate them.

8.3. **Embeddings and other properties.** Let us consider some other important properties of scales (8.2). Among these properties, there are the following embeddings.

Theorem 8.6. *Let $s \in \mathbb{R}$ and $\varphi, \varphi_1 \in \mathcal{M}$. The next assertions are true:*

i) *The set $C^\infty(\,\overline{\Omega}\,)$ is dense in $H^{s,\varphi}(\Omega)$, whereas the set $C^\infty_0(\Omega)$ is dense in $H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)$.*

ii) *For each $\varepsilon > 0$, the dense compact embeddings hold:*

$$H^{s+\varepsilon,\,\varphi_1}(\Omega) \hookrightarrow H^{s,\varphi}(\Omega), \quad H^{s+\varepsilon,\,\varphi_1}_{\overline{\Omega}}(\mathbb{R}^n) \hookrightarrow H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n). \tag{8.5}$$

iii) *The function φ/φ_1 is bounded in a neighbourhood of $+\infty$ if and only if the embeddings (8.5) are valid for $\varepsilon = 0$. The embeddings are continuous and dense. They are compact if and only if $\varphi(t)/\varphi_1(t) \to 0$ as $t \to +\infty$.*

iv) *For every fixed integer $k \geq 0$, the inequality* (3.6) *is equivalent to the embedding $H^{k+n/2,\varphi}(\Omega) \hookrightarrow C^k(\,\overline{\Omega}\,)$. This embedding is compact.*

Assertion i) can be deduced directly from the Sobolev case of $\varphi \equiv 1$ with the help of the interpolation Theorem 8.4. Assertions ii)–iv) follow from Theorem 3.7 with the exception of the statements on compactness. The compactness of the embeddings is a consequences of the boundedness of Ω and can be proved similarly to the argument of Theorem 5.6. The general analogs of assertions i)–iv) for the Hörmander inner product spaces parametrized by arbitrary weight functions were proved by L.R. Volevich and B.P. Paneach in [138, § 3, 7, and 8].

Further we examine the properties that exhibit a relation between the refined Sobolev scales over Ω and $\overline{\Omega}$. Denote by $H^{s,\varphi}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $H^{s,\varphi}(\Omega)$. We consider $H^{s,\varphi}_0(\Omega)$ as a Hilbert space with respect to the inner product in $H^{s,\varphi}(\Omega)$.

Theorem 8.7. *Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. The following assertions are true:*

i) *If $s < 1/2$, then $C^\infty_0(\Omega)$ is dense in $H^{s,\varphi}(\Omega)$, and therefore $H^{s,\varphi}(\Omega) = H^{s,\varphi}_0(\Omega)$.*

ii) *If $s > -1/2$ and $s + 1/2 \notin \mathbb{Z}$, then the restriction mapping $u \to u \restriction \Omega$, $u \in \mathcal{D}'(\mathbb{R}^n)$, establishes a homeomorphism of $H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)$ onto $H^{s,\varphi}_0(\Omega)$.*

iii) *The spaces $H^{s,\varphi}(\Omega)$ and $H^{-s,1/\varphi}_{\overline{\Omega}}(\mathbb{R}^n)$ are mutually dual with respect to the inner product in $L_2(\Omega)$.*

iv) *Suppose that $s < 1/2$ and $s - 1/2 \notin \mathbb{Z}$. Then the spaces $H^{s,\varphi}(\Omega)$ and $H^{-s,1/\varphi}_0(\Omega)$ are mutually dual, up to equivalence of norms, with respect to the inner product in $L_2(\Omega)$. Therefore the space $H^{s,\varphi}(\Omega)$ coincides, up to equivalence of norms, with the factor space $H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)/H^{s,\varphi}_{\partial\Omega}(\mathbb{R}^n)$ dual to $H^{-s,1/\varphi}_0(\Omega)$; i.e., $H^{s,\varphi}(\Omega) = \{u \restriction \Omega : \, u \in H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)\}$.*

This theorem is known in the Sobolev case of $\varphi \equiv 1$; see, e.g., [133, Sec. 4.3.2 and 4.8]. In general, assertion i) follows from the Sobolev case in view of Theorem 8.6 ii), where $\varphi_1 \equiv 1$; assertion ii) is deduced with the help of the interpolation Theorem 8.4; assertion iii) results from Theorem 3.7 iv); finally, assertion iv) follows from ii) and iii). To deduce assertion ii) we need, besides (8.4), the interpolation formula

$$\bigl[H^{s-\varepsilon}_0(\Omega), H^{s+\delta}_0(\Omega)\bigr]_\psi = H^{s,\varphi}(\Omega) \cap H^{s-\varepsilon}_0(\Omega) = H^{s,\varphi}_0(\Omega). \tag{8.6}$$

Here ε, δ are positive numbers such that $[s-\varepsilon, s+\delta]$ is disjoint from $\mathbb{Z} - 1/2$, and ψ is the interpolation parameter defined by (3.8). Formula (8.6) follows from (8.3) and Proposition 8.5 (interpolation of subspaces) because there exist a linear mapping that projects $H^\sigma(\Omega)$ onto $H^\sigma_0(\Omega)$ if σ runs over $[s-\varepsilon, s+\delta]$. The mapping is constructed in [133, Lemma 5.4.4 with regard for Theorem 4.7.1].

Note that if s is half-integer, then assertions ii) and iv) fail to hold at least for $\varphi \equiv 1$ [133, Sec. 4.3.2, Remark 2].

Remark 8.8. In the literature, there are three different definitions of the Sobolev space of negative order s over Ω. The first of them coincides with Definition 8.2 for $\varphi \equiv 1$ ([35, Sec. A.4] and [133, Sec. 4.2.1]). The second defines this space as the dual of $H_0^{-s}(\Omega)$ ([30, Ch. II, § 1, Sec. 5] and [61, Ch. 1, Sec. 12.1]), whereas the third defines it as the dual of $H^{-s}(\Omega)$ ([13, Ch. XIV, § 3] and [118, Sec. 1.10]), the both duality being with respect to the inner product in $L_2(\Omega)$. By Theorem 8.7 iii) and iv), the first and second definitions are tantamount if (and only if) $s-1/2 \notin \mathbb{Z}$, but the third gives $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ and, therefore, are not equivalent to them for $s < -1/2$. If $-1/2 < s < 0$, then all the three definitions are tantamount in view of Theorem 8.7 i) and ii). They are suitable in various situations appearing in the theory of elliptic boundary problems. The situations will occur below when we will investigate these problems in the refined Sobolev scale. We chose Definition 8.2 to introduce the Hörmander spaces over Ω because it is universal; i.e., it allows us to define the space $X(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ if we have an arbitrary function Banach space $X(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ instead of $H^{s,\varphi}(\mathbb{R}^n)$ (embeddings being continuous).

8.4. **Traces.** We now study the traces of functions $f \in H^{s,\varphi}(\Omega)$ and their normal derivatives on the boundary $\partial\Omega$. The traces belong to certain spaces from the refined Sobolev scale on $\partial\Omega$. This scale was defined in Section 5.1 because $\partial\Omega$ is a closed infinitely smooth oriented manifold (of dimension $n-1$). Further we use the notation $D_\nu := i\,\partial/\partial\nu$, where ν is the field of unit vectors of inner normals to the boundary $\partial\Omega$; this field is given in a neighbourhood of $\partial\Omega$. (For us it will be more suitable to use D_ν instead of $\partial/\partial\nu$; see Sec. 11 below.)

Theorem 8.9. *Let an integer $r \geq 1$, real number $s > r - 1/2$, and function $\varphi \in \mathcal{M}$ be given. Then the mapping*

$$R_r : u \mapsto \bigl((D_\nu^{k-1} u) \restriction \partial\Omega : \, k = 1, \ldots, r\bigr), \quad u \in C^\infty(\overline{\Omega}), \tag{8.7}$$

extends uniquely to a continuous linear operator

$$R_r : \, H^{s,\varphi}(\Omega) \to \bigoplus_{k=1}^{r} H^{s-k+1/2,\,\varphi}(\partial\Omega) =: \mathcal{H}^r_{s,\varphi}(\partial\Omega).$$

The operator (8.7) has a right inverse continuous linear operator $\Upsilon_r : \mathcal{H}^r_{s,\varphi}(\partial\Omega) \to H^{s,\varphi}(\Omega)$ such that the mapping Υ_r does not depend on s and φ.

Theorem 8.9 is known in the Sobolev case of $\varphi \equiv 1$; see, e.g., [61, Ch. 1, Sec. 9.2] or [133, Sec. 4.7.1]. For arbitrary $\varphi \in \mathcal{M}$, the theorem follows from this case by the interpolation Theorems 5.7 and 8.4.

It useful to note that

$$H_0^{s,\varphi}(\Omega) = \bigl\{u \in H^{s,\varphi}(\Omega) : \, R_r u = 0\bigr\} \quad \text{if} \quad r - 1/2 < s < r + 1/2; \tag{8.8}$$

here the integer $r \geq 1$. This formula is known in the $\varphi \equiv 1$ case, the equality $r + 1/2 = s$ being possible; see, e.g., [61, Ch. 1, Sec. 11.4] or [133, Sec. 4.7.1]. In general, (8.8) follows from the Sobolev case by (8.6).

If $s > 1/2$ and $\varphi \in \mathcal{M}$, then, by Theorem 8.9, a trace $u \restriction \partial\Omega := R_1 u$ of each function $u \in H^{s,\varphi}(\Omega)$ on the boundary $\partial\Omega$ exists and belongs to the space $H^{s-1/2,\varphi}(\partial\Omega)$. Moreover, we get the following description of this space in terms of traces. Put $\sigma := s - 1/2$.

Corollary 8.10. *Let $\sigma > 0$ and $\varphi \in \mathcal{M}$. Then*

$$H^{\sigma,\varphi}(\partial\Omega) = \{g := u\restriction\partial\Omega :\ u \in H^{\sigma+1/2,\varphi}(\Omega)\},$$
$$\|g\|_{H^{\sigma,\varphi}(\partial\Omega)} \asymp \inf\bigl\{\|u\|_{H^{\sigma+1/2,\varphi}(\Omega)} :\ u\restriction\partial\Omega = g\bigr\}.$$

If $s < 1/2$ and $\varphi \in \mathcal{M}$, then the trace mapping

$$R_1 : u \mapsto u\restriction\partial\Omega, \quad u \in C^\infty(\overline{\Omega}), \tag{8.9}$$

has not a continuous extension $R_1 : H^{s,\varphi}(\Omega) \to \mathcal{D}'(\partial\Omega)$. Indeed, if this extension existed, we would get, by Theorem 8.7 i), the equality $R_1 u = 0$ on $\partial\Omega$ for each $u \in H^{s,\varphi}(\Omega)$. But this equality fails to hold, e.g., for the function $u \equiv 1$.

In the limiting case of $s = 1/2$, we have the next criterion for the trace operator R_1 to be well defined on $H^{1/2,\varphi}(\Omega)$.

Theorem 8.11. *Let $\varphi \in \mathcal{M}$. The following assertions are true:*

i) *The function φ satisfies (3.6) if an only if the mapping (8.9) is a continuous operator from the space $C^\infty(\overline{\Omega})$ endowed with the topology of $H^{1/2,\varphi}(\Omega)$ to the space $\mathcal{D}'(\partial\Omega)$.*

ii) *Moreover, if φ satisfies (3.6), then the mapping (8.9) extends uniquely to a continuous linear operator $R_1 : H^{1/2,\varphi}(\Omega) \to H^{0,\varphi_0}(\partial\Omega)$, where $\varphi_0 \in \mathcal{M}$ is given by the formula*

$$\varphi_0(\tau) := \Biggl(\int_\tau^\infty \frac{dt}{t\,\varphi^2(t)}\Biggr)^{-1/2} \quad \text{for} \quad \tau \geq 1. \tag{8.10}$$

This operator has a right inverse continuous linear operator

$$\Upsilon_{1,\varphi} : H^{0,\varphi_0}(\partial\Omega) \to H^{1/2,\varphi}(\Omega),$$

the map $\Upsilon_{1,\varphi}$ depending on φ.

Theorem 8.11 follows from the trace theorems proved by L. Hörmander [43, Sec. 2.2, Theorem 2.2.8] and L.R. Volevich, B.P. Paneah [138, § 6, Theorems 6.1 and 6.2]. Indeed, consider a Hörmander space $B_{p,\mu}(\mathbb{R}^n)$ for some weight function μ, and let U be a neighbourhood of the origin in $\mathbb{R}^n$. We write points $x \in \mathbb{R}^n$ as $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. According to the trace theorems, the condition

$$\nu^{-2}(\xi\,') := \int_{-\infty}^{\infty} \mu^{-2}(\xi', \xi_n)\, d\xi_n < \infty \quad \text{for all} \quad \xi' \in \mathbb{R}^{n-1} \tag{8.11}$$

holds true if and only if the mapping $u(x', x_n) \to u(x', 0)$ is a continuous operator from the space $C_0^\infty(U)$ endowed with the topology of $B_{2,\mu}(\mathbb{R}^n)$ to the space $\mathcal{D}'(\mathbb{R}^{n-1})$. (In (8.11) the phrase 'for all' can be replaced with 'for a certain'.) Moreover, if (8.11) holds and $U = \mathbb{R}^n$, then this mapping extends by continuity to a bounded operator from $B_{2,\mu}(\mathbb{R}^n)$ to $B_{2,\nu}(\mathbb{R}^{n-1})$; the operator has a linear bounded right-inverse. Whence, by setting $\mu(\xi) := \langle\xi\rangle^{1/2}\varphi(\langle\xi\rangle)$ and observing that (8.11) $\Leftrightarrow$ (3.6) with $\nu(\xi') \asymp \varphi_0(\langle\xi'\rangle)$, we can deduce Theorem 8.11 with the help of the local coordinates method.

Let us note that the domain Ω is a special case of an infinitely smooth compact manifold with boundary. The refined Sobolev scale over such a manifold was introduced and investigated by authors in [77, Sec. 3]. Specifically, the interpolation Theorem 8.4 and the traces Theorems 8.9 (for $r = 1$) and 8.11 were proved.

8.5. **Riggings.** We recall the important notion of a Hilbert rigging, which has various applications, specifically, in the theory of elliptic operators; see, e.g., [11, Ch. I, § 1] and [13, Ch. XIV, § 1]. Let H and H_+ be Hilbert spaces such that the dense continuous embedding $H_+ \hookrightarrow H$ holds. Denote by H_- the completion of H with respect to the norm

$$\|f\|_{H_-} := \sup\left\{ \frac{|(f,u)_H|}{\|u\|_{H_+}} : u \in H_+ \right\}, \quad f \in H.$$

It is known the following: this norm and the space H_- are Hilbert; the spaces H_+ and H_- are mutually dual with respect to the inner product in H; the dense continuous embeddings $H_+ \hookrightarrow H \hookrightarrow H_-$ hold.

Definition 8.12. The chain $H_- \hookleftarrow H \hookleftarrow H_+$ is said to be a (Hilbert) rigging of H with H_+ and H_-. In this rigging, H_-, H and H_+ are called negative, zero, and positive spaces respectively.

According to Theorem 8.7 iii) we have the following rigging of $L_2(\Omega)$ with some Hörmander spaces:

$$H_{\overline{\Omega}}^{-s,1/\varphi}(\mathbb{R}^n) \hookleftarrow L_2(\Omega) \hookleftarrow H^{s,\varphi}(\Omega), \quad s > 0, \ \varphi \in \mathcal{M}. \tag{8.12}$$

Here we naturally identify $L_2(\Omega)$ with $H_{\overline{\Omega}}^{0}(\mathbb{R}^n)$ (extending the functions $f \in L_2(\Omega)$ by zero).

In some applications to elliptic problems, it is useful to have a scale consisting of both negative and positive spaces in (8.12). For this purpose we introduce the uniform notation

$$H^{s,\varphi,(0)}(\Omega) := \begin{cases} H^{s,\varphi}(\Omega) & \text{for } s \geq 0, \\ H_{\overline{\Omega}}^{s,\varphi}(\mathbb{R}^n) & \text{for } s < 0, \end{cases} \tag{8.13}$$

with $\varphi \in \mathcal{M}$, and form the scale of Hilbert spaces

$$\left\{ H^{s,\varphi,(0)}(\Omega) : s \in \mathbb{R}, \varphi \in \mathcal{M} \right\}. \tag{8.14}$$

In the Sobolev case of $\varphi \equiv 1$ the rigging (8.12) and the scale of spaces $H^{s,(0)}(\Omega) := H^{s,1,(0)}(\Omega)$, $s \in \mathbb{R}$, were used by Yu.M. Berezansky, S.G. Krein, Ya.A. Roitberg [11, 12, 13, 114] and M. Schechter [122] in the elliptic theory. (They also considered the Banach L_p-analogs of these spaces with $1 < p < \infty$ and denoted negative spaces in the same manner as positive ones but with negative index s, e.g. $H^s(\Omega)$; see Remark 8.8 above.)

Properties of the scale (8.14) are inherited from the refined Sobolev scales over Ω and $\overline{\Omega}$. Now we dwell on the properties that link negative and positive spaces to each other.

When dealing with the scale (8.14), it is suitable to identify each function $f \in C^\infty(\,\overline{\Omega}\,)$ with its extension by zero

$$\mathcal{O}f(x) := \begin{cases} f(x) & \text{for } \ x \in \overline{\Omega}, \\ 0 & \text{for } \ x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{8.15}$$

The extension is a regular distribution belonging to $H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)$ for $s<0$. Now the set $C^\infty(\,\overline{\Omega}\,)$ is dense in every space $H^{s,\varphi,(0)}(\Omega)$ from (8.14) in view of Theorem 8.6 i). This allow us to consider the continuous embeddings of spaces pertaining to (8.14) and viewed as the completions of the same linear manifold, $C^\infty(\,\overline{\Omega}\,)$, with respect to different norms. (The general theory of such embeddings is in [13, Ch. XIV, § 7]). So, by Theorem 8.6 ii) and formula (8.17) given below, we have the dense compact embeddings in the scale (8.14):

$$H^{s_1,\varphi_1,(0)}(\Omega) \hookrightarrow H^{s,\varphi,(0)}(\Omega), \quad -\infty < s < s_1 < \infty \ \text{ and } \ \varphi, \varphi_1 \in \mathcal{M}. \tag{8.16}$$

Note that in the $|s|<1/2$ case the spaces $H^{s,\varphi}(\Omega)$ and $H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)$ can be considered as completions of $C^\infty_0(\Omega)$ with respect to equivalent norms in view of Theorem 8.7 i) and ii). Hence, up to equivalence of norms,

$$H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n) = H^{s,\varphi,(0)}(\Omega) = H^{s,\varphi}(\Omega) \quad \text{for } \ |s|<1/2, \ \ \varphi \in \mathcal{M}. \tag{8.17}$$

It follows from this result and Theorem 8.7 iii) that the spaces $H^{s,\varphi,(0)}(\Omega)$ and $H^{-s,1/\varphi,(0)}(\Omega)$ are mutually dual with respect to the inner product in $L_2(\Omega)$ for every $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$, the duality being up to equivalence of norms if $s=0$.

