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SUBORDINATION PROPERTIES OF MULTIVALENT FUNCTIONS DEFINED BY CERTAIN INTEGRAL OPERATOR

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ABSTRACT. The object of this paper is to investigate some inclusion relationships and a number of other useful properties among certain subclasses of analytic and p-valent functions, which are defined here by certain integral operator.

1. Introduction and preliminaries

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p, n \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and p–valent in the unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in U, we say that f *is subordinate to* g, written symbolically as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a *Schwarz function* w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, $z \in \mathrm{U}$. In particular, if the function g is univalent in U, then we have the following equivalency (cf., e.g. [4]; see also [6, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

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For the functions $f \in A(p)$ given by (1.1) and $g \in A(p)$ defined by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

the *Hadamard (or convolution) product* of f and g is given by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z).$$

Motivated essentially by Jung et al. [2], Shams et al. [10] introduced the operator $\mathrm{I}_p^\alpha : A(p) \to A(p)$ as follows:

$$\mathrm{I}_p^\alpha f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+1}{k+p+1}\right)^\alpha a_{k+p} z^{k+p}, \quad \alpha \in \mathbb{R}.$$

Using the above definition relation, it is easy verify that the operator becomes an integral operator

$$\mathrm{I}_p^\alpha f(z) = \frac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\, dt, \quad \text{for} \quad \alpha > 0,$$
$$\mathrm{I}_p^0 f(z) = f(z), \quad \text{for} \quad \alpha = 0,$$

and, moreover

$$z\left(\mathrm{I}_p^\alpha f(z)\right)' = (p+1)\, \mathrm{I}_p^{\alpha-1} f(z) - \mathrm{I}_p^\alpha f(z), \quad \text{for} \quad \alpha \in \mathbb{R}. \tag{1.2}$$

We mention that the one-parameter family of integral operator $\mathrm{I}^\alpha \equiv \mathrm{I}_1^\alpha$ was defined by Jung et al. [2].

Definition 1.1. For fixed parameters A and B $(-1 \le B < A \le 1)$ and $\beta \in [0, p)$, we say that a function $f \in A(p)$ is in the class $S_p^\alpha(\beta; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\beta}\left(\frac{z(\mathrm{I}_p^\alpha f(z))'}{\mathrm{I}_p^\alpha f(z)} - \beta\right) \prec \frac{1+Az}{1+Bz}.$$

In particular, for $A = 1$ and $B = -1$ we write $S_p^\alpha(\beta; 1, -1) = S_p^\alpha(\beta)$, where

$$S_p^\alpha(\beta) = \left\{ f \in A(p) : \operatorname{Re} \frac{z\left(\mathrm{I}_p^\alpha f(z)\right)'}{\mathrm{I}_p^\alpha f(z)} > \beta,\ z \in \mathrm{U} \right\}.$$

2. Preliminaries

We begin by recalling each of the following lemmas which will be required in our present investigation.

Lemma 2.1. [5],[6] *Let a function h be analytic and convex (univalent) in* U, *with $h(0) = 1$. Suppose also that the function φ given by*

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \dots. \tag{2.1}$$

is analytic in U. *If*

$$\varphi(z)+\frac{z\varphi'(z)}{c}\prec h(z)\quad (c\neq 0,\ \operatorname{Re}c\geq 0), \tag{2.2}$$

then

$$\varphi(z)\prec\Psi(z)=\frac{c}{z^c}\int_0^z t^{c-1}h(t)\,dt\prec h(z),$$

and Ψ *is the best dominant of* (2.2).