The scale (8.14) has an interpolation property analogous to that stated in Theorem 8.4.

Theorem 8.13. *Under the conditions of Theorem 8.4 we have*

$$\bigl[H^{s-\varepsilon,(0)}(\Omega), H^{s+\delta,(0)}(\Omega)\bigr]_\psi = H^{s,\varphi,(0)}(\Omega) \quad \textit{for all } \ s\in\mathbb{R} \tag{8.18}$$

up to equivalence of norms in the spaces.

If $s-\varepsilon>-1/2$ or $s+\delta<1/2$, then (8.18) holds by Theorem 8.4 and (8.17). If $\varphi\equiv 1$, then (8.18) is proved in [61, Ch. 1, Sec. 12.5, Theorem 12.5]. The general case can be reduced to the previous ones by the reiterated interpolation.

9. Elliptic boundary-value problems on the one-sided scale

In Sections 9–12, we will investigate regular elliptic boundary-value problems on various scales of Hörmander spaces. We begin with the one-sided refined Sobolev scale consisting of the spaces $H^{s,\varphi}(\Omega)$ for sufficiently large s.

9.1. **The statement of the boundary-value problem.** Recall that Ω is a bounded domain in $\mathbb{R}^n$ with $n\geq 2$, and its boundary $\partial\Omega$ is an infinitely smooth closed manifold of the dimension $n-1$. Let $\nu(x)$ denote the unit vector of the inner normal to $\partial\Omega$ at a point $x\in\partial\Omega$.

We consider the following boundary-value problem in Ω:

$$L\,u \equiv \sum_{|\mu|\leq 2q} l_{\mu}(x)\, D^{\mu}u = f \quad \text{in} \;\; \Omega, \tag{9.1}$$

$$B_j\,u \equiv \sum_{|\mu|\leq m_j} b_{j,\mu}(x)\, D^{\mu}u = g_j \quad \text{on} \;\; \partial\Omega, \quad j = 1,\dots,q. \tag{9.2}$$

Here $L = L(x,D)$, $x \in \overline{\Omega}$, and $B_j = B_j(x,D)$, $x \in \partial\Omega$, are linear partial differential expressions with complex-valued coefficients $l_{\mu} \in C^{\infty}(\,\overline{\Omega}\,)$ and $b_{j,\mu} \in C^{\infty}(\partial\Omega)$. We suppose that $\operatorname{ord} L = 2q$ is an even positive number and $\operatorname{ord} B_j = m_j \leq 2q-1$ for all $j = 1,\dots,q$. Set $B := (B_1,\dots,B_q)$ and $m := \max\{m_1,\dots,m_q\}$.

Note that we use the standard notation in 9.1 and 9.2; namely, for a multi-index $\mu = (\mu_1,\dots,\mu_n)$ we let $|\mu| := \mu_1 + \ldots + \mu_n$ and $D^{\mu} := D_1^{\mu_1}\ldots D_n^{\mu_n}$, with $D_k := i\,\partial/\partial x_k$ for $k = 1,\dots,n$ and $x = (x_1,\dots,x_n) \in \mathbb{R}^n$.

Lemma 9.1. *Let $s > m + 1/2$ and $\varphi \in \mathcal{M}$. Then the mapping*

$$(L,B):\; u \to (Lu, B_1u,\dots,B_qu), \quad u \in C^{\infty}(\,\overline{\Omega}\,), \tag{9.3}$$

extends uniquely to a continuous linear operator

$$(L,B):\; H^{s,\varphi}(\Omega) \to H^{s-2q,\,\varphi}(\Omega) \oplus \bigoplus_{j=1}^{q} H^{s-m_j-1/2,\,\varphi}(\partial\Omega) =: \mathcal{H}_{s,\varphi}(\Omega,\partial\Omega). \tag{9.4}$$

Note the differential operator L maps $H^{s,\varphi}(\Omega)$ continuously into $H^{s-2g,\varphi}(\Omega)$ for each real s, whereas the boundary differential operator B_j maps $H^{s,\varphi}(\Omega)$ continuously into $H^{s-m_j-1/2,\varphi}(\partial\Omega)$ provided that $s > m_j + 1/2$. This is well known in the Sobolev case of $\varphi \equiv 1$ (see, e.g., [46, Sec. B.2]), whence the case of an arbitrary $\varphi \in \mathcal{M}$ is got by the interpolation in view of Theorems 5.7 and 8.4. The mentioned continuity of B_j also results from the trace Theorem 8.9 (the $r = 1$ case). If $s \leq m + 1/2$, then Lemma 9.1 fails to hold (see Section 8.4). In the limiting case of $s = m + 1/2$ an analog of the lemma is valid provided the function φ satisfies (3.6) and we exchange the space $\mathcal{H}_{s,\varphi}(\Omega,\partial\Omega)$ for another (see Section 9.3 below).

We will investigate properties of the operator (9.4) under the assumption that the boundary-value problem (9.1), (9.2) is regular elliptic in Ω. Recall some relevant notions; see, e.g., [30, Ch. III, § 6] or [61, Ch. 2, Sec. 1 and 2].

The principal symbols of the partial differential expressions $L(x,D)$, with $x \in \overline{\Omega}$, and $B_j(x,D)$, with $x \in \partial\Omega$, are defined as follows:

$$L^{(0)}(x,\xi) := \sum_{|\mu|=2q} l_{\mu}(x)\,\xi^{\mu}, \qquad B_j^{(0)}(x,\xi) := \sum_{|\mu|=m_j} b_{j,\mu}(x)\,\xi^{\mu}.$$

They are homogeneous polynomials in $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{C}^n$; here as usual $\xi^{\mu} := \xi_1^{\mu_1}\ldots\xi_n^{\mu_n}$.

Definition 9.2. The boundary-value problem (9.1), (9.2) is said to be regular (or normal) elliptic in Ω if the following conditions are satisfied:

i) The expression L is proper elliptic on $\overline{\Omega}$; i.e., for each point $x \in \overline{\Omega}$ and for all linearly independent vectors $\xi', \xi'' \in \mathbb{R}^n$, the polynomial $L^{(0)}(x, \xi' + \tau\xi'')$ in $\tau \in \mathbb{C}$ has q roots $\tau_j^+(x; \xi', \xi'')$, $j = 1, \ldots, q$, with positive imaginary part and q roots with negative imaginary part, each root being taken the number of times equal to its multiplicity.

ii) The system $\{B_1, \ldots, B_q\}$ satisfies the Lopatinsky condition with respect to L on $\partial\Omega$; i.e., for an arbitrary point $x \in \partial\Omega$ and for each vector $\xi \neq 0$ tangent to $\partial\Omega$ at x, the polynomials $B_j^{(0)}(x, \xi + \tau\nu(x))$, $j = 1, \ldots, q$, in τ are linearly independent modulo $\prod_{j=1}^{q}\bigl(\tau - \tau_j^+(x; \xi, \nu(x))\bigr)$.

iii) The system $\{B_1, \ldots, B_q\}$ is normal; i.e., the orders m_j are pairwise distinct, and $B_j^{(0)}(x, \nu(x)) \neq 0$ for each $x \in \partial\Omega$.

Remark 9.3. It follows from condition i) that $L^{(0)}(x, \xi) \neq 0$ for each point $x \in \overline{\Omega}$ and nonzero vector $\xi \in \mathbb{R}^n$, i.e. L is elliptic on $\overline{\Omega}$. If $n \geq 3$, then the ellipticity condition is equivalent to i). The equivalence also holds if $n = 2$ and all coefficients of L are real-valued. Not more that there are various equivalent statements of the Lopatinsky condition; see [8, Sec. 1.2 and 1.3].

Example 9.4. Let L satisfy condition i), and let $B_j u := \partial^{k+j-1}u/\partial\nu^{k+j-1}$, with $j = 1, \ldots, q$, for some $k \in \mathbb{Z}$, $0 \leq k \leq q$. Then the boundary-value problem (9.1), (9.2) is regular elliptic. If $k = 0$, we have the Dirichlet boundary-value problem.

In what follows the boundary-value problem (9.1), (9.2) is supposed to be regular elliptic in Ω.

To describe the range of the operator (9.4) we consider the boundary-value problem

$$L^+ v \equiv \sum_{|\mu| \leq 2q} D^\mu(\overline{l_\mu(x)}\, v) = \omega \quad \text{in } \ \Omega, \tag{9.5}$$

$$B_j^+ v = h_j \quad \text{on } \ \partial\Omega, \quad j = 1, \ldots, q, \tag{9.6}$$

that is formally adjoint to the problem (9.1), (9.2) with respect to the Green formula

$$(Lu, v)_\Omega + \sum_{j=1}^{q} (B_j u,\, C_j^+ v)_{\partial\Omega} = (u, L^+ v)_\Omega + \sum_{j=1}^{q} (C_j u,\, B_j^+ v)_{\partial\Omega}, \tag{9.7}$$

where $u, v \in C^\infty(\overline{\Omega})$. Here $\{B_j^+\}$, $\{C_j\}$, $\{C_j^+\}$ are some normal systems of linear partial differential boundary expressions with coefficients from $C^\infty(\partial\Omega)$; the orders of expressions $B_j^\pm$, $C_j^\pm$, $j = 1, \ldots, q$, run over the set $\{0, 1, \ldots, 2q-1\}$ and satisfy the equality

$$\operatorname{ord} B_j + \operatorname{ord} C_j^+ = \operatorname{ord} C_j + \operatorname{ord} B_j^+ = 2q - 1.$$

We denote $m_j^+ := \operatorname{ord} B_j^+$. In (9.7) and bellow, the notations $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\partial\Omega}$ stand for the inner products in the spaces $L_2(\Omega)$ and $L_2(\partial\Omega)$ respectively, and for extensions by continuity of these products as well.

The expression L^+ is said to be formally adjoint to L, whereas the system $\{B_j^+\}$ is said to be adjoint to $\{B_j\}$ with respect to L. Note that $\{B_j^+\}$ is not uniquely defined.

We set

$$\mathcal{N} := \{u \in C^\infty(\overline{\Omega}) : \; Lu = 0 \text{ in } \Omega, \; B_j u = 0 \text{ on } \partial\Omega \text{ for all } j = 1, \ldots, q\},$$
$$\mathcal{N}^+ := \{v \in C^\infty(\overline{\Omega}) : \; L^+ v = 0 \text{ in } \Omega, \; B_j^+ v = 0 \text{ on } \partial\Omega \text{ for all } j = 1, \ldots, q\}.$$

Since both the problems (9.1), (9.2) and (9.5), (9.6) are regular elliptic, both the spaces $\mathcal{N}$ and $\mathcal{N}^+$ are finite dimensional. Note that the space $\mathcal{N}^+$ does not not depend on the choice of the system $\{B_j^+\}$ adjoint to $\{B_j\}$.

Example 9.5. If the boundary-value problem (9.1), (9.2) is Dirichlet, then the formally adjoint problem is also Dirichlet, with $\dim \mathcal{N} = \dim \mathcal{N}^+$.

9.2. **The operator properties.** Now we investigate properties of the operator (9.4) corresponding to the regular elliptic boundary-value problem (9.1), (9.2).

Theorem 9.6. *Let $s > m + 1/2$ and $\varphi \in \mathcal{M}$. Then the bounded linear operator (9.4) is Fredholm, its kernel coincides with $\mathcal{N}$, and its range consists of all the vectors $(f, g_1, \ldots, g_q) \in \mathcal{H}_{s,\varphi}(\Omega, \partial\Omega)$ such that*

$$(f, v)_\Omega + \sum_{j=1}^{q} (g_j, C_j^+ v)_{\partial\Omega} = 0 \quad \textit{for all} \quad v \in \mathcal{N}^+. \tag{9.8}$$

The index of (9.4) is $\dim \mathcal{N} - \dim \mathcal{N}^+$ and does not depend on s and φ.

In the Sobolev case of $\varphi \equiv 1$ Theorem 9.6 is a classical result if $s \geq 2q$; see, e.g., [30, Ch. III, § 6, Subsec. 4] or [61, Ch. 2, Sec. 5.4]. If $m + 1/2 < s < 2q$, then this theorem is also true [25, Ch. III, Sec. 2.2]. For an arbitrary $\varphi \in \mathcal{M}$ the theorem can be deduced from the Sobolev case by the interpolation with a function parameter if we apply Proposition 6.5 and Theorems 5.7 and 8.4.

Remark 9.7. G. Slenzak [129] proved an analog of Theorem 9.6 for a different scale of Hörmander inner product spaces. These spaces are not attached to Sobolev spaces with the number parameter; the class of the weight functions used by Slenzak is not described constructively.

Theorem 9.8. *Let $s > m + 1/2$, $\varphi \in \mathcal{M}$, and $\sigma < s$. Then the following a priori estimate holds:*

$$\|u\|_{H^{s,\varphi}(\Omega)} \leq c \left(\|(L, B)u\|_{\mathcal{H}_{s,\varphi}(\Omega,\partial\Omega)} + \|u\|_{H^{\sigma,\varphi}(\Omega)} \right) \quad \textit{for all} \quad u \in H^{s,\varphi}(\Omega);$$

here the number $c > 0$ is independent of u.

This theorem follows from the Fredholm property of the operator (9.4) in view of Proposition 6.7 and the compactness of the embedding $H^{s,\varphi}(\Omega) \hookrightarrow H^{\sigma,\varphi}(\Omega)$ for $\sigma < s$.

Now we study a local smoothness of a solution u to the boundary-value problem (9.1), (9.2) in the refined Sobolev scale. The relevant property will be formulated as a theorem on the local increase in smoothness.

Let U be an open set in $\mathbb{R}^n$; we put $\Omega_0 := U \cap \Omega \neq \varnothing$ and $\Gamma_0 := U \cap \partial\Omega$ (the case were $\Gamma_0 = \varnothing$ is possible). We introduce the following local analog of the space $H^{s,\varphi}(\Omega)$ with $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$:

$$H^{s,\varphi}_{\mathrm{loc}}(\Omega_0, \Gamma_0) := \bigl\{u \in \mathcal{D}'(\Omega) : \chi\, u \in H^{s,\varphi}(\Omega) \\ \text{for all } \ \chi \in C^\infty(\overline{\Omega}) \ \text{ with } \ \operatorname{supp}\chi \subseteq \Omega_0 \cup \Gamma_0\bigr\}.$$

The other local space $H^{s,\varphi}_{\mathrm{loc}}(\Gamma_0)$, which we need, was defined in (6.3).

Theorem 9.9. *Let $s > m + 1/2$ and $\eta \in \mathcal{M}$. Suppose that the distribution $u \in H^{s,\eta}(\Omega)$ is a solution to the boundary-value problem (9.1), (9.2), with*

$$f \in H^{s-2q+\varepsilon,\varphi}_{\mathrm{loc}}(\Omega_0, \Gamma_0) \quad \text{and} \quad g_j \in H^{s-m_j-1/2+\varepsilon,\varphi}_{\mathrm{loc}}(\Gamma_0), \quad j = 1, \ldots, q,$$

for some $\varepsilon \geq 0$ and $\varphi \in \mathcal{M}$. Then $u \in H^{s+\varepsilon,\varphi}_{\mathrm{loc}}(\Omega_0, \Gamma_0)$.

In the special case where $\Omega_0 = \Omega$ and $\Gamma_0 = \partial\Omega$ we have the global smoothness increase (i.e. the increase in the domain Ω up to its boundary). This case follows from Theorem 9.6. Indeed, since the vector $(f, g) \in \mathcal{H}_{s+\varepsilon,\varphi}(\Omega, \partial\Omega)$ satisfies (9.8), we can write $(L, B)v = (f, g)$ for some $v \in H^{s+\varepsilon,\varphi}(\Omega)$, whence $u - v \in \mathcal{N}$ and $u \in H^{s+\varepsilon,\varphi}(\Omega)$; here $g := (g_1, \ldots, g_q)$. In general, we can deduce Theorem 9.9 from the above case reasoning similar to the proof of Theorem 4.4. Note, if $\Gamma_0 = \varnothing$, then we get an interior smoothness increase (in neighbourhoods of interior points of Ω).

Theorem 9.9 specifies, with regard to the refined Sobolev scale, the classical results on a local smoothness of solutions to elliptic boundary-value problems [11, Ch. 3, Sec. 4], [16, 102, 121].

Theorems 9.9 and 8.6 iv) imply the following sufficient condition for the solution u to be classical.

Theorem 9.10. *Let $s > m + 1/2$ and $\eta \in \mathcal{M}$. Suppose that the distribution $u \in H^{s,\eta}(\Omega)$ is a solution to the boundary-value problem (9.1), (9.2), where*

$$f \in H^{n/2,\varphi}_{\mathrm{loc}}(\Omega, \varnothing) \cap H^{m-2q+n/2,\varphi}(\Omega),$$
$$g_j \in H^{m-m_j+(n-1)/2,\varphi}(\partial\Omega), \quad j = 1, \ldots, q,$$

for some $\varphi \in \mathcal{M}$. If φ satisfies (3.6), then the solution u is classical, i.e. $u \in C^{2q}(\Omega) \cap C^m(\overline{\Omega})$.

Note that the condition (3.6) not only is sufficient for u to be a classical solution but also is necessary on the class of all the considered solutions u. This follows from Theorem 3.3.

9.3. **The limiting case.** This case is $s = m + 1/2$. We study it, e.g., for the Dirichlet problem for the Laplace equation:

$$\Delta\, u = f \ \text{ in } \ \Omega, \quad R_1 u := u \restriction \partial\Omega = g.$$

This problem is regular elliptic, with $m = 0$. Let $\varphi \in \mathcal{M}$. By Theorem 8.11, the mapping $u \mapsto (\Delta\, u, R_1 u)$, $u \in C^\infty(\overline{\Omega})$, extends uniquely to a continuous linear operator from $H^{1/2,\varphi}(\Omega)$ to $H^{-3/2,\varphi}(\Omega) \times \mathcal{D}'(\partial\Omega)$ if and only if φ satisfies (3.6).

Suppose the inequality (3.6) is fulfilled, and $\varphi_0 \in \mathcal{M}$ is defined by (8.10). Then we get the bounded linear operator

$$(\Delta, R_1) : \, H^{1/2,\varphi}(\Omega) \to H^{-3/2,\varphi}(\Omega) \oplus H^{0,\varphi_0}(\partial\Omega) =: \mathcal{H}(\Omega, \partial\Omega), \qquad (9.9)$$

with $R_1(H^{1/2,\varphi}(\Omega))$ being equal to $H^{0,\varphi_0}(\partial\Omega)$. It is reasonable to ask whether this operator is Fredholm or not. The answer is no because the range of (9.9) is not closed in $\mathcal{H}(\Omega, \partial\Omega)$.

To prove this let us suppose the contrary, i.e., the range of (9.9) to be closed in $\mathcal{H}(\Omega, \partial\Omega)$. Then the restriction of (9.9) to the subspace

$$K_{\Delta}^{1/2,\varphi}(\Omega) := \big\{ u \in H^{1/2,\varphi}(\Omega) : \, \Delta\, u = 0 \;\text{ in }\; \Omega \big\}$$

has a closed range in $H^{0,\varphi_0}(\partial\Omega)$. But, according to Theorem 10.1 given below in Section 10.1, this restriction establishes a homeomorphism of $K_{\Delta}^{1/2,\varphi}(\Omega)$ onto $H^{0,\varphi}(\partial\Omega)$. Hence, $H^{0,\varphi}(\partial\Omega)$ is a (closed) subspace of $H^{0,\varphi_0}(\partial\Omega)$, so that $H^{0,\varphi}(\partial\Omega) = H^{0,\varphi_0}(\partial\Omega)$. We arrive at a contradiction if we note that $\varphi_0(t)/\varphi(t) \to 0$ as $t \to +\infty$ and use Theorem 5.6 ii). Thus our hypothesis is false.

Given a general elliptic boundary-value problem (9.1), (9.2), the reasoning is similar. If $s = m + 1/2$, φ satisfies (3.6), and φ_0 is defined by (8.10), then we get the bounded linear operator (9.4) providing the space $H^{s-m_j-1/2,\varphi}(\partial\Omega) = H^{0,\varphi}(\partial\Omega)$ is replaced by $H^{0,\varphi_0}(\partial\Omega)$ for j such that $m_j = m$. This operator has a nonclosed range and therefore is not Fredholm.

10. Semihomogeneous elliptic problems

As we have mentioned, the results of the previous section are not valid for $s \leq m+1/2$. But if the boundary-value problem (9.1), (9.2) is semihomogeneous (i.e., $f \equiv 0$ or all $g_j \equiv 0$), it establishes a bounded and Fredholm operator on the two-sided refined Sobolev scale, in which the number parameter s runs over the whole real axis. In this section we separately consider the case of the homogeneous elliptic equation (9.1) and the case of the homogeneous boundary conditions (9.2). In what follows we focuss our attention on analogs of Theorem 9.6 on the Fredholm property of (L, B). Counterparts of Theorems 9.8–9.10 can be derived from the analogs similarly to the reasoning outlined in Section 9 (for details, see [75, 79, 80]).

10.1. **A boundary-value problem for a homogeneous elliptic equation.**

Let us consider the regular elliptic boundary-value problem (9.1), (9.2) provided that $f \equiv 0$, namely

$$Lu = 0 \;\text{ in }\; \Omega, \quad B_j u = g_j \;\text{ on }\; \partial\Omega, \;\; j = 1, \ldots, q. \qquad (10.1)$$

We connect the following linear spaces with this problem:

$$K_L^{\infty}(\Omega) := \big\{ u \in C^{\infty}(\,\overline{\Omega}\,) : \, L\,u = 0 \;\text{ in }\; \Omega \big\},$$
$$K_L^{s,\varphi}(\Omega) := \big\{ u \in H^{s,\varphi}(\Omega) : \, L\,u = 0 \;\text{ in }\; \Omega \big\}$$

for $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. Here the equality $L\,u = 0$ is understood in the distribution theory sense. It follows from a continuity of the embedding $H^{s,\varphi}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$

that $K^{s,\varphi}_{L}(\Omega)$ is a (closed) subspace in $H^{s,\varphi}(\Omega)$. We consider $K^{s,\varphi}_{L}(\Omega)$ as a Hilbert space with respect to the inner product in $H^{s,\varphi}(\Omega)$.

Theorem 10.1. *Let $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$. Then the set $K^{\infty}_{L}(\Omega)$ is dense in the space $K^{s,\varphi}_{L}(\Omega)$, and the mapping*

$$u\mapsto Bu=(B_1u,\ldots,B_qu),\quad u\in K^{\infty}_{L}(\Omega),$$

extends uniquely to a continuous linear operator

$$B:\,K^{s,\varphi}_{L}(\Omega)\to\bigoplus_{j=1}^{q}H^{s-m_j-1/2,\,\varphi}(\partial\Omega)=:\mathcal{H}_{s,\varphi}(\partial\Omega). \tag{10.2}$$

This operator is Fredholm. Its kernel coincides with $\mathcal{N}$, whereas its range consists of all the vectors $(g_1,\ldots,g_q)\in\mathcal{H}_{s,\varphi}(\partial\Omega)$ such that

$$\sum_{j=1}^{q}(g_j,C^{+}_{j}v)_{\partial\Omega}=0\quad\textit{for all}\quad v\in\mathcal{N}^{+}.$$

The index of the operator (10.2) is equal to $\dim\mathcal{N}-\dim\mathcal{G}^{+}$, with

$$\mathcal{G}^{+}:=\bigl\{\,(C^{+}_{1}v,\ldots,C^{+}_{q}v):\,v\in\mathcal{N}^{+}\,\bigr\},$$

and does not depend on s and φ.

Theorem 10.1 was proved in [75, Sec. 6]. In the $s>m+1/2$ case the theorem follows plainly from Lemma 9.1 and Theorem 9.6. If $s\leq m+1/2$, then the ellipticity condition is essential for the continuity of the operator (10.2). Note that $\dim\mathcal{G}^{+}\leq\dim\mathcal{N}^{+}$, the strict inequality being possible [45, Theorem 13.6.15].

Theorem 10.1 can be regarded as a certain analog of the Harnack theorem on convergence of sequences of harmonic functions (see, e.g., [90, Ch. 11, § 9]), however we use the metric in $H^{s,\varphi}(\Omega)$ instead of the uniform metric. Here it is relevant to mention R. Seeley's investigation [124] of the Cauchy data of solutions to a homogeneous elliptic equation in the two-sided Sobolev scale; see also the survey [8, Sec. 5.4 b].

Let us outline the proof of Theorem 10.1. For the sake of simplicity, we suppose that both $\mathcal{N}$ and $\mathcal{N}^{+}$ are trivial. Let $s<2q$ and $\varphi\in\mathcal{M}$. Chose an integer $r\geq1$ such that $2q(1-r)<s<2q$. We need the following Hilbert space

$$D^{s,\varphi}_{L}(\Omega):=\bigl\{u\in H^{s,\varphi}(\Omega):\,L\,u\in L_2(\Omega)\bigr\},$$
$$(u_1,u_2)_{D^{s,\varphi}_{L}(\Omega)}:=(u_1,u_2)_{H^{s,\varphi}(\Omega)}+(L\,u_1,L\,u_2)_{L_2(\Omega)}.$$

The mapping (9.3) extends uniquely to the homeomorphisms

$$(L,B):\,D^{2q(1-r)}_{L}(\Omega)\leftrightarrow L_2(\Omega)\oplus\mathcal{H}_{2q(1-r)}(\partial\Omega),$$
$$(L,B):\,H^{2q}(\Omega)\leftrightarrow L_2(\Omega)\oplus\mathcal{H}_{2q}(\partial\Omega).$$

The first of them follows from the Lions–Magenes theorems [61] stated below in Section 11.1, whereas the second is a special case of Theorem 9.6. (Recall that we omit φ in the notations if $\varphi\equiv1$.) Applying the interpolation with the function parameter ψ defined by (3.8) with $\varepsilon := s - 2q(1-r)$ and $\delta := 2q - s$ we get another homeomorphism

$$(L,B):\ \bigl[D_L^{2q(1-r)}(\Omega), H^{2q}(\Omega)\bigr]_{\psi} \leftrightarrow L_2(\Omega) \oplus \mathcal{H}_{s,\varphi}(\partial\Omega). \tag{10.3}$$

Now if we prove that $Z_{\psi} := \bigl[D_L^{2q(1-r)}(\Omega), H^{2q}(\Omega)\bigr]_{\psi}$ coincides with $D_L^{s,\varphi}(\Omega)$ up to equivalence of norms, then the restriction of (10.3) to $K_L^{s,\varphi}(\Omega)$ will give the homeomorphism (10.2).

The continuous embedding $Z_{\psi} \hookrightarrow D_L^{s,\varphi}(\Omega)$ is evident. The inverse can be proved by the following modification of the reasoning used by Lions and Magenes [61, Ch. 2, Sec. 7.2] for $r = 1$ and power parameter ψ. In view of Theorem 9.6 we have the homeomorphism

$$L^r L^{r+} + I:\ \bigl\{u \in H^{\sigma}(\Omega) : (D_{\nu}^{j-1}u) \restriction \partial\Omega = 0\ \ \forall\, j = 1,\ldots,r\bigr\} \leftrightarrow H^{\sigma-4qr}(\Omega)$$

for each $\sigma \geq 2qr$. Here L^r is the r-th iteration of L, L^{r+} is the formally adjoint to L^r, and I is the identity operator. We regard the domain of $L^r L^{r+} + I$ as a subspace of $H^{\sigma}(\Omega)$. Consider the bounded linear inverse operators

$$(L^r L^{r+} + I)^{-1}:\ H^{\sigma}(\Omega) \to H^{\sigma+4qr}(\Omega), \quad \sigma \geq -2qr.$$

Set $R := L^{r-1}L^{r+}(L^r L^{r+} + I)^{-1}$ and $P := -RL + I$. Since

$$LPu = (L^r L^{r+} + I)^{-1} Lu \in L_2(\Omega) \quad \text{for each} \quad u \in H^{2q(1-r)}(\Omega),$$

the operator P maps continuously $H^{\sigma}(\Omega) \to D_L^{\sigma}(\Omega)$ with $\sigma \geq 2q(1-r)$. Therefore, by the interpolation, we get the bounded operator

$$P:\ H^{s,\varphi}(\Omega) = \bigl[H^{2q(1-r)}(\Omega), H^{2q}(\Omega)\bigr]_{\psi} \to \bigl[D_L^{2q(1-r)}(\Omega), H^{2q}(\Omega)\bigr]_{\psi} = Z_{\psi}.$$

Now, for each $u \in D_L^{s,\varphi}(\Omega)$, we have $u = Pu + RLu$, with $Pu \in Z_{\psi}$ and $RLu \in H^{2q}(\Omega) \subset Z_{\psi}$. So $D_L^{s,\varphi}(\Omega) \subseteq Z_{\psi}$, and our reasoning is complete.

Note that in the Sobolev case of $\varphi \equiv 1$ Theorem 10.1 is a consequence of the above-mentioned Lions–Magenes theorems provided s is negative and not half-integer. If negative s is half-integer, then Theorem 10.1 is new even in the Sobolev case.

10.2. **An elliptic problem with homogeneous boundary conditions.** Now we consider the regular elliptic boundary-value problem (9.1), (9.2) provided that all $g_j \equiv 0$, namely

$$Lu = f \ \text{ in } \ \Omega, \quad B_j u = 0 \ \text{ on } \ \partial\Omega, \ \ j = 1,\ldots,q. \tag{10.4}$$

Let us introduce some function spaces related to the boundary-value problem (10.4). For the sake of brevity, we denote by (b.c.) the homogeneous boundary conditions in (10.4). In addition, we denote by (b.c.)$^+$ the homogeneous adjoint boundary conditions (9.6):

$$B_j^+ v = 0 \ \text{ on } \ \partial\Omega, \ \ j = 1,\ldots,q.$$

We set

$$C^{\infty}(\text{b.c.}) := \bigl\{u \in C^{\infty}(\overline{\Omega}):\ B_j u = 0 \ \text{ on } \ \partial\Omega \ \ \forall\ \ j = 1,\ldots,q\bigr\},$$
$$C^{\infty}(\text{b.c.})^+ := \bigl\{v \in C^{\infty}(\overline{\Omega}):\ B_j^+ v = 0 \ \text{ on } \ \partial\Omega \ \ \forall\ \ j = 1,\ldots,q\bigr\}.$$

Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. We introduce the separable Hilbert spaces $H^{s,\varphi}(\text{b.c.})$ and $H^{s,\varphi}(\text{b.c.})^{+}$ formed by distributions satisfying the homogeneous boundary conditions (b.c.) and (b.c.)$^{+}$ respectively.

Definition 10.2. If $s \notin \{m_j + 1/2 : j = 1, \ldots, q\}$, then $H^{s,\varphi}(\text{b.c.})$ is defined to be the closure of $C^{\infty}(\text{b.c.})$ in $H^{s,\varphi,(0)}(\Omega)$, the space $H^{s,\varphi}(\text{b.c.})$ being regarded as a subspace of $H^{s,\varphi,(0)}(\Omega)$. If $s \in \{m_j + 1/2 : j = 1, \ldots, q\}$, then the space $H^{s,\varphi}(\text{b.c.})$ is defined by means of the interpolation with the power parameter $\psi(t) = t^{1/2}$:

$$H^{s,\varphi}(\text{b.c.}) := \bigl[H^{s-1/2,\varphi}(\text{b.c.}), H^{s+1/2,\varphi}(\text{b.c.})\bigr]_{t^{1/2}}. \tag{10.5}$$

Changing (b.c.) for (b.c.)$^{+}$, and m_j for m_j^{+} in the last two sentences, we have the definition of the space $H^{s,\varphi}(\text{b.c.})^{+}$.

The space $C^{\infty}(\text{b.c.})^{+}$ and therefore $H^{s,\varphi}(\text{b.c.})^{+}$ are independent of the choice of the system $\{B_j^{+}\}$ adjoint to $\{B_j\}$; see, e.g., [61, Ch. 2, Sec. 2.5].

Note that the case of $s \in \{m_j + 1/2 : j = 1, \ldots, q\}$ is special in the definition of $H^{s,\varphi}(\text{b.c.})$. We have to resort to the interpolation formula (10.5) to get the spaces for which the main result of the subsection, Theorem 10.4, will be valid. In this case the norms in the spaces $H^{s,\varphi}(\text{b.c.})$ and $H^{s,\varphi,(0)}(\Omega)$ are not equivalent. The analogous fact is true for $H^{s,\varphi}(\text{b.c.})^{+}$. Providing $\varphi \equiv 1$, this was proved in [33, 125] (see also [133, Sec. 4.3.3]).

The spaces just introduced admit the following constructive description.

Theorem 10.3. *Let $s \in \mathbb{R}$, $s \neq m_j + 1/2$ for all $j = 1, \ldots, q$, and $\varphi \in \mathcal{M}$. If $s > 0$, then the space $H^{s,\varphi}(\text{b.c.})$ consists of the functions $u \in H^{s,\varphi}(\Omega)$ such that $B_j u = 0$ on $\partial\Omega$ for all indices $j = 1, \ldots, q$ satisfying $s > m_j + 1/2$. If $s < 1/2$, then $H^{s,\varphi}(\text{b.c.}) = H^{s,\varphi,(0)}(\Omega)$. This proposition remains true if one changes m_j for m_j^{+}, (b.c.) for (b.c.)$^{+}$, and B_j for B_j^{+}.*

Theorem 10.3 is known in the Sobolev case of $\varphi \equiv 1$ [118, Sec. 5.5.2]. In general, we can deduce it by means of the interpolation with a function parameter. Here we only need to treat the case where $m_k + 1/2 < s < m_{k+1} + 1/2$ for some $k = 1, \ldots, q$, with $m_1 < m_2 < \ldots < m_q$ and $m_{q+1} := \infty$. Chose $\varepsilon > 0$ such that $m_k + 1/2 < s \mp \varepsilon < m_{k+1} + 1/2$. Then the space $H^{s\mp\varepsilon}(\text{b.c.})$ consists of the functions $u \in H^{s\mp\varepsilon}(\Omega)$ satisfying the condition $B_j u = 0$ on $\partial\Omega$ for all $j = 1, \ldots, k$. So there exists a projector P_k of $H^{s\mp\varepsilon}(\Omega)$ onto $H^{s\mp\varepsilon}(\text{b.c.})$; it is constructed in [133, the proof of Lemma 5.4.4]. Hence, by Proposition 8.5 and Theorem 8.4 with $\varepsilon = \delta$, we get that $Y_{\psi} := [H^{s-\varepsilon}(\text{b.c.}), H^{s+\varepsilon}(\text{b.c.})]_{\psi}$ is the subspace $H^{s,\varphi}(\Omega) \cap H^{s-\varepsilon}(\text{b.c.})$ of $H^{s,\varphi}(\Omega)$. Now since $C^{\infty}(\text{b.c.})$ is dense in Y_{ψ}, we have

$$\bigl[H^{s-\varepsilon}(\text{b.c.}), H^{s+\varepsilon}(\text{b.c.})\bigr]_{\psi} = H^{s,\varphi}(\text{b.c.}) \tag{10.6}$$

with equivalence of norms, so that $H^{s,\varphi}(\text{b.c.})$ admits the description stated in Theorem 10.1.

The following theorem is about the Fredholm property of the boundary-value problem (10.4) in the two-sided refined Sobolev scale.