Lemma 2.2. [6] *Suppose that the function* $\Psi:\mathbb{C}^2\times\mathrm{U}\to\mathbb{C}$ *satisfies the following condition*

$$\operatorname{Re}\Psi(ix,y;z)\leq\varepsilon,$$

for all $x\in\mathbb{R}$ *and* $y\leq-\frac{1}{2}(1+x^2)$, *and for all* $z\in\mathrm{U}$. *If the function* φ *of the form* (2.1) *is analytic in* U *and*

$$\operatorname{Re}\Psi(\varphi(z),z\varphi'(z);z)>\varepsilon,\ z\in\mathrm{U},$$

then

$$\operatorname{Re}\varphi(z)>0,\ z\in\mathrm{U}.$$

Note that a more general form of this Lemma is given by the first part of Theorem 2.3i [6, p. 35].

Lemma 2.3. [3] *Let* $\lambda\neq 0$ *be a real number,* $\frac{a}{\lambda}>0$ *and* $0\leq\beta<1$. *Suppose also that the function* $\Psi(z)=1+c_nz^n+c_{n+1}z^{n+1}+\ldots$ *is analytic in* U *and that*

$$\Psi(z)\prec 1+\frac{aM}{n\lambda+a}z,\quad (n\in\mathbb{N}),$$

where

$$M=\frac{(1-\beta)\,|\lambda|\left(1+\frac{n\lambda}{a}\right)}{|1-\lambda+\lambda\beta|+\sqrt{1+\left(1+\frac{n\lambda}{a}\right)^2}}.$$

If the function $\theta(z)=1+e_nz^n+e_{n+1}z^{n+1}+\ldots$ *is analytic in* U *and satisfies the following subordination condition:*

$$\Psi(z)\Big\{1-\lambda+\lambda\left[(1-\beta)\theta(z)+\beta\right]\Big\}\prec 1+Mz,$$

then

$$\operatorname{Re}\theta(z)>0,\ z\in\mathrm{U}.$$

With a view to stating a well-known result (Lemma 2.4 below), we denote by $P(\gamma)$ the class of function φ given by (2.1) which are analytic in U and satisfy the inequality

$$\operatorname{Re}\varphi(z)>\gamma,\ z\in\mathrm{U}\quad(\gamma<1).$$

Lemma 2.4. [8] *Let the function φ given by (2.1) be in the class $P(\gamma)$. Then*

$$\operatorname{Re}\varphi(z) \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|},\ z \in \mathrm{U} \quad (\gamma < 1).$$

Lemma 2.5. [11] *For $0 \leq \gamma_1 < \gamma_2 < 1$,*

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3), \quad \textit{where} \quad \gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2).$$

The result is the best possible.

For any complex numbers $\alpha_1, \alpha_2, \beta_1$ ($\beta_1 \notin \mathbb{Z}_0^-$), the *Gauss hypergeometric function* is defined by

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = 1 + \frac{\alpha_1\alpha_2}{\beta_1}\frac{z}{1!} + \frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)}{\beta_1(\beta_1+1)}\frac{z^2}{2!} + \dots.$$

The above series converges absolutely for all $z \in \mathrm{U}$, and hence represents an analytic function in the unit disc U (see, for details, [12, Chapter 14]).

Each of the identities asserted by Lemma below is well-known (cf., e.g. [12, Chapter 14]).

Lemma 2.6. [12] *For any complex parameters $\alpha_1, \alpha_2, \beta_1$ ($\beta_1 \notin \mathbb{Z}_0^-$), the next equalities hold:*

$$\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-zt)^{-\alpha_1}dt$$

$$= \frac{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1)}\, {}_2F_1(\alpha_1, \alpha_2; \beta_1; z), \quad \operatorname{Re}\beta_1 > \operatorname{Re}\alpha_2 > 0; \tag{2.3}$$

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = {}_2F_1(\alpha_2, \alpha_1; \beta_1; z); \tag{2.4}$$

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = (1-z)^{-\alpha_1}\, {}_2F_1\left(\alpha_1, \beta_1-\alpha_2; \beta_1; \frac{z}{z-1}\right). \tag{2.5}$$