Theorem 10.4. *Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. Then the mapping $u \mapsto Lu$, with $u \in C^\infty(\mathrm{b.c.})$, extends uniquely to a continuous linear operator*

$$L : H^{s,\varphi}(\mathrm{b.c.}) \to (H^{2q-s,1/\varphi}(\mathrm{b.c.})^+)'. \tag{10.7}$$

Here Lu is interpreted as the functional $(Lu, \cdot)_\Omega$, and $(H^{2q-s,\,1/\varphi}(\mathrm{b.c.})^+)'$ denotes the antidual space to $H^{2q-s,1/\varphi}(\mathrm{b.c.})^+$ with respect to the inner product in $L_2(\Omega)$. The operator (10.7) is Fredholm. Its kernel coincides with $\mathcal{N}$, whereas its range consists of all the functionals $f \in (H^{2q-s,\,1/\varphi}(\mathrm{b.c.})^+)'$ such that $(f, v)_\Omega = 0$ for all $v \in \mathcal{N}^+$. The index of (10.7) is $\dim \mathcal{N} - \dim \mathcal{N}^+$ and does not depend on s and φ.

For the Sobolev scale, where $\varphi \equiv 1$, this theorem was proved by Yu. M. Berezansky, S.G. Krein, and Ya.A. Roitberg ([12] and [11, Ch. III, § 6, Sec. 10]) in the case of integral s and by Roitberg [118, Sec. 5.5.2] for all real s; see also the textbook [13, Ch. XVI, § 1] and the survey [8, Sec. 7.9 c]. They formulated the theorem in an equivalent form of a homeomorphism theorem. Note that if $s \le m + 1/2$, then the ellipticity condition is essential for the continuity of the operator (10.7).

For arbitrary $\varphi \in \mathcal{M}$, Theorem 10.4 follows from the Sobolev case by Proposition 6.5 if we apply the interpolation formulas (10.6), (10.5) and their counterparts for $H^{2q-s,1/\varphi}(\mathrm{b.c.})^+$. First we should use (10.6) for $s \notin \{j - 1/2 : j = 1, \dots, 2q\}$ and a sufficiently small $\varepsilon > 0$, then should apply (10.5) for the rest of s. Moreover, we have to resort to the interpolation duality formula $[X_1', X_0']_\psi = [X_0, X_1]_\chi'$, where $X := [X_0, X_1]$ is an admissible couple of Hilbert spaces and $\chi(t) := t/\psi(t)$ for $t > 0$. The formula follows directly from the definition of X_ψ; see, e.g., [81, Sec. 2.4].

10.3. **On a connection between nonhomogeneous and semihomogeneous elliptic problems.** Here, for the sake of simplicity, we suppose that $\mathcal{N} = \mathcal{N}^+ = \{0\}$. Let $s > m + 1/2$ and $\varphi \in \mathcal{M}$. It follows from Theorems 9.6 and 10.3 that the space $H^{s,\varphi}(\Omega)$ is the direct sum of the subspaces $K_L^{s,\varphi}(\Omega)$ and $H^{s,\varphi}(\mathrm{b.c.})$. Therefore Theorem 9.6 are equivalent to Theorems 10.1 and 10.4 taken together; note that the antidual space $(H^{2q-s,1/\varphi}(\mathrm{b.c.})^+)'$ coincides with $H^{s-2q,\varphi}(\Omega)$. Thus the nonhomogeneous problem (9.1), (9.2) can be reduced immediately to the semihomogeneous problems (10.1) and (10.4) provided $s > m + 1/2$.

This reduction fails for $s < m + 1/2$. Indeed, if $0 \le s < m + 1/2$, then the operator (L, B) cannot be well defined on $K_L^{s,\varphi}(\Omega) \cup H^{s,\varphi}(\mathrm{b.c.})$ because $K_L^{s,\varphi}(\Omega) \cap H^{s,\varphi}(\mathrm{b.c.}) \ne \varnothing$. This inequality follows from Theorems 10.1 and 10.3 if we note that the boundary-value problem (10.1), with $g_q \equiv 1$ and $g_j \equiv 0$ for $j < q$, has a nonzero solution $u \in K_L^\infty(\Omega)$ belonging to $H^{s,\varphi}(\mathrm{b.c.})$. Here we may suppose that $m_q = m$.

So much the more, the above reduction is impossible for negative s. Note if $s < -1/2$, then solutions to the semihomogeneous problems pertain to the spaces of distributions of the different nature. Namely, the solutions to the problem (10.1) belong to $K_L^{s,\varphi}(\Omega) \subset H^{s,\varphi}(\Omega)$ and are distributions given in the open domain Ω, whereas the solutions to the problem (10.4) belong to $H^{s,\varphi}(\mathrm{b.c.}) \subset H^{s,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)$ and are distributions supported on the closed domain $\overline{\Omega}$.

The same conclusions are valid in general, for nontrivial $\mathcal{N}$ and/or $\mathcal{N}^+$.

11. Generic theorems for elliptic problems in two-sided scales

Let us return to the nonhomogeneous regular elliptic boundary-value problem (9.1), (9.2). We aim to prove analogs of Theorem 9.6 for *arbitrary* real s. To get the bounded operator (L,B) for such s we have to chose another space instead of $H^{s,\varphi}(\Omega)$ as a domain of the operator. There are known two essentially different ways to construct the domain. They were suggested by Ya.A. Roitberg [114, 115, 118] and J.-L. Lions, E. Magenes [59, 60, 61, 62] in the Sobolev case. These ways lead to different kinds of theorems on the Fredholm property of (L,B); we name them generic and individual theorems. In generic theorems, the domain of (L,B) does not depend on the coefficients of the elliptic expression L and is generic for all boundary-value problems of the same order. Note that Theorem 9.6 is generic. In individual theorems, the domain depends on coefficients of L, even on the coefficients of lover order derivatives.

In this section we realize Roitberg's approach with regard to the refined Sobolev scale; namely, we modify this scale by Roitberg and prove a generic theorem about the Fredholm property of (L,B) on the two-sided modified scale. The results of the section were obtained by the authors in [82]. Lions and Magenes's approach led to individual theorems will be considered below in Section 12.

11.1. **The modification of the refined Sobolev scale.** Let $s\in\mathbb{R}$, $\varphi\in\mathcal{M}$, and integer $r>0$. We set $E_r:=\{k-1/2:k=1,\ldots,r\}$. Note that $D_\nu:=i\,\partial/\partial\nu$, where ν is the field of unit vectors of inner normals to $\partial\Omega$. Let us define the separable Hilbert spaces $H^{s,\varphi,(r)}(\Omega)$, which form the modified scale.

Definition 11.1. If $s\in\mathbb{R}\setminus E_r$, then the space $H^{s,\varphi,(r)}(\Omega)$ is defined to be the completion of $C^\infty(\,\overline{\Omega}\,)$ with respect to the Hilbert norm

$$\|u\|_{H^{s,\varphi,(r)}(\Omega)}:=\biggl(\,\|u\|^2_{H^{s,\varphi,(0)}(\Omega)}+\sum_{k=1}^{r}\bigl\|(D_\nu^{k-1}u)\restriction\partial\Omega\,\bigr\|^2_{H^{s-k+1/2,\varphi}(\partial\Omega)}\biggr)^{1/2}. \quad (11.1)$$

If $s\in E_r$, then the space $H^{s,\varphi,(r)}(\Omega)$ is defined by means of the interpolation with the power parameter $\psi(t)=t^{1/2}$, namely

$$H^{s,\varphi,(r)}(\Omega):=\bigl[\,H^{s-1/2,\varphi,(r)}(\Omega),H^{s+1/2,\varphi,(r)}(\Omega)\,\bigr]_{t^{1/2}}. \quad (11.2)$$

In the Sobolev case of $\varphi\equiv1$ the space $H^{s,\varphi,(r)}(\Omega)$ was introduced and investigated by Ya.A. Roitberg; see [114, 115] and [118, Ch. 2]. As usual, we put $H^{s,(r)}(\Omega):=H^{s,1,(r)}(\Omega)$.

Note that the case of $s\in E_r$ is special in Definition 11.1 because the norm in $H^{s,\varphi,(r)}(\Omega)$ is defined by the interpolation formula (11.2) instead of (11.1). These formulas give nonequivalent norms. As in Subsection 10.2, we have to resort to the interpolation in the mentioned case to get the spaces for which the main result of this section, Theorem 11.4, will be true.

Definition 11.2. The class of Hilbert spaces

$$\{H^{s,\varphi,(r)}(\Omega):\ s\in\mathbb{R},\,\varphi\in\mathcal{M}\} \quad (11.3)$$

is called the refined Sobolev scale modified by Roitberg. The number r is called the order of the modification.

The scale (11.3) is found fruitful in the theory of boundary-value problems because the trace mapping (8.7) extends uniquely to an operator R_r mapping continuously $H^{s,\varphi,(r)}(\Omega) \to \mathcal{H}^r_{s,\varphi}(\partial\Omega)$ for all real s. It is useful to compare this fact with Theorem 8.9, in which the condition $s > r - 1/2$ cannot be neglected. Note that

$$H^{s,\varphi,(r)}(\Omega) = H^{s,\varphi}(\Omega) \quad \text{if} \quad s > r - 1/2 \tag{11.4}$$

because the spaces in (11.4) are completions of $C^\infty(\,\overline{\Omega}\,)$ with equivalence norms due to Theorem 8.9.

The spaces $H^{s,\varphi,(r)}(\Omega)$ admit the following isometric representation. We let $K_{s,\varphi,(r)}(\Omega, \partial\Omega)$ denote the linear space of the vectors

$$(u_0, u_1, \ldots, u_r) \in H^{s,\varphi,(0)}(\Omega) \oplus \bigoplus_{k=1}^{r} H^{s-k+1/2,\,\varphi}(\partial\Omega) =: \Pi_{s,\varphi,(r)}(\Omega, \partial\Omega) \tag{11.5}$$

such that $u_k = (D_\nu^{k-1} u_0) \restriction \partial\Omega$ for each integer $k = 1, \ldots r$ satisfying $s > k - 1/2$. By Theorem 8.9 we may regard $K_{s,\varphi,(r)}(\Omega, \partial\Omega)$ as a subspace of $\Pi_{s,\varphi,(r)}(\Omega, \partial\Omega)$.

Theorem 11.3. *The mapping*

$$T_r : u \mapsto \big(u, u \restriction \partial\Omega, \ldots, (D_\nu^{r-1} u) \restriction \partial\Omega\,\big), \quad u \in C^\infty(\,\overline{\Omega}\,),$$

extends uniquely to a continuous linear operator

$$T_r :\; H^{s,\varphi,(r)}(\Omega) \to K_{s,\varphi,(r)}(\Omega, \partial\Omega) \tag{11.6}$$

for all $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. This operator is injective. Moreover, if $s \notin E_r$, then (11.6) is an isometric isomorphism.

We need only to argue that (11.6) is surjective if $s \notin E_r$. For $\varphi \equiv 1$ this property is proved by Ya.A. Roitberg; see, e.g., [118, Sec. 2.2]. In general, the proof is quite similar provided we apply Theorem 8.9 and (8.8).

Note that we have the following dense compact embeddings in the modified scale (11.3):

$$H^{s_1,\varphi_1,(r)}(\Omega) \hookrightarrow H^{s,\varphi,(r)}(\Omega), \quad -\infty < s < s_1 < \infty \;\text{ and }\; \varphi, \varphi_1 \in \mathcal{M}. \tag{11.7}$$

They results from (8.16) and Theorem 5.6 (i) by Theorem 11.3 and are understood as embeddings of spaces which are completions of the same set, $C^\infty(\,\overline{\Omega}\,)$, with different norms. Suppose that $s = s_1$, then the continuous embedding (11.7) holds if and only if φ/φ_1 is bounded in a neighbourhood of $+\infty$; the embedding is compact if and only if $\varphi(t)/\varphi_1(t) \to 0$ as $t \to +\infty$. This follows from the relevant properties of the refined Sobolev scales over Ω and $\partial\Omega$.

11.2. **Roitberg's type generic theorem.** The main result of the section is the following generic theorem about properties of the operator (L, B) on the two-sided scale (11.3) with $r = 2q$.

Theorem 11.4. *Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. The mapping (9.3) extends uniquely to a continuous linear operator*

$$(L,B):\ H^{s,\varphi,(2q)}(\Omega)\to H^{s-2q,\varphi,(0)}(\Omega)\oplus\bigoplus_{j=1}^{q}H^{s-m_j-1/2,\varphi}(\partial\Omega)\\ =:\mathcal{H}_{s,\varphi,(0)}(\Omega,\partial\Omega). \tag{11.8}$$

This operator is Fredholm. Its kernel coincides with $\mathcal{N}$, and its range consists of all the vectors $(f,g_1,\dots,g_q)\in\mathcal{H}_{s,\varphi,(0)}(\Omega,\partial\Omega)$ that satisfy (9.8). The index of (11.8) is $\dim\mathcal{N}-\dim\mathcal{N}^+$ and does not depend on s and φ.

Note that Theorem 11.4 is generic because the domain of the operator (11.8), the space $H^{s,\varphi,(2q)}(\Omega)$, is independent of L due to Definition 11.1. If $s>2q-1/2$, then generic Theorems 9.6 and 11.4 are tantamount in view of (11.4) and (8.17).

For the modified Sobolev scale, with $\varphi\equiv 1$, Theorem 11.4 was proved by Ya. A. Roitberg [114, 115], [118, Ch. 4 and Sec. 5.3]; see also the monograph [11, Ch. 3, Sec. 6, Theorem 6.9], the handbook [30, Ch. III, § 6, Sec. 5], and the survey [8, Sec. 7.9].

For arbitrary $\varphi\in\mathcal{M}$ we can deduce Theorem 11.4 from the $\varphi\equiv 1$ case with the help of the interpolation in the following way. First assume that $s\notin E_{2q}$ and let $\varepsilon>0$. We have the Fredholm bounded operators on the modified Sobolev scale

$$(L,B):\ H^{s\mp\varepsilon,(2q)}(\Omega)\to\mathcal{H}_{s\mp\varepsilon,(0)}(\Omega,\partial\Omega). \tag{11.9}$$

They possess the common kernel $\mathcal{N}$ and the common index $\varkappa:=\dim\mathcal{N}-\dim\mathcal{N}^+$. Applying the interpolation with the function parameter ψ defined by (3.8) for $\varepsilon=\delta$, we get by Proposition 6.5 and Theorems 5.7, 8.13 that (11.9) implies the boundedness and the Fredholm property of the operator

$$(L,B):\ \bigl[H^{s-\varepsilon,(2q)}(\Omega),H^{s+\varepsilon,(2q)}(\Omega)\bigr]_{\psi}\to\mathcal{H}_{s,\varphi,(0)}(\Omega,\partial\Omega).$$

It remains to prove the interpolation formula

$$\bigl[H^{s-\varepsilon,(2q)}(\Omega),H^{s+\varepsilon,(2q)}(\Omega)\bigr]_{\psi}=H^{s,\varphi,(2q)}(\Omega), \tag{11.10}$$

where the equality of spaces is up to equivalence of norms.

Let an index p be such that $s\in\alpha_p$, where $\alpha_0:=(-\infty,1/2)$, $\alpha_k:=(k-1/2,\,k+1/2)$ with $k=1,\dots,2q-1$, and $\alpha_{2q}:=(2q-1/2,\infty)$. We chose $\varepsilon>0$ satisfying $s\mp\varepsilon\in\alpha_p$. By Theorem 11.3, the mapping

$$T_{2q,p}:\ u\mapsto\bigl(u,\ \{(D_\nu^{k-1}u)\!\restriction\!\partial\Omega:\ p+1\le k\le 2q\}\bigr)$$

establishes the homeomorphisms

$$T_{2q,p}:\ H^{s,\varphi,(2q)}(\Omega)\leftrightarrow H^{s,\varphi,(0)}(\Omega)\oplus\bigoplus_{p+1\le k\le 2q}H^{s-k+1/2,\varphi}(\partial\Omega)=:K^{p}_{s,\varphi,(2q)}(\Omega,\partial\Omega), \tag{11.11}$$

$$T_{2q,p}:\ H^{s\mp\varepsilon,(2q)}(\Omega)\leftrightarrow K^{p}_{s\mp\varepsilon,(2q)}(\Omega,\partial\Omega). \tag{11.12}$$

Applying the interpolation with ψ, we deduce another homeomorphism from (11.12):

$$T_{2q,p}:\ \bigl[H^{s-\varepsilon,(2q)}(\Omega),H^{s+\varepsilon,(2q)}(\Omega)\bigr]_{\psi}\leftrightarrow K^{p}_{s,\varphi,(2q)}(\Omega,\partial\Omega). \tag{11.13}$$

Now (11.11) and (11.13) imply the required formula (11.10).

In the remaining case of $s\in E_{2q}$, we deduce Theorem 11.4 from the $s\notin E_{2q}$ case by the interpolation with the power parameter $\psi(t)=t^{1/2}$ if we apply (11.2) and the counterparts of Theorem 3.9 for the refined Sobolev scales over Ω and $\partial\Omega$.

Note that the continuity of the operator (11.8) holds without the assumption about the regular ellipticity of the boundary-value problem (9.1), (9.2).

11.3. **Roitberg's interpretation of generalized solutions.** Using Theorem 10.1, we can give the following interpretation of a solution $u\in H^{s,\varphi,(2q)}(\Omega)$ to the boundary-value problem (9.1), (9.2) in the framework of the distribution theory.

Let us write down the differential expressions L and B_j in a neighbourhood of $\partial\Omega$ in the form

$$L=\sum_{k=0}^{2q}\,L_k\,D_\nu^k,\quad B_j=\sum_{k=0}^{m_j}\,B_{j,k}\,D_\nu^k. \tag{11.14}$$

Here L_k and $B_{j,k}$ are certain tangent differential expression. Integrating by parts, we arrive at the (special) Green formula

$$(Lu,v)_\Omega=(u,L^{+}v)_\Omega-i\sum_{k=1}^{2q}\,(D_\nu^{k-1}u,L^{(k)}v)_{\partial\Omega},\quad u,v\in C^\infty(\,\overline{\Omega}\,).$$

Here $L^{(k)}:=\sum_{r=k}^{2q}D_\nu^{r-k}L_r^{+}$, where L_r^{+} is the tangent differential expression formally adjoint to L_r. By passing to the limit and using the notation

$$(u_0,u_1,\ldots,u_{2q}):=T_{2q}u\in K_{s,\varphi,(2q)}(\Omega,\partial\Omega), \tag{11.15}$$

we get the next equality for $u\in H^{s,\varphi,(2q)}(\Omega)$:

$$(Lu,v)_\Omega=(u_0,L^{+}v)_\Omega-i\sum_{k=1}^{2q}\,(u_k,L^{(k)}v)_{\partial\Omega},\quad v\in C^\infty(\,\overline{\Omega}\,). \tag{11.16}$$

Now it follows from (11.14) and (11.16) that the element $u\in H^{s,\varphi,(2q)}(\Omega)$ is a solution to the boundary-value problem (9.1), (9.2) with $f\in H^{s-2q,\varphi,(0)}(\Omega)$, $g_i\in H^{s-m_j-1/2,\,\varphi}(\partial\Omega)$ if and only if

$$(u_0,L^{+}v)_\Omega-i\sum_{k=1}^{2q}\,(u_k,L^{(k)}v)_{\partial\Omega}=(f,v)_\Omega\quad\text{for all}\quad v\in C^\infty(\,\overline{\Omega}\,), \tag{11.17}$$

$$\sum_{k=0}^{m_j}\,B_{j,k}\,u_{k+1}=g_j\ \ \text{on}\ \ \partial\Omega,\quad j=1,\ldots,q. \tag{11.18}$$

Note that these equalities have meaning for arbitrary distributions

$$u_0 \in \mathcal{D}'(\mathbb{R}^n), \ \ \operatorname{supp} u_0 \subseteq \overline{\Omega}, \quad u_1, \dots, u_{2q} \in \mathcal{D}'(\Gamma), \tag{11.19}$$

$$f \in \mathcal{D}'(\mathbb{R}^n), \ \ \operatorname{supp} f \subseteq \overline{\Omega}, \quad g_1, \dots, g_q \in \mathcal{D}'(\Gamma). \tag{11.20}$$

Therefore it is useful to introduce the following notion.