3. Properties involving the operator I_p^α

Unless otherwise mentioned, we assume throughout this paper that $-1 \leq B < A \leq 1$, $\lambda > 0$ and $p \in \mathbb{N}$. By using the above lemmas, first we will prove the next result:

Theorem 3.1. *Let $\lambda > 0$, $\alpha \in \mathbb{R}$ and $-1 \leq B_j < A_j \leq 1$, $j = 1, 2$. If the functions $f_j \in A(p)$ satisfy the following subordination condition:*

$$(1-\lambda)\frac{\mathrm{I}_p^\alpha f_j(z)}{z^p} + \lambda\frac{\mathrm{I}_p^{\alpha-1} f_j(z)}{z^p} \prec \frac{1+A_j z}{1+B_j z},\ j = 1, 2, \tag{3.1}$$

then

$$(1-\lambda)\frac{\mathrm{I}_p^\alpha F(z)}{z^p} + \lambda\frac{\mathrm{I}_p^{\alpha-1} F(z)}{z^p} \prec \frac{1+(1-2\eta_0)z}{1-z},$$

where

$$F = \mathrm{I}_p^\alpha(f_1 * f_2)$$

and

$$\eta_0 = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[1 - \frac{1}{2}\,{}_2F_1\left(1, 1; \frac{p+1}{\lambda} + 1; \frac{1}{2}\right)\right]. \tag{3.2}$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. Suppose that the functions $f_j \in A(p)$, $j = 1, 2$, satisfy the condition (3.1). Setting

$$\varphi_j(z) = (1 - \lambda)\frac{\mathrm{I}_p^\alpha\, f_j(z)}{z^p} + \lambda\frac{\mathrm{I}_p^{\alpha-1}\, f_j(z)}{z^p},\ j = 1, 2, \tag{3.3}$$

we see that

$$\varphi_j \in P(\gamma_j), \quad \text{where} \quad \gamma_j = \frac{1 - A_j}{1 - B_j},\ j = 1, 2.$$

Thus, by making use of the identity (1.2) and (3.3), we obtain

$$\mathrm{I}_p^\alpha\, f_j(z) = \frac{p+1}{\lambda} z^{p - \frac{p+1}{\lambda}} \int\limits_0^z t^{\frac{p+1}{\lambda} - 1} \varphi_j(t)\, dt,\ j = 1, 2. \tag{3.4}$$

Now, if we let

$$F(z) = \mathrm{I}_p^\alpha (f_1 * f_2)(z),$$

then, by using (3.4) and the fact that

$$\mathrm{I}_p^\alpha\, F(z) = \mathrm{I}_p^\alpha \left(\mathrm{I}_p^\alpha\, (f_1 * f_2)\,(z)\right) = \mathrm{I}_p^\alpha (f_1)(z) * \mathrm{I}_p^\alpha (f_2)(z),$$

a simple computation shows that

$$\mathrm{I}_p^\alpha\, F(z) = \frac{p+1}{\lambda} z^{p - \frac{p+1}{\lambda}} \int\limits_0^z t^{\frac{p+1}{\lambda} - 1} \varphi_0(t)\, dt,$$

where

$$\begin{aligned} \varphi_0(z) &= (1 - \lambda)\frac{\mathrm{I}_p^\alpha\, F(z)}{z^p} + \lambda\frac{\mathrm{I}_p^{\alpha-1}\, F(z)}{z^p} \\ &= \frac{p+1}{\lambda} z^{-\frac{p+1}{\lambda}} \int\limits_0^z t^{\frac{p+1}{\lambda} - 1} (\varphi_1 * \varphi_2)(t)\, dt. \end{aligned} \tag{3.5}$$