Definition 11.5. Suppose that (11.19) and (11.20) are fulfilled. Then the vector $u := (u_0, u_1, \dots, u_{2q})$ is called a generalized solution in Roitberg's sense to the boundary-value problem (9.1), (9.2) if the conditions (11.17) and (11.18) are valid.

This interpretation of a generalized solution is suggested by Roitberg; see, e.g, his monograph [118, Sec. 2.4].

Thus, Theorem 11.4 can be regarded as a statement about the solvability of the boundary-value problem (9.1), (9.2) in the class of generalized solutions in Roitberg's sense provided that we identify solutions $u \in H^{s,\varphi,(2q)}(\Omega)$ with vectors (11.15).

Roitberg's interpretation of a generalized solution and the relevant Theorem 11.4 have been found fruitful in the theory of elliptic boundary-value problems. Analogs of this theorem were proved by Roitberg for nonregular elliptic boundary-value problems and for general elliptic systems of differential equations, the modified scale of the L_p-type Sobolev spaces with $1 < p < \infty$ being used. In the literature [30, 118, 119], Theorem 11.4 and its analogs are known as theorems on a complete collection of homeomorphisms. They have various applications; among them are the theorems on an increase in smoothness of solutions up to the boundary, application to the investigation of Green functions of elliptic boundary-value problems, applications to elliptic problems with power singularities, to the transmission problem, the Odhnoff problem, and others. The investigations of Ya.A. Roitberg, Z.G. Sheftel' and their disciples into this subject were summed up in Roitberg's monographs [118].

Note that, in the most general form, the theorem on a complete collection of homeomorphisms was proved by A. Kozhevnikov [56] for general elliptic pseudodifferential boundary-value problems. Analogs of Theorem 11.4 were obtained in [92, 93] for some non-Sobolev Banach spaces parametrized by collections of numbers; the case of a scalar elliptic equation was treated therein. We also remark applications of the concept of a generalized solution and relevant modified two-sided scale in the theory of elliptic boundary-value problems in nonsmooth domains [57] and in the theory of parabolic [26] and hyperbolic [119] equations.

12. Individual theorems for elliptic problems

In this section, we generalize J.-L. Lions and E. Magenes's method [59, 60, 61, 62] for constructing of the domain of the operator (L, B). We prove new theorems on the Fredholm property of the operator on scales of Sobolev inner product spaces and some Hörmander spaces. These theorems has an individual character because the domain of (L, B) depends on coefficients of elliptic expression L, as distinguished from generic Theorems 9.6 and 11.4. Moreover, in the individual theorems the operator (L, B) acts on the spaces consisting of distributions given in the domain Ω, so that we do not need to modify the refined Sobolev scale as this was done for Theorem 11.4.

The section is organized in the following manner. First, for the sake of the reader's convenience, we recall Lions and Magenes's theorems about elliptic boundary-value problems. Then we prove a certain general form of the Lions–Magenes theorems; we call it the key theorem. Namely, we find a general condition on the space of right-hand sides of the elliptic equation $Lu = f$ under which the operator (L, B) is bounded and Fredholm on the corresponding pairs of Sobolev inner product spaces of negative order. Extensive classes of the spaces satisfying this condition will be constructed; they contain the spaces used by Lions and Magenes and many others spaces. These results motivate statements and proofs of individual theorems on the Fredholm property of the operator (L, B) on some Hilbert Hörmander spaces.

12.1. **The Lions–Magenes theorems.** As we have mentioned in Remark 8.8, J.-L. Lions and E. Magenes used a definition of the Sobolev space of negative order s over Ω which is different from our Definition 8.2 for $\varphi \equiv 1$. Namely, they defined this space as the dual of $H_0^{-s}(\Omega)$ with respect to the inner product in $L_2(\Omega)$. We use this definition throughout Section 12. To distinguish the Sobolev spaces $H^s(\Omega)$ introduced above by Definition 8.2 from ones used here, we resort to the somewhat different notation $\mathrm{H}^s(\Omega)$, where the letter H is not slanted.

Thus we put

$$\mathrm{H}^s(\Omega) := \begin{cases} H^s(\Omega) & \text{for } s \geq 0, \\ (H_0^{-s}(\Omega))' & \text{for } s < 0. \end{cases}$$

Here $(H_0^{-s}(\Omega))'$ denotes the Hilbert space antidual to $H_0^{-s}(\Omega)$ with respect to the inner product in $L_2(\Omega)$.

The antilinear continuous functionals from $\mathrm{H}^s(\Omega)$ with $s < 0$ are defined uniquely by their values on the functions in $C_0^\infty(\Omega)$. Therefore it is reasonable to identify these functionals with distributions given in Ω. In so doing, we have [61, Ch. 1, Remark 12.5]

$$\mathrm{H}^s(\Omega) = H^s_{\overline{\Omega}}(\mathbb{R}^n)/H^s_{\partial\Omega}(\mathbb{R}^n) = \bigl\{ w \restriction \Omega : \, w \in H^s_{\overline{\Omega}}(\mathbb{R}^n) \bigr\} \quad \text{for} \quad s < 0. \tag{12.1}$$

It is remarkable that the spaces $\mathrm{H}^s(\Omega)$ and $H^s(\Omega)$, with $s < 0$, coincide up to equivalence of norms provided $s + 1/2 \notin \mathbb{Z}$; see, e.g., [133, Sec. 4.8.2]. If s is half-integer, then $\mathrm{H}^s(\Omega)$ is narrower than $H^s(\Omega)$. Note also that

$$-1/2 \leq s < 0 \;\Rightarrow\; \mathrm{H}^s(\Omega) = H^{s,(0)}(\Omega) \;\text{ with equality of norms.} \tag{12.2}$$

This fact follows, by the duality, from the equality $H_0^{-s}(\Omega) = H^{-s}(\Omega)$; see, e.g., [133, Sec. 4.7.1].

Lions and Magenes consider the operator

$$(L, B) : \, D^{\sigma+2q}_{L,X}(\Omega) \to X^\sigma(\Omega) \oplus \bigoplus_{j=1}^{q} H^{\sigma+2q-m_j-1/2}(\partial\Omega) =: \mathbf{X}_\sigma(\Omega, \partial\Omega), \tag{12.3}$$

with $\sigma \in \mathbb{R}$. Here $X^\sigma(\Omega)$ is a certain Hilbert space consisting of distributions in Ω and embedded continuously in $\mathcal{D}'(\Omega)$. The domain of the operator (12.3) is the

Hilbert space

$$D^{\sigma+2q}_{L,X}(\Omega):=\bigl\{u\in \mathrm{H}^{\sigma+2q}(\Omega):\, Lu\in X^{\sigma}(\Omega)\bigr\}$$

endowed with the graph inner product

$$(u_1,u_2)_{D^{\sigma+2q}_{L,X}(\Omega)}:=(u_1,u_2)_{\mathrm{H}^{\sigma+2q}(\Omega)}+(Lu_1,Lu_2)_{X^{\sigma}(\Omega)}.$$

In the case where $s:=\sigma+2q>m+1/2$ we may set $X^{\sigma}(\Omega):=H^{\sigma}(\Omega)$ that leads us to Theorem 9.6 for $\varphi\equiv 1$. But in the case where $s\leq m+1/2$ we cannot do so if we want to have the well-defined operator (12.3). The space $X^{\sigma}(\Omega)$ must be narrower than $H^{\sigma}(\Omega)$.

Lions and Magenes found some important spaces $X^{\sigma}(\Omega)$ with $\sigma<0$ such that the operator (12.3) is bounded and Fredholm; see [59, 60] and [61, Ch. 2, Sec. 6.3]. We state their results in the form of two individual theorems on elliptic boundary-value problems.

Theorem 12.1 (the first Lions–Magenes theorem [59, 60])**.** *Let $\sigma<0$ and $X^{\sigma}(\Omega):=L_2(\Omega)$. Then the mapping (9.3) extends uniquely to the continuous linear operator (12.3). This operator is Fredholm. Its kernel coincides with $\mathcal{N}$, and its range consists of all the vectors $(f,g_1,\ldots,g_q)\in\mathbf{X}_{\sigma}(\Omega,\partial\Omega)$ satisfying (9.8). The index of (12.3) is $\dim\mathcal{N}-\dim\mathcal{N}^{+}$ and does not depend on σ.*

Remark 12.2. Here, the $\sigma=-2q$ case is important in the spectral theory of elliptic operators with general boundary conditions [34, 35, 70, 71]; see also the survey [8, Sec. 7.7 and 9.6]. Then the space $D^{0}_{A,L_2}(\Omega)=\{u\in L_2(\Omega):\, Au\in L_2(\Omega)\}$ is the domain of the maximal operator $A_{\max}$ corresponding to the differential expression A. Even when all coefficients of A are constant, this space depends essentially on each of them [42, Sec. 3.1, Theorem 3.1].

To formulate the second Lions–Magenes theorem, we need the next weighted space

$$\varrho\mathrm{H}^{\sigma}(\Omega):=\{f=\varrho v:\; v\in \mathrm{H}^{\sigma}(\Omega)\,\},\quad (f_1,f_2)_{\varrho\mathrm{H}^{\sigma}(\Omega)}:=(\varrho^{-1}f_1,\varrho^{-1}f_2)_{\mathrm{H}^{\sigma}(\Omega)},$$

with $\sigma<0$ and a positive function $\varrho\in C^{\infty}(\Omega)$. The space $\varrho\mathrm{H}^{\sigma}(\Omega)$ is Hilbert and imbedded continuously in $\mathcal{D}'(\Omega)$. Consider a weight function $\varrho:=\varrho_1^{-\sigma}$ such that

$$\varrho_1\in C^{\infty}(\,\overline{\Omega}\,),\;\; \varrho_1>0\;\text{ in }\Omega,\;\; \varrho_1(x)=\mathrm{dist}(x,\partial\Omega)\;\text{ near by }\partial\Omega. \tag{12.4}$$

Theorem 12.3 (the second Lions–Magenes theorem [61])**.** *Let $\sigma<0$ and*

$$X^{\sigma}(\Omega):=\left\{\begin{array}{ll}\varrho_1^{-\sigma}\mathrm{H}^{\sigma}(\Omega) & \text{if }\ \sigma+1/2\notin\mathbb{Z},\\ \bigl[\,\varrho_1^{-\sigma+1/2}\,\mathrm{H}^{\sigma-1/2}(\Omega),\;\varrho_1^{-\sigma-1/2}\,\mathrm{H}^{\sigma+1/2}(\Omega)\bigr]_{t^{1/2}} & \text{if }\ \sigma+1/2\in\mathbb{Z}.\end{array}\right. \tag{12.5}$$

Then the conclusion of Theorem 12.1 remains true.

Remark 12.4. In the cited monograph [61, Ch. 2, Sec. 6.3], Lions and Magenes introduced the space $X^{\sigma}(\Omega)$ in a way different from (12.5) and designated $X^{\sigma}(\Omega)$ as $\Xi^{\sigma}(\Omega)$. Namely, for an integer $\sigma\geq 0$, the space $\Xi^{\sigma}(\Omega)$ is defined to consists of all $f\in\mathcal{D}'(\Omega)$ such that $\varrho_1^{|\mu|}D^{\mu}f\in L_2(\Omega)$ for each multi-index μ with $|\mu|\leq\sigma$, and $\Xi^{\sigma}(\Omega)$ is endowed with the Hilbert norm $\sum_{|\mu|\leq\sigma}\|\varrho_1^{|\mu|}D^{\mu}f\|_{L_2(\Omega)}$. Then, $\Xi^{\sigma}(\Omega):=[\Xi^{[\sigma]}(\Omega),\Xi^{[\sigma]+1}(\Omega)]_{t^{\{\sigma\}}}$ for fractional $\sigma>0$, with $\sigma=[\sigma]+\{\sigma\}$ and $[\sigma]$

being the integral part of σ. Finally, $\Xi^{\sigma}(\Omega) := (\Xi^{-\sigma}(\Omega))'$ for $\sigma < 0$, the duality being with respect to the inner product in $L_2(\Omega)$. It follows from the result of Lions and Magenes [61, Ch. 2, Sec. 7.1, Corollary 7.4] that, for every $\sigma < 0$, the space $\Xi^{\sigma}(\Omega)$ coincides with (12.5) up to equivalence of norms.

12.2. **An extension of the Lions–Magenes theorems.** The results presented here are got by the second author in [100]. First, we establish the key theorem, which is a certain generalization of the Lions–Magenes theorems stated above. The key theorem asserts that the operator (12.3) is well defined, bounded, and Fredholm for $\sigma < 0$ provided that a Hilbert space $X^{\sigma}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ satisfies the following condition.

Condition 12.5 (we name it as I_σ)**.** The set $X^{\infty}(\Omega) := X^{\sigma}(\Omega) \cap C^{\infty}(\,\overline{\Omega}\,)$ is dense in $X^{\sigma}(\Omega)$, and there exists a number $c > 0$ such that $\|\mathcal{O}f\|_{H^{\sigma}(\mathbb{R}^n)} \leq c\,\|f\|_{X^{\sigma}(\Omega)}$ for all $f \in X^{\infty}(\Omega)$, where $\mathcal{O}f$ is defined by (8.15).

Note that the smaller σ is, the weaker Condition 12.5 (I_σ) will be for the same space $X^{\sigma}(\Omega)$.

The spaces $X^{\sigma}(\Omega)$ appearing in Theorems 12.1 and 12.3 satisfy Condition 12.5. This is evident for the first theorem, whereas, for the second one, this follows from the dense continuous imbedding $\mathrm{H}^{-\sigma}(\Omega) \hookrightarrow \Xi^{-\sigma}(\Omega)$ by the duality in view of Theorem 8.7 iii) and Remark 12.4.

Our key theorem is the following.

Theorem 12.6. *Let $\sigma < 0$ and $X^{\sigma}(\Omega)$ be an arbitrary Hilbert space imbedded continuously in $\mathcal{D}'(\Omega)$ and satisfying Condition 12.5 (I_σ). Then:*

- i) *The set $D^{\infty}_{L,X}(\Omega) := \{u \in C^{\infty}(\,\overline{\Omega}\,) : Lu \in X^{\sigma}(\Omega)\}$ is dense in $D^{\sigma+2q}_{L,X}(\Omega)$.*
- ii) *The mapping $u \to (Lu, Bu)$, with $u \in D^{\infty}_{L,X}(\Omega)$, extends uniquely to the continuous linear operator (12.3).*
- iii) *The operator (12.3) is Fredholm. Its kernel is $\mathcal{N}$, and its range consists of all the vectors $(f, g_1, \ldots, g_q) \in \mathbf{X}_{\sigma}(\Omega, \partial\Omega)$ that satisfy (9.8).*
- iv) *If $\mathcal{O}(X^{\infty}(\Omega))$ is dense in $H^{\sigma}_{\overline{\Omega}}(\mathbb{R}^n)$, then the index of (12.3) is $\dim \mathcal{N} - \dim \mathcal{N}^{+}$.*

Let us outline the proof of Theorem 12.6. The main idea is to derive this theorem from Roitberg's generic theorem, i.e. from Theorem 11.4 considered in the $\varphi \equiv 1$ case. For the sake of simplicity, suppose that $\mathcal{N} = \mathcal{N}^{+} = \{0\}$.

We get from Condition 12.5 (I_σ) that the mapping $f \mapsto \mathcal{O}f$, $f \in X^{\infty}(\Omega)$, extends by a continuity to a bounded linear injective operator $\mathcal{O} : X^{\sigma}(\Omega) \to H^{\sigma}_{\overline{\Omega}}(\mathbb{R}^n)$. This operator defines the continuous imbedding $X^{\sigma}(\Omega) \hookrightarrow H^{\sigma,(0)}(\Omega)$. Hence, by Theorem 11.4 a restriction of (11.8) establishes a homeomorphism

$$(L, B) : D^{\sigma+2q,(2q)}_{L,X}(\Omega) \leftrightarrow \mathbf{X}_{\sigma}(\Omega, \partial\Omega). \tag{12.6}$$

Its domain is the Hilbert space

$$D^{\sigma+2q,(2q)}_{L,X}(\Omega) := \{u \in H^{\sigma+2q,(2q)}(\Omega) : Lu \in X^{\sigma}(\Omega)\},$$
$$\|u\|^2_{D^{\sigma+2q,(2q)}_{L,X}(\Omega)} := \|u\|^2_{H^{\sigma+2q,(2q)}(\Omega)} + \|Lu\|^2_{X^{\sigma}(\Omega)}.$$

It follows from (12.6) that $D^{\infty}_{L,X}(\Omega)$ is dense in $D^{\sigma+2q,(2q)}_{L,X}(\Omega)$.