Since $\varphi_j \in P(\gamma_j)$, $j = 1, 2$, it follows from Lemma 2.5 that

$$\varphi_1 * \varphi_2 \in P(\gamma_3), \quad \text{with} \quad \gamma_3 = 1 - 2(1 - \gamma_1)n(1 - \gamma_2),$$

and the bound γ_3 is the best possible. Hence, by using Lemma 2.4 in (3.5), we deduce that

$$\begin{aligned}\operatorname{Re}\varphi_0(z) &= \frac{p+1}{\lambda}\int_0^1 u^{\frac{p+1}{\lambda}-1}\operatorname{Re}(p_1 * p_2)(uz)\,du \\ &\geq \frac{p+1}{\lambda}\int_0^1 u^{\frac{p+1}{\lambda}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u\,|z|}\right)du \\ &> \frac{p+1}{\lambda}\int_0^1 u^{\frac{p+1}{\lambda}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right)du\end{aligned}$$

$$= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1 - \frac{p+1}{\lambda}\int_0^1 \frac{u^{\frac{p+1}{\lambda}-1}}{1+u}\,du\right) = \eta_0,\ z \in \mathrm{U},$$

where η_0 is given by (3.2).

When $B_1 = B_2 = -1$, we consider the functions $f_j \in A(p)$ which satisfy the hypothesis (3.1) and are given by

$$\mathrm{I}_p^{\alpha} f_j(z) = \frac{p+1}{\lambda} z^{p-\frac{p+1}{\lambda}} \int_0^z t^{\frac{p+1}{\lambda}-1}\left(\frac{1+A_j t}{1-t}\right)dt,\ j = 1, 2.$$

Then it follows from (3.5) and Lemma 2.4 that

$$\varphi_0(z) = \frac{p+1}{\lambda}\int_0^1 u^{\frac{p+1}{\lambda}-1}\left(1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right)du$$

$$= 1 - (1+A_1)(1+A_2) + (1+A_1)(1+A_2)(1-z)^{-1}\,{}_2F_1\left(1, 1; \frac{p+1}{\lambda}+1; \frac{z}{z-1}\right)$$

$$\to 1 - (1+A_1)(1+A_2) + \frac{1}{2}(1+A_1)(1+A_2)\,{}_2F_1\left(1, 1; \frac{p+1}{\lambda}+1; \frac{1}{2}\right),$$

as $z \to -1$, which completes the proof of Theorem 3.1. □

Putting $A_j = 1 - 2\eta_j$, $B_j = -1$ $(0 \leq \eta_j < 1,\ j = 1, 2)$ and $\alpha = 1$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let $\lambda > 0$ and let the functions $f_j \in A(p)$, $j = 1, 2$, satisfy the following inequality:*

$$\operatorname{Re}\left[(1-\lambda)\frac{p+1}{z^{p+1}}\int_0^z f_j(t)\,dt + \lambda\frac{f_j(z)}{z^p}\right] > \eta_j,\ z \in \mathrm{U}\ (0 \leq \eta_j < 1).$$

Then

$$\operatorname{Re}\left[(1-\lambda)\frac{p+1}{z^{p+1}}\int_0^z (f_1 * f_2)(t)\,dt + \lambda\frac{(f_1 * f_2)(z)}{z^p}\right] > \eta_0^*,\ z \in \mathrm{U},$$

where

$$\eta_0^* = 1 - 4(1-\eta_1)(1-\eta_2)\left[1 - \frac{1}{2}\,{}_2F_1\left(1, 1; \frac{p+1}{\lambda} + 1; \frac{1}{2}\right)\right],$$

and the result is the best possible.

Putting $A_j = 1 - 2\eta_j$, $B_j = -1$ $(0 \le \eta_j < 1,\ j = 1, 2)$, and $\alpha = \lambda = 1$ in Theorem 3.1, we obtain the following result.

Corollary 3.3. *If the functions $f_j \in A(p)$, $j = 1, 2$, satisfy the following inequality:*

$$\operatorname{Re}\frac{f_j(z)}{z^p} > \eta_j,\ z \in \mathrm{U}\ (0 \le \eta_j < 1),$$

then

$$\begin{aligned} &\operatorname{Re}\left[\frac{p+1}{z^{p+1}}\int_0^z (f_1 * f_2)(t)\,dt\right] \\ &> 1 - 4(1-\eta_1)(1-\eta_2)\left[1 - \frac{1}{2}\,{}_2F_1\left(1, 1; p+2; \frac{1}{2}\right)\right],\ z \in \mathrm{U}, \end{aligned}$$

and the result is the best possible.

Remark 3.4. (i) We note that this result was also obtained by Patel et al. [9, Corollary 5].

(ii) It is easy to see that $\widetilde{F}(p) := {}_2F_1\left(1, 1; p+2; \frac{1}{2}\right)$ is a decreasing function in p, and $1 < \widetilde{F}(p) \le \widetilde{F}(1) = 4(1 - \log 2) = 1.2274\ldots$. Moreover, $\widetilde{F}(p)$ can be computed explicitly for each $p \in \mathbb{N}$, that is

$$\begin{aligned} \widetilde{F}(p) &= 2(p+1)\int_0^1 \frac{u^p}{1+u}\,du = 2(p+1)\int_1^2 \frac{(x-1)^p}{x}\,dx \\ &= 2(p+1)\left[\sum_{k=1}^{p}\binom{p}{k}\frac{(-1)^{p-k}}{k}\left(2^k - 1\right) + (-1)^p \log 2\right]. \end{aligned}$$

Theorem 3.5. *Let $\lambda > 0$, $\alpha \in \mathbb{R}$, $-1 \le B \le 1$ and $B < A$. If $f \in A(p)$ satisfy the following subordination condition:*

$$(1-\lambda)\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p} + \lambda\frac{\mathrm{I}_p^{\alpha-1} f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \tag{3.6}$$

then

$$\operatorname{Re}\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p} > \zeta,\ z \in \mathrm{U}, \tag{3.7}$$

where

$$\zeta = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1, 1; \dfrac{p+1}{\lambda} + 1; \dfrac{B}{B-1}\right), & \text{if } B \ne 0 \\ 1 - \dfrac{p+1}{p+1+\lambda}A, & \text{if } B = 0. \end{cases}$$

The result is the best possible.

Proof. If we let

$$\varphi(z) = \frac{\mathrm{I}_p^{\alpha} f(z)}{z^p}, \tag{3.8}$$

then φ is of the form (2.1) and is analytic in U. Differentiating (3.8) with respect to z and using the identity (1.2), we obtain

$$\frac{\mathrm{I}_p^{\alpha-1} f(z)}{z^p} = \varphi(z) + \frac{z\varphi'(z)}{p+1}. \tag{3.9}$$

From (3.6), (3.8) and (3.9), we get

$$\varphi(z) + \frac{\lambda}{p+1} z\varphi'(z) \prec \frac{1+Az}{1+Bz}.$$

Now, by applying Lemma 2.1, we get

$$\varphi(z) \prec Q(z) = \frac{p+1}{\lambda} z^{-\frac{p+1}{\lambda}} \int_0^z t^{\frac{p+1}{\lambda}-1} \frac{1+At}{1+Bt}\, dt$$

$$= \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1, 1; \dfrac{p+1}{\lambda}+1; \dfrac{Bz}{Bz+1}\right), & \text{if} \quad B \neq 0 \\ 1 + \dfrac{p+1}{p+1+\lambda} Az, & \text{if} \quad B = 0, \end{cases}$$

where we have also made a change of variables followed by the use of the identities (2.3), (2.4) and (2.5). Next we will show that

$$\inf\left\{ \operatorname{Re} Q(z) : |z| < 1 \right\} = Q(-1). \tag{3.10}$$

We have

$$\operatorname{Re} \frac{1+Az}{1+Bz} \geq \frac{1-Ar}{1-Br}, \ |z| = r < 1,$$

and setting

$$g(s,z) = \frac{1+Azs}{1+Bzs} \quad (0 \leq s \leq 1) \quad \text{and} \quad dv(s) = \frac{(p+1)s^{\frac{p+1}{\lambda}}}{\lambda}\, ds,$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z) = \int_0^1 g(s,z)\, dv(s),$$

so that

$$\operatorname{Re} Q(z) \geq \int_0^1 \frac{1-Asr}{1-Bsr}\, dv(s) = Q(-r), \ |z| = r < 1.$$