According to Ya.A. Roitberg [118, Sec. 6.1, Theorem 6.1.1] we have the equivalence of norms

$$\|u\|_{H^{\sigma+2q,(2q)}(\Omega)} \asymp \big(\,\|u\|^2_{H^{\sigma+2q,(0)}(\Omega)} + \|Lu\|^2_{H^{\sigma,(0)}(\Omega)}\,\big)^{1/2}, \quad u \in C^{\infty}(\,\overline{\Omega}\,). \qquad (12.7)$$

This result and the continuous imbedding $X^{\sigma}(\Omega) \hookrightarrow H^{\sigma,(0)}(\Omega)$ imply

$$\|u\|_{D^{\sigma+2q,(2q)}_{L,X}(\Omega)} \asymp \big(\,\|u\|^2_{H^{\sigma+2q,(0)}(\Omega)} + \|Lu\|^2_{X^{\sigma}(\Omega)}\,\big)^{1/2}, \quad u \in C^{\infty}(\,\overline{\Omega}\,). \qquad (12.8)$$

Thus, $D^{\sigma+2q,(2q)}_{L,X}(\Omega)$ is the completion of $D^{\infty}_{L,X}(\Omega)$ with respect to the norm which is the right-hand side of (12.8).

Consider the mapping $u \mapsto u_0$ that takes each $u \in D^{\sigma+2q,(2q)}_{L,X}(\Omega)$ to the initial component $u_0 \in H^{\sigma+2q,(0)}(\Omega)$ of the vector $T_{2q}u$. Here the operator T_{2q} is that in Theorem 11.3 for $r = 2q$.

If $-2q - 1/2 \leq \sigma < 0$, then $H^{\sigma+2q,(0)}(\Omega) = \mathrm{H}^{\sigma+2q}(\Omega)$ by (12.2). Now, we may assert that the mapping $u \mapsto u_0$ establishes a homeomorphism of $D^{\sigma+2q,(2q)}_{L,X}(\Omega)$ onto $D^{\sigma+2q}_{L,X}(\Omega)$. Hence, (12.6) implies the required homeomorphism

$$(L, B) : D^{\sigma+2q}_{L,X}(\Omega) \leftrightarrow \mathbf{X}_{\sigma}(\Omega, \partial\Omega). \qquad (12.9)$$

Further, if $\sigma < -2q - 1/2$, then $H^{\sigma+2q,(0)}(\Omega) = H^{\sigma+2q}_{\overline{\Omega}}(\mathbb{R}^n)$. Then using (12.1) and Roitberg's result [118, Sec. 6.2, Theorem 6.2] we can prove that the mapping $u \mapsto u_0 \restriction \Omega$ establishes a homeomorphism of $D^{\sigma+2q,(2q)}_{L,X}(\Omega)$ onto $D^{\sigma+2q}_{L,X}(\Omega)$. Hence, (12.6) implies (12.9) in this case as well. See [100, Sec. 4] for more details.

Remark 12.7. A proposition similar to Theorem 12.6 was proved in Magenes's survey [62, Sec. 6.10] for non half-integer $\sigma \leq -2q$ and the Dirichlet problem, the space $X^{\sigma}(\Omega)$ obeying some different conditions depending on the problem. Our Condition 12.5 (I_{σ}) does not depend on it.

Remark 12.8. Ya.A. Roitberg [117, Sec. 2.4] considered a condition on the space $X^{\sigma}(\Omega)$, which was somewhat stronger than Condition 12.5 (I_{σ}). He required additionally that $C^{\infty}(\,\overline{\Omega}\,) \subset X^{\sigma}(\Omega)$. Under this stronger condition, Roitberg [117, Sec. 2.4], [118, Sec. 6.2, p. 190] proved the boundedness of the operator (12.3) for all $\sigma < 0$. Homeomorphism Theorem for this operator was formulated in the survey [8, Sec. 7.9, p. 85] provided that $-2q \leq s \leq 0$ and $\mathcal{N} = \mathcal{N}^{+} = \{0\}$. We also mention the analogs of Theorem 12.6 proved by Yu.V. Kostarchuk and Ya.A. Roitberg [52, Theorem 4], [119, Sec. 1.3.8]. In these analogs, Roitberg's condition is used, but solutions of an elliptic boundary-value problem are considered in $H^{\sigma+2q,(2q)}(\Omega)$. Note that Roitberg's condition does not include the important case where $X^{\sigma}(\Omega) = \{0\}$ and does not cover some weighted spaces $X^{\sigma}(\Omega) = \varrho\mathrm{H}^{\sigma}(\Omega)$, which we consider.

Let us consider some applications of Theorem 12.6 caused by a particular choice of the space $X^{\sigma}(\Omega)$. Apparently, the space $X^{\sigma}(\Omega) := \{0\}$ satisfies Condition 12.5 (I_{σ}). In this case, Theorem 12.6 coincides with Theorem 10.1 for $s := \sigma + 2q < 2q$. It is remarkable that, despite $\mathrm{H}^{s}(\Omega) \neq H^{s}(\Omega)$ for half-integer $s < 0$, we have

$$\{u \in \mathrm{H}^{s}(\Omega) :\ Lu = 0 \ \text{ in } \ \Omega\} = \{u \in H^{s}(\Omega) :\ Lu = 0 \ \text{ in } \ \Omega\}, \qquad (12.10)$$

the norms in $\mathrm{H}^s(\Omega)$ and $H^s(\Omega)$ being equivalent on the distributions u appearing in (12.10).

It is also evident that the space $X^\sigma(\Omega) := L_2(\Omega)$ satisfies Condition 12.5 (I_σ) for every $\sigma < 0$. In this important case, Theorem 12.6 coincides with Theorem 12.1.

We can describe all the Sobolev inner product spaces satisfying Condition 12.5.

Lemma 12.9. *Let $\sigma < 0$ and $\lambda \in \mathbb{R}$. The space $X^\sigma(\Omega) := \mathrm{H}^\lambda(\Omega)$ satisfies Condition 12.5 (I_σ) if and only if $\lambda \geq \max\{\sigma, -1/2\}$.*

Indeed, we can restrict ourselves to the $\lambda < 0$ case. Then the space $X^\sigma(\Omega) := \mathrm{H}^\lambda(\Omega)$ satisfies Condition 12.5 (I_σ) if and only if the mapping $\mathcal{O}$ establishes the dense continuous embedding $\mathrm{H}^\lambda(\Omega) \hookrightarrow H^\sigma_{\overline{\Omega}}(\mathbb{R}^n)$. By the duality, this embedding is equivalent to the dense continuous embedding $H^{-\sigma}(\Omega) \hookrightarrow H_0^{-\lambda}(\Omega)$, which is valid if and only if $-\sigma \geq -\lambda$ and $H_0^{-\lambda}(\Omega) = H^{-\lambda}(\Omega)$. Since the latter equality $\Leftrightarrow -\lambda \leq 1/2$, the lemma is proved.

The next individual theorem results from Theorem 12.6 and Lemma 12.9.

Theorem 12.10. *Let $\sigma < 0$ and $\lambda \geq \max\{\sigma, -1/2\}$. Then the mapping $u \mapsto (Lu, Bu)$, with $u \in C^\infty(\overline{\Omega})$, extends uniquely to a continuous linear operator*

$$(L, B) : \{u \in \mathrm{H}^{\sigma+2q}(\Omega) : Lu \in \mathrm{H}^\lambda(\Omega)\} \to \mathrm{H}^\lambda(\Omega) \oplus \bigoplus_{j=1}^{q} H^{\sigma+2q-m_j-1/2}(\partial\Omega) \tag{12.11}$$

provided that its domain is endowed with the norm

$$\big(\,\|u\|^2_{\mathrm{H}^{\sigma+2q}(\Omega)} + \|Lu\|^2_{\mathrm{H}^\lambda(\Omega)}\big)^{1/2}.$$

The domain is a Hilbert space with respect to this norm. Moreover, the operator (12.11) is Fredholm, and its index is $\dim\mathcal{N} - \dim\mathcal{N}^+$.

Here, it is useful to discuss the special case where $\lambda = \sigma$. If $-1/2 < \lambda = \sigma < 0$, then the domain of (12.11) coincides with $H^{\sigma+2q}(\Omega)$ and we arrive at Theorem 9.6 for $s = \sigma + 2q$ and $\varphi \equiv 1$. If $\lambda = \sigma = -1/2$, then the domain is narrower than $\mathrm{H}^{2q-1/2}(\Omega)$ and is equal to $H^{2q-1/2,(2q)}(\Omega)$ in view of (12.8) and (12.2) so that we get Theorem 11.4 for $s = 2q - 1/2$ and $\varphi \equiv 1$.

In Theorem 12.10, we always have $X^\sigma(\Omega) \subseteq \mathrm{H}^{-1/2}(\Omega)$. But we can get a space $X^\sigma(\Omega)$ containing an extensive class of distributions $f \notin \mathrm{H}^{-1/2}(\Omega)$ and satisfying Condition 12.5 (I_σ) if we use certain weighted spaces $\varrho\mathrm{H}^\sigma(\Omega)$.

In this connection, recall the following.

Definition 12.11. Let $X(\Omega)$ be a Banach space lying in $\mathcal{D}'(\Omega)$. A function ϱ given in Ω is called a multiplier in $X(\Omega)$ if the operator of multiplication by ϱ is defined and bounded on $X(\Omega)$.

Let $\sigma < -1/2$ and consider the next condition.

Condition 12.12 (we name it as II_σ)**.** The function ϱ is a multiplier in $H^{-\sigma}(\Omega)$, and

$$D_\nu^j\, \varrho = 0 \;\text{ on }\; \partial\Omega \;\text{ for every }\; j \in \mathbb{Z} \;\text{ such that }\; 0 \leq j < -\sigma - 1/2. \tag{12.12}$$

Note if ϱ is a multiplier in $H^{-\sigma}(\Omega)$, then evidently $\varrho \in H^{-\sigma}(\Omega)$ so that, by Theorem 8.9, the trace of $D_\nu^j \varrho$ on $\partial\Omega$ is well defined in (12.12). A description of the set of all multipliers in $H^{-\sigma}(\Omega)$ is given in [66, Sec. 9.3.3].

Using Condition 12.12 (II_σ), we can describe the class of all weighted Sobolev inner product spaces of order σ that satisfies Condition 12.5 (I_σ).

Lemma 12.13. *Let $\sigma < -1/2$, and let a function $\varrho \in C^\infty(\Omega)$ be positive. The space $X^\sigma(\Omega) := \varrho\mathrm{H}^\sigma(\Omega)$ satisfies Condition 12.5 (I_σ) if and only if ϱ meets Condition 12.12 (II_σ).*

Indeed, using the intrinsic description of $H_0^{-\sigma}(\Omega)$ mentioned in Subsection 8.4, we can prove that ϱ satisfies Condition 12.12 (II_σ) if and only if the multiplication by ϱ is a bounded operator $M_\varrho : H^{-\sigma}(\Omega) \to H_0^{-\sigma}(\Omega)$. The latter is equivalent, by the duality, to the boundedness of the operator $M_\varrho : \mathrm{H}^\sigma(\Omega) \to H^\sigma_{\overline{\Omega}}(\mathbb{R}^n)$. Note that the mapping $f \mapsto \varrho^{-1} f$ establishes the homeomorphism $M_{\varrho^{-1}} : \varrho\mathrm{H}^\sigma(\Omega) \leftrightarrow \mathrm{H}^\sigma(\Omega)$. Therefore, we conclude that ϱ satisfies Condition 12.12 (II_σ) if and only if the identity operator $M_\varrho M_{\varrho^{-1}}$ establishes a continuous embedding $\mathcal{O} : \varrho\mathrm{H}^\sigma(\Omega) \to H^\sigma_{\overline{\Omega}}(\mathbb{R}^n)$. The embedding means that the space $X^\sigma(\Omega) = \varrho\mathrm{H}^\sigma(\Omega)$ satisfies Condition 12.5 (I_σ).

The next individual theorem results from Theorem 12.6 and Lemma 12.13.

Theorem 12.14. *Let $\sigma < -1/2$, and let a positive function $\varrho \in C^\infty(\Omega)$ satisfy Condition 12.12 (II_σ). Then the mapping $u \to (Lu, Bu)$, with $u \in C^\infty(\overline{\Omega})$, $Lu \in \varrho\mathrm{H}^\sigma(\Omega)$, extends uniquely to a continuous linear operator*

$$(L,B) : \bigl\{u \in \mathrm{H}^{\sigma+2q}(\Omega) : Lu \in \varrho\mathrm{H}^\sigma(\Omega)\bigr\} \to \varrho\mathrm{H}^\sigma(\Omega) \oplus \bigoplus_{j=1}^{q} H^{\sigma+2q-m_j-1/2}(\partial\Omega) \tag{12.13}$$

provided that its domain is endowed with the norm

$$\bigl(\|u\|^2_{\mathrm{H}^{\sigma+2q}(\Omega)} + \|\varrho^{-1}Lu\|^2_{\mathrm{H}^\sigma(\Omega)}\bigr)^{1/2}.$$

The domain is a Hilbert space with respect to this norm. Moreover, the operator (12.13) is Fredholm, and its index is $\dim\mathcal{N} - \dim\mathcal{N}^+$.

We give an important example of a function ϱ satisfying Condition 12.12 (II_σ) for fixed $\sigma < -1/2$ if we set $\varrho := \varrho_1^\delta$ provided that ϱ_1 meets (12.4) and that $\delta \geq -\sigma - 1/2 \in \mathbb{Z}$ or $\delta > -\sigma - 1/2 \notin \mathbb{Z}$.

It is useful to compare Theorem 12.3 (the second Lions–Magenes theorem) with Theorems 12.10 and 12.14. For non half-integer $\sigma < -1/2$, Theorem 12.3 is the special case of Theorem 12.14, where $\varrho := \varrho_1^{-\sigma}$. For the half-integer values of $\sigma < -1/2$, Theorem 12.3 follows from this case by the interpolation with the power parameter $t^{1/2}$. Finally, if $-1/2 \leq \sigma < 0$, then Theorem 12.3 is a consequence of Theorem 12.10, in which we can take the space $X^\sigma(\Omega) := \mathrm{H}^\sigma(\Omega)$ containing $\varrho_1^{-\sigma}\mathrm{H}^\sigma(\Omega)$.

12.3. **Individual theorems on classes of Hörmander spaces.** Here we give analogs of Theorems 12.6, 12.10, and 12.14 for some classes of Hörmander spaces. The proofs of the analogs are similar to those outlined in the previous subsection.

First, we state the key theorem, an analog of Theorems 12.6. Let $\sigma < 0$ and $\varphi \in \mathcal{M}$. Suppose that a Hilbert space $X^{\sigma,\varphi}(\Omega)$ is embedded continuously in $\mathcal{D}'(\Omega)$. Consider the following analog of Condition 12.5 (I_σ).

Condition 12.15 (we name it as $\mathrm{I}_{\sigma,\varphi}$)**.** The set $X^\infty(\Omega) := X^{\sigma,\varphi}(\Omega) \cap C^\infty(\overline{\Omega})$ is dense in $X^{\sigma,\varphi}(\Omega)$, and there exists a number $c > 0$ such that $\|\mathcal{O}f\|_{H^{\sigma,\varphi}(\mathbb{R}^n)} \leq c\,\|f\|_{X^{\sigma,\varphi}(\Omega)}$ for all $f \in X^\infty(\Omega)$, where $\mathcal{O}f$ is defined by (8.15).

The domain of (L, B) is defined by the formula

$$D^{\sigma+2q,\varphi}_{L,X}(\Omega) := \{u \in H^{\sigma+2q,\varphi}(\Omega) : \, Lu \in X^{\sigma,\varphi}(\Omega)\}$$

and endowed with the graph inner product

$$(u_1, u_2)_{D^{\sigma+2q}_{L,X}(\Omega)} := (u_1, u_2)_{H^{\sigma+2q}(\Omega)} + (Lu_1, Lu_2)_{X^\sigma(\Omega)}.$$

The space $D^{\sigma+2q,\varphi}_{L,X}(\Omega)$ is Hilbert.

Our key theorem on classes of Hörmander spaces is the following.

Theorem 12.16. *Let $\varphi \in \mathcal{M}$, and let a number $\sigma < 0$ be such that*

$$\sigma + 2q \neq 1/2 - k \quad \text{for every integer} \quad k \geq 1. \tag{12.14}$$

Suppose that $X^{\sigma,\varphi}(\Omega)$ is an arbitrary Hilbert space imbedded continuously in $\mathcal{D}'(\Omega)$ and satisfying Condition 12.15 ($\mathrm{I}_{\sigma,\varphi}$). Then:

i) *The set $D^\infty_{L,X}(\Omega) := \{u \in C^\infty(\overline{\Omega}) : Lu \in X^{\sigma,\varphi}(\Omega)\}$ is dense in $D^{\sigma+2q,\varphi}_{L,X}(\Omega)$.*

ii) *The mapping $u \to (Lu, Bu)$, with $u \in D^\infty_{L,X}(\Omega)$, extends uniquely to a continuous linear operator*

$$(L, B) : \, D^{\sigma+2q,\varphi}_{L,X}(\Omega) \to X^{\sigma,\varphi}(\Omega) \oplus \bigoplus_{j=1}^{q} H^{\sigma+2q-m_j-1/2,\varphi}(\partial\Omega) =: \mathbf{X}_{\sigma,\varphi}(\Omega, \partial\Omega), \tag{12.15}$$

iii) *The operator (12.15) is Fredholm. Its kernel is $\mathcal{N}$, and its range consists of all the vectors $(f, g_1, \ldots, g_q) \in \mathbf{X}_{\sigma,\varphi}(\Omega, \partial\Omega)$ that satisfy (9.8).*

iv) *If $\mathcal{O}(X^\infty(\Omega))$ is dense in $H^{\sigma,\varphi}_{\overline{\Omega}}(\mathbb{R}^n)$, then the index of (12.15) is $\dim \mathcal{N} - \dim \mathcal{N}^+$.*

Note that the condition (12.14) is stipulated by that, in definition of $D^{\sigma+2q,\varphi}_{L,X}(\Omega)$, we use the space $H^{\sigma+2q,\varphi}(\Omega)$, rather than an appropriate analog of $\mathrm{H}^{\sigma+2q}(\Omega)$, which is different from $H^{\sigma+2q,\varphi}(\Omega)$ if $\sigma + 2q$ is negative and half-integer.

The following two individual theorems result from the key theorem. The first of them is for nonweighted Hörmander spaces $X^{\sigma,\varphi}(\Omega) := H^{\lambda,\eta}(\Omega)$. In view of Theorem 9.6, we can confine ourselves to the $\sigma < -1/2$ case.

Theorem 12.17. *Let $\sigma < -1/2$, the condition (12.14) be fulfilled, $\lambda > -1/2$, and $\varphi, \eta \in \mathcal{M}$. Then the mapping $u \mapsto (Lu, Bu)$, with $u \in C^\infty(\overline{\Omega})$, extends uniquely to a continuous linear operator*

$$\begin{gathered}(L, B) : \, \{u \in H^{\sigma+2q,\varphi}(\Omega) : Lu \in H^{\lambda,\eta}(\Omega)\} \to \\ H^{\lambda,\eta}(\Omega) \oplus \bigoplus_{j=1}^{q} H^{\sigma+2q-m_j-1/2,\varphi}(\partial\Omega)\end{gathered} \tag{12.16}$$

provided that its domain is endowed with the norm

$$\bigl(\,\|u\|^2_{H^{\sigma+2q,\varphi}(\Omega)}+\|Lu\|^2_{H^{\lambda,\eta}(\Omega)}\bigr)^{1/2}.$$

The domain is a Hilbert space with respect to this norm. Moreover, the operator (12.16) *is Fredholm, and its index is* $\dim\mathcal{N}-\dim\mathcal{N}^{+}$*.*

It is remarkable that, in this individual theorem, the solution and right-hand side of the elliptic equation $Lu=f$ can be of different supplementary smoothness, φ and η.

The second individual theorem is for weighted Hörmander spaces $X^{\sigma,\varphi}(\Omega):=\varrho H^{\sigma,\varphi}(\Omega)$, namely

$$\varrho H^{\sigma,\varphi}(\Omega):=\{\,f=\varrho v:\,v\in H^{\sigma,\varphi}(\Omega)\,\},$$
$$(f_1,f_2)_{\varrho H^{\sigma,\varphi}(\Omega)}:=(\varrho^{-1}f_1,\varrho^{-1}f_2)_{H^{\sigma,\varphi}(\Omega)}.$$

Here $\sigma<-1/2$, $\varphi\in\mathcal{M}$, and the function $\varrho\in C^{\infty}(\Omega)$ is positive. The space $\varrho H^{\sigma,\varphi}(\Omega)$ is Hilbert.

Theorem 12.18. *Let* $\sigma<-1/2$*, the condition* (12.14) *be valid, and* $\varphi\in\mathcal{M}$*. Suppose that a positive function* $\varrho\in C^{\infty}(\Omega)$ *is a multiplier in* $H^{-\sigma,1/\varphi}(\Omega)$ *and satisfies* (12.12)*. Then the mapping* $u\rightarrow(Lu,Bu)$*, with* $u\in C^{\infty}(\,\overline{\Omega}\,)$*,* $Lu\in\varrho H^{\sigma,\varphi}(\Omega)$*, extends uniquely to a continuous linear operator*

$$\begin{aligned}(L,B):\bigl\{u\in H^{\sigma+2q,\varphi}(\Omega):Lu\in\varrho H^{\sigma,\varphi}(\Omega)\bigr\}\rightarrow\\ \varrho H^{\sigma,\varphi}(\Omega)\oplus\bigoplus_{j=1}^{q}H^{\sigma+2q-m_j-1/2,\varphi}(\partial\Omega)\end{aligned}\tag{12.17}$$

provided that its domain is endowed with the norm

$$\bigl(\,\|u\|^2_{H^{\sigma+2q,\varphi}(\Omega)}+\|\varrho^{-1}Lu\|^2_{H^{\sigma,\varphi}(\Omega)}\bigr)^{1/2}.$$

The domain is a Hilbert space with respect to this norm. Moreover, the operator (12.18) *is Fredholm, and its index is* $\dim\mathcal{N}-\dim\mathcal{N}^{+}$*.*

We get a wide enough class of weight functions ϱ satisfying the condition of this theorem if we set $\varrho:=\varrho_1^{\delta}$, where ϱ_1 is subject to (12.4) and $\delta>-\sigma-1/2$.

13. Other results

In this section, we outline applications of Hörmander spaces to other classes of elliptic problems, namely to nonregular elliptic boundary-value problems, parameter-elliptic problems, mixed elliptic problems, and elliptic systems. We recall the statements of these problems and formulate theorems about properties of the correspondent operators. As for Sobolev spaces, the Fredholm property and its implications will be preserved for some classes of Hörmander spaces. The theorems stated below are deduced from the Sobolev case with the help of the interpolation with an appropriate function parameter. We will not sketch the proofs and only will refer to the authors' relevant papers.

13.1. **Nonregular elliptic boundary-value problems.** Here we suppose that the boundary-value problem (9.1), (9.2) is elliptic in Ω but can be nonregular. This means that it satisfies conditions i) and ii) of Definition 9.2 but need not meet condition iii). Theorems 9.6–9.10 remain valid for this boundary-value problem except for the description of the operator range and the index formula given in Theorem 9.6. The exception is caused by that the boundary-value problem need not have a formally adjoint boundary-value problem in the class of differential equations. A version of Theorem 9.6 in this situation is the following.

Theorem 13.1. *Let $s > m + 1/2$ and $\varphi \in \mathcal{M}$. Then the bounded linear operator (9.4) is Fredholm. Its kernel coincides with $\mathcal{N}$, whereas its range consists of all the vectors $(f, g_1, \ldots, g_q) \in \mathcal{H}_{s,\varphi}(\Omega, \partial\Omega)$ such that the equality in (9.8) is fulfilled for each $v \in W$. Here W is a certain finite-dimensional space that lies in $C^\infty(\overline{\Omega}) \times (C^\infty(\partial\Omega))^q$ and does not depend on s and φ. The index of (9.4) is $\dim \mathcal{N} - \dim W$ and is also independent of s, φ.*

The proof is given in [77, Sec. 4]. Recall, if the boundary-value problem (9.1), (9.2) is regular elliptic, then $W = \mathcal{N}^+$

Example 13.2. The oblique derivative problem for the Laplace equation:

$$\Delta u = f \ \text{ in } \ \Omega, \qquad \frac{\partial u}{\partial \eta} = g \ \text{ on } \ \partial\Omega. \tag{13.1}$$

Here η is an infinitely smooth field of unit vectors $\eta(x)$, $x \in \partial\Omega$. Suppose that $\dim \Omega = 2$, then the boundary-value problem (13.1) is elliptic in Ω, but it is nonregular provided $\partial\Omega_\eta \neq \varnothing$. Here $\partial\Omega_\eta$ denotes the set of all $x \in \partial\Omega$ such that $\eta(x)$ is tangent to $\partial\Omega$. If $\overline{\Omega}$ is a disk, then the correspondent operator index equals $2 - \delta(\eta)/\pi$, where $\delta(\eta)$ is the increment of the angle between $i := (1, 0)$ and $\eta(x)$ when x goes counterclockwise around $\partial\Omega$; see, e.g., [90, Ch. 19, § 4]. Note if $\dim \Omega \geq 3$ and $\partial\Omega_\eta \neq \varnothing$, then the boundary-value problem (13.1) is not elliptic at all.

Other examples of nonregular elliptic boundary-value problems are given in [116, Sec. 4].

At the end of this subsection, we recall the following important result concerning an arbitrary boundary-value problem (9.1), (9.1) (see, e.g., [8, Sec. 2.4]). If the corresponding operator (9.4) is Fredholm for certain $s \geq 2q$ with $\varphi \equiv 1$, then this problem is elliptic in Ω, i.e., the above-mentioned conditions i) and ii) are satisfied.

13.2. **Parameter-elliptic problems.** Such problems were distinguished by S. Agmon and L. Nirenberg [1, 4], M.S. Agranovich and M.I. Vishik [9] as a class of elliptic boundary-value problems that depend on a complex-valued parameter, say λ, and possess the following remarkable property. Providing $|\lambda| \gg 1$, the operator correspondent to the problem establishes a homeomorphism on appropriate pairs of Sobolev spaces, and moreover the operator norm admits a two-sided a priory estimate with constants independent of λ. Parameter-elliptic problems were applied to the spectral theory of elliptic operators and to parabolic equations. Some wider classes of parameter-elliptic operators and boundary-value problems were investigated by M.S. Agranovich [5, 6], R. Denk, R. Mennicken and L.R. Volevich [19, 20], G. Grubb [35, Ch. 2], A.N. Kozhevnikov [53, 54, 55] (see also the surveys [7, 8]).

In this subsection, we give an application of Hörmander spaces to parameter-elliptic boundary-value problems considered by Agmon, Nirenberg, and Agranovich, Vishik. Namely, we state a homeomorphism theorem on a class of Hörmander spaces and give a correspondent two-sided a priory estimate for the operator norm.

Recall the definition of the parameter-elliptic boundary-value problem. We consider the nonhomogeneous boundary-value problem

$$L(\lambda)\,u=f\quad\text{in}\quad\Omega,\qquad B_j(\lambda)\,u=g_j\quad\text{on}\quad\partial\Omega,\quad j=1,\dots,q,\tag{13.2}$$

that depends on the parameter $\lambda\in\mathbb{C}$ as follows:

$$L(\lambda):=\sum_{r=0}^{2q}\lambda^{2q-r}L_r,\qquad B_j(\lambda):=\sum_{r=0}^{m_j}\lambda^{m_j-r}B_{j,r}.\tag{13.3}$$

Here $L_r=L_r(x,D)$, $x\in\overline{\Omega}$, and $B_{j,r}=B_{j,r}(x,D)$, $x\in\partial\Omega$, are linear partial differential expressions of order $\leq r$ and with complex-valued infinitely smooth coefficients. As above, the integers q and m_j satisfy the equalities $q\geq 1$ and $0\leq m_j\leq 2q-1$. Note that $L(0)=L_{2q}$ and $B_j(0)=B_{j,m_j}$.

We associate certain homogeneous polynomials in $(\xi,\lambda)\in\mathbb{C}^{n+1}$ with partial differential expressions (13.3). Namely, we set

$$L^{(0)}(x;\xi,\lambda):=\sum_{r=0}^{2q}\lambda^{2q-r}L_r^{(0)}(x,\xi),\quad\text{with}\quad x\in\overline{\Omega},\ \xi\in\mathbb{C}^n,\ \lambda\in\mathbb{C}.$$

Here $L_r^{(0)}(x,\xi)$ is the principal symbol of $L_r(x,D)$ provided $\operatorname{ord}L_r=r$, or $L_r^{(0)}(x,\xi)\equiv 0$ if $\operatorname{ord}L_r<r$. Similarly, for $j=1,\dots,q$, we put

$$B_j^{(0)}(x;\xi,\lambda):=\sum_{r=0}^{m_j}\lambda^{m_j-r}B_{j,r}^{(0)}(x,\xi),\quad\text{with}\quad x\in\partial\Omega,\ \xi\in\mathbb{C}^n,\ \lambda\in\mathbb{C}.$$

Here $B_{j,r}^{(0)}(x,\xi)$ is the principal symbol of $B_{j,r}(x,D)$ provided $\operatorname{ord}B_{j,r}=r$, or $B_{j,r}^{(0)}(x,\xi)\equiv 0$ if $\operatorname{ord}B_{j,r}<r$. Note that $L^{(0)}(x;\xi,\lambda)$ and $B_j^{(0)}(x;\xi,\lambda)$ are homogeneous polynomials in (ξ,λ) of the orders $2q$ and m_j respectively.

Let K be a fixed closed angle on the complex plain with vertex at the origin; here we admits the case where K degenerates into a ray.

Definition 13.3. The boundary-value problem (13.2) is called parameter-elliptic in the angle K if the following conditions are satisfied:

i) $L^{(0)}(x;\xi,\lambda)\neq 0$ for each $x\in\overline{\Omega}$, $\xi\in\mathbb{R}^n$, and $\lambda\in K$ whenever $|\xi|+|\lambda|\neq 0$.

ii) Let $x\in\partial\Omega$, $\xi\in\mathbb{R}^n$, and $\lambda\in K$ be such that ξ is tangent to $\partial\Omega$ at x and that $|\xi|+|\lambda|\neq 0$. Then the polynomials $B_j^{(0)}(x;\xi+\tau\nu(x),\lambda)$ in τ, $j=1,\dots,q$, are linearly independent modulo $\prod_{j=1}^{q}(\tau-\tau_j^{+}(x;\xi,\lambda))$. Here $\tau_1^{+}(x;\xi,\lambda),\dots,\tau_q^{+}(x;\xi,\lambda)$ are all the τ-roots of $L^{(0)}(x;\xi+\tau\nu(x),\lambda)$

with $\operatorname{Im}\tau > 0$, each root being taken the number of times equal to its multiplicity.

Remark 13.4. Condition ii) of Definition 13.3 is well stated in the sense that, for the polynomial $L^{(0)}(x;\xi+\tau\nu(x),\lambda)$, the numbers of the τ-roots with $\operatorname{Im}\tau > 0$ and of those with $\operatorname{Im}\tau < 0$ are the same and equal to q if we take into account the roots multiplicity. Indeed, it follows from condition i) that the partial differential expression

$$L(x;D,D_t) := \sum_{r=0}^{2q} D_t^{2q-r} L_r(x,D), \quad x \in \overline{\Omega},$$

is elliptic. Since the expression includes the derivation with respect to $n+1 \geq 3$ real arguments $x_1,\ldots,x_n,t$, its ellipticity is equivalent to the proper ellipticity condition (see Remark 9.3). So, the τ-roots of $L^{(0)}(x;\xi+\tau\nu(x),\lambda)$ satisfy the indicated property.

Let us give some instances of parameter-elliptic boundary-value problems [8, Sec. 3.1 b)].

Example 13.5. Let differential expression $L(\lambda)$ satisfy condition i) of Definition 13.3. Then the Dirichlet boundary-value problem for the equation $L(\lambda) = f$ is parameter-elliptic in the angle K. Here the boundary conditions do not depend on the parameter λ.

Example 13.6. The boundary-value problem

$$\Delta u + \lambda^2 u = f \ \text{ in } \ \Omega, \qquad \frac{\partial u}{\partial \nu} - \lambda u = g \ \text{ on } \ \partial\Omega$$

is parameter-elliptic in each angle $K_\varepsilon := \{\lambda \in \mathbb{C} : \varepsilon \leq |\operatorname{Im}\lambda| \leq \pi - \varepsilon\}$, with $0 < \varepsilon < \pi/2$, if the complex plane is slitted along the negative semiaxis.

Further in this subsection the boundary-value problem (13.2) is supposed to be parameter-elliptic in the angle K.

It follows from Definition 13.3 in view of Remark 13.4 that the boundary-value problem (13.2) is elliptic in Ω (and need not be regular) provided $\lambda = 0$. Since λ is contained only in the lover order terms of differential expressions $L(\lambda)$ and $B_j(\lambda)$, the problem is elliptic in Ω for every $\lambda \in \mathbb{C}$. So, by Theorem 9.6, we have the Fredholm bounded operator

$$(L(\lambda), B(\lambda)) : \ H^{s,\varphi}(\Omega) \to \mathcal{H}_{s,\varphi}(\Omega,\partial\Omega) \tag{13.4}$$

for each $s > m + 1/2$, $\varphi \in \mathcal{M}$, and $\lambda \in \mathbb{C}$. The operator index does not depend on s, φ, and on λ because λ influences only the lover order terms; see, e.g., [46, Sec. 20.1, Theorem 20.1.8]. Moreover, since the boundary-value problem (13.2) is parameter-elliptic in K, the operator (13.4) possesses the following additional properties.

Theorem 13.7. i) *There exists a number $\lambda_0 > 0$ such that for each $\lambda \in K$ with $|\lambda| \geq \lambda_0$ and for any $s > m + 1/2$, $\varphi \in \mathcal{M}$, the operator (13.2) is a homeomorphism of $H^{s,\varphi}(\Omega)$ onto $\mathcal{H}_{s,\varphi}(\Omega,\partial\Omega)$.*

ii) *Suppose that $s > 2q$ and $\varphi \in \mathcal{M}$, then there is a number $c = c(s, \varphi) \geq 1$ such that, for each $\lambda \in K$, with $|\lambda| \geq \max\{\lambda_0, 1\}$, and for every $u \in H^{s,\varphi}(\Omega)$, we have the following two-sided estimate*

$$\begin{aligned} & c^{-1}\big(\,\|u\|_{H^{s,\varphi}(\Omega)} + |\lambda|^{s}\varphi(|\lambda|)\,\|u\|_{L_2(\Omega)}\,\big) \\ \leq\; & \|L(\lambda)u\|_{H^{s-2q,\varphi}(\Omega)} + |\lambda|^{s-2q}\varphi(|\lambda|)\,\|L(\lambda)u\|_{L_2(\Omega)} \\ & + \sum_{j=1}^{q}\big(\,\|B_j(\lambda)u\|_{H^{s-m_j-1/2,\varphi}(\partial\Omega)} \\ & + |\lambda|^{s-m_j-1/2}\varphi(|\lambda|)\,\|B_j(\lambda)u\|_{L_2(\partial\Omega)}\,\big) \\ \leq\; & c\,\big(\,\|u\|_{H^{s,\varphi}(\Omega)} + |\lambda|^{s}\varphi(|\lambda|)\,\|u\|_{L_2(\Omega)}\,\big). \end{aligned} \tag{13.5}$$

Here c does not depend on u and λ.

We should comment on assertion ii) of this theorem. For fixed λ, the estimate (13.5) is written for the norms, non-Hilbert, that are equivalent to $\|u\|_{H^{s,\varphi}(\Omega)}$ and $\|(L(\lambda), B(\lambda))u\|_{\mathcal{H}_{s,\varphi}(\Omega,\partial\Omega)}$ respectively. The non-Hilbert norms are used to avoid cumbersome expressions. To have the finite norm $\|L(\lambda)u\|_{L_2(\Omega)}$ in (13.5), we suppose that $s > 2q$ is fulfilled instead of the condition $s > m + 1/2$ used in assertion i). Finally, the supplement condition $|\lambda| \geq 1$ is caused by that the function $\varphi(t)$ is defined for $t \geq 1$. Note the estimate (13.5) is of interest for $|\lambda| \gg 1$ only.

In the Sobolev case where $s \geq 2q$ and $\varphi \equiv 1$, Theorem 13.7 was proved by M.S. Agranovich and M.I. Vishik [9, § 4 and 5]; see also [8, Sec. 3.2]. In general, the theorem is proved in [78, Sec. 7]. Note that the right-hand side of the estimate (13.5) is valid without the assumption about the parameter-ellipticity of (13.2). Analogs of Theorem 13.7 for parameter-elliptic operators, scalar or matrix, are proved in [96, 97].

We note an important consequence of Theorem 13.7 i). Suppose that the boundary-value problem (13.2) is parameter-elliptic on a certain ray $K := \{\lambda \in \mathbb{C} : \arg\lambda = \mathrm{const}\}$. Then the operator (13.4) is of zero index for each $s > m+1/2$, $\varphi \in \mathcal{M}$, and $\lambda \in \mathbb{C}$.

13.3. **Mixed elliptic problems.** Here we consider a certain class of elliptic boundary-value problems in multiply connected bonded domains. As distinguished from the above, we allow the orders of the boundary differential expressions to be distinct on different connected components of the boundary. For instance, studying the Laplace equation in a ring, one may set the Dirichlet condition on a chosen connected component of the ring boundary and the Neumann condition on the other component. The problems under consideration relate to the mixed elliptic boundary-value problems [107, 120, 128, 137]. They have not investigated so completely as the unmixed elliptic problems. This is concerned with some difficulties, that appear when one reduces the mixed problem to a pseudodifferential operator on the boundary; see, e.g., [128]. In the problems we consider, the portions of boundary on which the boundary expression has distinct orders do not adjoin to each other. These problems are called formally mixed. They can be reduced locally to a model elliptic problem in the half-space [94].

In this subsection, we suppose that the boundary of Ω consists of $r \geq 2$ nonempty connected components $\Gamma_1, \ldots, \Gamma_r$. Fix an integer $q \geq 1$ and consider a formally mixed boundary-value problem

$$L\,u = f \quad \text{in } \Omega, \quad B_j^{(k)} u = g_{k,j} \text{ on } \Gamma_k, \;\; j = 1, \ldots, q, \;\; k = 1, \ldots, r. \tag{13.6}$$

Here the partial differential expression $L = L(x, D)$, $x \in \overline{\Omega}$, of order $2q$, is the same as in Section 9, whereas $B^{(k)} := \{B_j^{(k)} : j = 1, \ldots, q\}$ is a system of boundary linear partial differential expressions given on the component Γ_k. Suppose that the coefficients of the expressions $B_j^{(k)} = B_j^{(k)}(x, D)$, $x \in \Gamma_k$, are infinitely smooth complex-valued functions and that all $m_j^{(k)} := \operatorname{ord} B_j^{(k)} \leq 2q - 1$. We denote

$$\Lambda := (L, B_1^{(1)}, \ldots, B_q^{(1)}, \ldots, B_1^{(r)}, \ldots, B_q^{(r)}),$$
$$\mathcal{N}_\Lambda := \{u \in C^\infty(\,\overline{\Omega}\,) : \, \Lambda u = 0\},$$
$$m := \max\,\{\operatorname{ord} B_j^{(k)} : \, j = 1, \ldots, q, \;\; k = 1, \ldots, r\}.$$

The mapping $u \mapsto \Lambda u$, $u \in C^\infty(\,\overline{\Omega}\,)$, extends uniquely to a bounded linear operator

$$\begin{aligned} \Lambda : \, H^{s,\varphi}(\Omega) \to H^{s-2q,\,\varphi}(\Omega) \oplus \bigoplus_{k=1}^{r} \bigoplus_{j=1}^{q} H^{s-m_j^{(k)}-1/2,\varphi}(\Gamma_k) \\ =: \mathcal{H}_{s,\varphi}(\Omega, \Gamma_1, \ldots, \Gamma_r) \end{aligned} \tag{13.7}$$

for each $s > m + 1/2$ and $\varphi \in \mathcal{M}$.

Definition 13.8. The formally mixed boundary-value problem (13.6) is called elliptic in the multiply connected domain Ω if L is proper elliptic on $\overline{\Omega}$ and if, for each $k = 1, \ldots, r$, the system $B^{(k)}$ satisfies the Lopatinsky condition with respect to L on Γ_k.

Suppose the mixed boundary-value problem (13.6) is elliptic in Ω. Then it has the following properties [94].

Theorem 13.9. *Let $s > m + 1/2$ and $\varphi \in \mathcal{M}$. Then the bounded linear operator (13.7) is Fredholm. Its kernel coincides with $\mathcal{N}_\Lambda$, whereas its range consists of all the vectors*

$$(f, g_{1,1}, \ldots, g_{1,q}, \ldots, g_{r,1}, \ldots, g_{r,q}) \in \mathcal{H}_{s,\varphi}(\Omega, \Gamma_1, \ldots, \Gamma_r)$$

such that

$$(f, w_0)_\Omega + \sum_{k=1}^{r} \sum_{j=1}^{q} (g_{k,j}, w_{k,j})_{\Gamma_k} = 0 \tag{13.8}$$

for each vector-valued function

$$(w_0, w_{1,1}, \ldots, w_{1,q}, \ldots, w_{r,1}, \ldots, w_{r,q}) \in W_\Lambda.$$

Here W_Λ is a certain finite-dimensional space that lies in

$$C^\infty(\,\overline{\Omega}\,) \times \prod_{j=1}^{r} (C^\infty(\Gamma_j))^q$$

and does not depend on s and φ. The index of (13.7) *is* $\dim\mathcal{N} - \dim W_\Lambda$ *and is also independent of* s, φ.

It is self-clear that, in (13.8), the notation $(\cdot,\cdot)_{\Gamma_k}$ stands for the inner product in $L_2(\Gamma_k)$.

13.4. **Elliptic systems.** Extensive classes of elliptic systems of linear partial differential equations were introduced and investigated by I.G. Petrovskii [109] and A. Douglis, L. Nirenberg [22]. For pseudodifferential equations, general elliptic systems were studied by L. Hörmander [44, Sec. 1.0]. He proved a priori estimates for solutions of these systems in appropriate couples of Sobolev inner product spaces of arbitrary real orders. If the system is given on a closed smooth manifold, then the estimate is equivalent to the Fredholm property of the correspondent elliptic matrix PsDO; see, e.g., the monograph [46, Ch. 19], and the survey [7, Sec. 3.2]. This fact is of great importance in the theory of elliptic boundary-value problems because each of these problems can be reduced to an elliptic system of pseudodifferential equations on the boundary of the domain; see, e.g., [46, Ch. 20] and [139, Part IV].

In this subsection, we examine the Petrovskii elliptic systems on the refined Sobolev scale over a closed smooth manifold Γ and generalize the results of Subsection 6.1 to these systems.

Let us consider a system of $p \geq 2$ linear equations

$$\sum_{k=1}^{p} A_{j,k}\, u_k = f_j \quad \text{on} \quad \Gamma, \quad j = 1, \ldots, p. \tag{13.9}$$

Here $A_{j,k}$, $j,k = 1,\ldots,p$, are scalar classical pseudodifferential operators of arbitrary real orders defined on Γ. We consider equations (13.9) in the sense of the distribution theory so that $u_k,\ f_j \in \mathcal{D}'(\Gamma)$. Put $m_k := \max\{\operatorname{ord} A_{1,k}, \ldots, \operatorname{ord} A_{p,k}\}$.

Let us rewrite the system (13.9) in the matrix form: $Au = f$ on Γ, where $A := (A_{j,k})$ is a square matrix of order p, and $u = \operatorname{col}(u_1,\ldots,u_p)$, $f = \operatorname{col}(f_1,\ldots,f_p)$ are functional columns. The mapping $u \mapsto Au$ is a linear continuous operator on the space $(\mathcal{D}'(\Gamma))^p$. By lemma 6.1, a restriction of the mapping sets a bounded linear operator

$$A : \bigoplus_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma) \to (H^{s,\varphi}(\Gamma))^p \tag{13.10}$$

for each $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$.

Definition 13.10. The system (13.9) and the matrix PsDO A are called Petrovskii elliptic on Γ if $\det\bigl(a^{(0)}_{j,k}(x,\xi)\bigr)_{j,k=1}^{p} \neq 0$ for each point $x \in \Gamma$ and covector $\xi \in T^*_x\Gamma \setminus \{0\}$. Here $a^{(0)}_{j,k}(x,\xi)$ is the principal symbol of $A_{j,k}$ provided $\operatorname{ord} A_{j,k} = m_k$; otherwise $a^{(0)}_{j,k}(x,\xi) \equiv 0$.

We suppose that the system $Au = f$ is elliptic on Γ. Then both the spaces

$$N := \bigl\{\, u \in (C^\infty(\Gamma))^p :\ Au = 0 \ \text{ on } \ \Gamma \,\bigr\},$$
$$N^+ := \bigl\{\, v \in (C^\infty(\Gamma))^p :\ A^+v = 0 \ \text{ on } \ \Gamma \,\bigr\}$$

are finite-dimensional [7, Sec. 3.2]. Here A^{+} is the matrix pseudodifferential operator formally adjoint to A with respect to the inner product in $(L_2(\Gamma))^p$.

Theorem 13.11. *The operator (13.10) corresponding to the elliptic system is Fredholm for each $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. Its kernel coincides with N, whereas its range consists of all the vectors $f \in (H^{s,\varphi}(\Gamma))^p$ such that $\sum_{j=1}^{p} (f_j, v_j)_\Gamma = 0$ for each $(v_1, \ldots, v_p) \in N^{+}$. The index of (13.10) is equal to $\dim N - \dim N^{+}$ and independent of s and φ.*

This theorem is proved in [83] together with other properties of the system (13.9). They are similar to that given in Subsection 6.1, in which the scalar case is treated. We also refer to the second author's papers [97, 98, 99] devoted to various classes of elliptic systems in Hörmander spaces.

13.5. **Boundary-value problems for elliptic systems.** Boundary-value problems for various classes of elliptic systems of linear partial differential equations were investigated by S. Agmon, A. Douglis, and L. Nirenberg, M.S. Agranovich and A.S. Dynin, L. Hörmander, L.N. Slobodetskii, V.A. Solonnikov, L.R. Volevich; see the until now unique monograph [139] devoted especially to these problems, the survey [8, § 6] and the references given therein. It was proved that the operator correspondent to the problem is Fredholm on appropriate pairs of the positive order Sobolev spaces. Regarding the boundary-value problems for Petrovskii elliptic systems, we extend this result over the one-sided refined Sobolev scale.

Let us consider a system of $p \geq 2$ partial differential equations

$$\sum_{k=1}^{p} L_{j,k}\, u_k = f_j \quad \text{in} \quad \Omega, \quad j = 1, \ldots, p. \tag{13.11}$$

Here $L_{j,k} = L_{j,k}(x, D)$, $x \in \overline{\Omega}$, $j, k = 1, \ldots, p$, are scalar linear partial differential expressions given on $\overline{\Omega}$. The expression $L_{j,k}$ is of an arbitrary finite order, the coefficients of $L_{j,k}$ are supposed to be complex-valued and infinitely smooth on $\overline{\Omega}$. Put $m_k := \max\{\operatorname{ord} L_{1,k}, \ldots, \operatorname{ord} L_{p,k}\}$ so that m_k is the maximal order of derivative of the unknown function u_k. Suppose that all $m_k \geq 1$ and that $\sum_{k=1}^{p} m_k$ is even, say $2q$.

We consider the solutions of (13.11) that satisfy the boundary conditions

$$\sum_{k=1}^{p} B_{j,k}\, u_k = g_j \quad \text{on} \quad \partial\Omega, \quad j = 1, \ldots, q. \tag{13.12}$$

Here $B_{j,k} = B_{j,k}(x, D)$, with $x \in \partial\Omega$, $j = 1, \ldots, q$, and $k = 1, \ldots, p$, are boundary linear partial differential expressions with infinitely smooth coefficients. We suppose $\operatorname{ord} B_{j,k} \leq m_k - 1$ and set $r_j := \min\{m_k - \operatorname{ord} B_{j,k} :\ k = 1, \ldots, p\}$ admitting $\operatorname{ord} B_{j,k} := -\infty$ for $B_{j,k} \equiv 0$; thus $\operatorname{ord} B_{j,k} \leq m_k - r_j$.

Let us write the boundary-value problem (13.11), (13.12) in the matrix form

$$Lu = f \ \text{ in } \ \Omega, \quad Bu = g \ \text{ on } \ \partial\Omega.$$

Here $L := (L_{j,k})_{j,k=1}^{p}$ and $B := (B_{j,k})_{\substack{j=1,\ldots,q\\ k=1,\ldots,p}}$ are matrix differential expressions, whereas $u := \mathrm{col}\,(u_1,\ldots,u_p)$, $f := \mathrm{col}\,(f_1,\ldots,f_p)$, and $g := \mathrm{col}\,(g_1,\ldots,g_q)$ are function columns.

It follows from Lemma 9.1 that the mapping $u \mapsto (Lu, Bu)$, $u \in (C^{\infty}(\,\overline{\Omega}\,))^p$, extends uniquely to a continuous linear operator

$$(L,B) : \bigoplus_{k=1}^{p} H^{s+m_k,\varphi}(\Omega) \to (H^{s,\varphi}(\Omega))^p \oplus \bigoplus_{j=1}^{q} H^{s+r_j-1/2,\varphi}(\partial\Omega) \qquad (13.13)$$
$$=: \mathbf{H}_{s,\varphi}(\Omega,\partial\Omega)$$

for each $s > -r + 1/2$ and $\varphi \in \mathcal{M}$, with $r := \min\{r_1,\ldots,r_q\} \geq 1$. We are interested in properties of this operator provided the boundary-value problem is elliptic in the Petrovskii sense. Recall the ellipticity definition.

With L and B we associate the matrixes of homogeneous polynomials

$$L^{(0)}(x,\xi) := \big(L^{(0)}_{j,k}(x,\xi)\big)_{j,k=1}^{p}, \quad B^{(0)}(x,\xi) := \big(B^{(0)}_{j,k}(x,\xi)\big)_{\substack{j=1,\ldots,q\\ k=1,\ldots,p}}.$$

Here $L^{(0)}_{j,k}(x,\xi)$, $x \in \overline{\Omega}$, $\xi \in \mathbb{C}^n$, is the principal symbol of $L_{j,k}(x,D)$ provided $\mathrm{ord}\,L_{j,k} = m_k$; otherwise $L^{(0)}_{j,k}(x,\xi) \equiv 0$. Similarly, $B^{(0)}_{j,k}(x,\xi)$, $x \in \partial\Omega$, $\xi \in \mathbb{C}^n$, is the principal symbol of $B_{j,k}(x,D)$ provided $\mathrm{ord}\,B_{j,k} = m_k - r_j$; otherwise $B^{(0)}_{j,k}(x,\xi) \equiv 0$.

Definition 13.12. The boundary-value problem (13.11), (13.12) is called Petrovskii elliptic in Ω if the following conditions are satisfied:

i) The system (13.11) is proper elliptic on $\overline{\Omega}$; i.e., condition i) of Definition 9.2 is fulfilled, with the notation $\det L^{(0)}(x,\xi'+\tau\xi'')$ being placed instead of $L^{(0)}(x,\xi'+\tau\xi'')$.

ii) The relations (13.12) satisfies the Lopatinsky condition with respect to (13.11) on $\partial\Omega$; i.e., for an arbitrary point $x \in \partial\Omega$ and for each vector $\xi \neq 0$ tangent to $\partial\Omega$ at x, the rows of the matrix $B^{(0)}(x,\xi+\tau\nu(x)) \times L^{(0)}_{\mathrm{c}}(x,\xi+\tau\nu(x))$ are linearly independent polynomials, in $\tau \in \mathbb{R}$, modulo $\prod_{j=1}^{q}\big(\tau - \tau_j^{+}(x;\xi,\nu(x))\big)$. Here $L^{(0)}_{\mathrm{c}}(x,\xi)$ is the transpose of the matrix composed by the cofactors of the matrix $L^{(0)}(x,\xi)$ elements.

Note, if condition i) is satisfied, then the system (13.11) is Petrovskii elliptic on $\overline{\Omega}$, i.e. $\det L^{(0)}(x,\xi) \neq 0$ for each $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. The converse is true provided that $\dim \Omega \geq 3$; see [8, Sec. 6.1 a)].

Example 13.13. The elliptic boundary-value problem for the Cauchy-Riemann system:

$$\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = f_1, \quad \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = f_2 \quad \text{in} \quad \Omega,$$
$$u_1 + u_2 = g \quad \text{on} \quad \partial\Omega.$$

Here $n = p = 2$ and $m_1 = m_2 = 1$, so that $q = 1$. The Cauchy-Riemann system is an instance of homogeneous elliptic systems, which satisfy Definition 13.12 with $m_1 = \ldots = m_p$.

Example 13.14. The Petrovskii elliptic boundary-value problem

$$\frac{\partial u_1}{\partial x_1} - \frac{\partial^3 u_2}{\partial x_2^3} = f_1, \quad \frac{\partial u_1}{\partial x_2} + \frac{\partial^3 u_2}{\partial x_1^3} = f_2 \quad \text{in} \quad \Omega,$$

$$u_1 = g_1, \quad u_2 \left(\text{or } \frac{\partial u_2}{\partial \nu}, \text{ or } \frac{\partial^2 u_2}{\partial \nu^2} \right) = g_2 \quad \text{on} \quad \partial\Omega.$$

Here $n = p = 2$, $m_1 = 1$, and $m_2 = 3$ so that $q = 2$. This system is not homogeneous elliptic.

Other examples of elliptic systems, of various kinds, are given in [8, § 6.2].

Suppose the boundary-value problem (13.11), (13.12) is Petrovskii elliptic in Ω. Then it has the following properties [95].

Theorem 13.15. *Let $s > -r + 1/2$ and $\varphi \in \mathcal{M}$. Then the bounded linear operator (13.13) is Fredholm. The kernel $\mathcal{N}$ of (13.13) lies in $(C^\infty(\overline{\Omega}))^p$ and does not depend on s and φ. The range of (13.13) consists of all the vectors $(f_1, \ldots, f_p; g_1, \ldots, g_q) \in \mathbf{H}_{s,\varphi}(\Omega, \partial\Omega)$ such that*

$$\sum_{j=1}^{p} (f_j, w_j)_\Omega + \sum_{j=1}^{q} (g_j, h_j)_{\partial\Omega} = 0$$

for each vector-valued function $(w_1, \ldots, w_p;\, h_1, \ldots, h_q) \in W$. Here W is a certain finite-dimensional space that lies in $(C^\infty(\overline{\Omega}))^p \times (C^\infty(\Gamma))^q$. The index of the operator (13.13) is $\dim \mathcal{N} - \dim W$ and independent of s, φ.

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