Upon letting $r \to 1^-$ in the above inequality, we obtain the assertion (3.10). Now, the estimation (3.7) follows directly from (3.10).

In order to show that the estimate (3.7) is the best possible, we consider the function $f \in A(p)$ defined by

$$\frac{\mathrm{I}_p^\alpha f(z)}{z^p} = \frac{p+1}{\lambda} \int\limits_0^1 u^{\frac{p+1}{\lambda}-1} \frac{1+Auz}{1+Buz}\, du,$$

from which it is easily seen that

$$(1-\lambda)\frac{\mathrm{I}_p^\alpha f(z)}{z^p} + \lambda \frac{\mathrm{I}_p^{\alpha-1} f(z)}{z^p} = \frac{1+Az}{1+Bz}$$

and that

$$\frac{\mathrm{I}_p^\alpha f(z)}{z^p} \to \frac{p+1}{\lambda} \int\limits_0^1 u^{\frac{p+1}{\lambda}-1} \frac{1-Au}{1-Bu}\, du$$

$$= \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\, {}_2F_1\left(1, 1; \dfrac{p+1}{\lambda}+1; \dfrac{B}{B-1}\right), & \text{if} \quad B \neq 0 \\ 1 - \dfrac{p+1}{p+1+\lambda} A, & \text{if} \quad B = 0, \end{cases}$$

as $z \to -1$, and the proof of Theorem 3.5 is thus completed. □

Remark 3.6. With the aid of the elementary inequality

$$\operatorname{Re} w^\gamma \geq (\operatorname{Re} w)^\gamma, \quad (\operatorname{Re} w > 0,\ 0 < \gamma \leq 1),$$

we could similarly prove that the assumptions of the above theorem implies

$$\operatorname{Re} \left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\gamma > \zeta^\gamma,\ z \in \mathrm{U} \quad (0 < \gamma \leq 1),$$

whenever $\zeta \geq 0$.

Putting $A = 1 - 2\eta$ ($\eta < 1$), $B = -1$ and $\alpha = 1$ in Theorem 3.5, we obtain the following result.

Corollary 3.7. *Let $\lambda > 0$ and let a function $f \in A(p)$ satisfy the following inequality:*

$$\operatorname{Re}\left[(1-\lambda)\frac{p+1}{z^{p+1}} \int_0^z f(t)\, dt + \lambda \frac{f(z)}{z^p}\right] > \eta,\ z \in \mathrm{U} \quad (\eta < 1).$$

Then

$$\operatorname{Re}\left[\frac{p+1}{z^{p+1}} \int_0^z f(t)\, dt\right] > \eta + (1-\eta)\left[{}_2F_1\left(1, 1; \frac{p+1}{\lambda}+1; \frac{1}{2}\right) - 1\right],\ z \in \mathrm{U},$$

and the result is the best possible.

Putting $\lambda = \gamma = 1$ in Theorem 3.5, for the special case $A = 1 - 2\gamma$ ($\gamma < 1$), and $B = -1$ we obtain the following result.

Corollary 3.8. *If $f \in A(p)$ satisfies the following condition:*

$$\operatorname{Re}\frac{\mathrm{I}_p^{\alpha-1} f(z)}{z^p} > \gamma,\ z \in \mathrm{U} \quad (\gamma < 1,\ \alpha \in \mathbb{R}),$$

then

$$\operatorname{Re}\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p} > \gamma + (1-\gamma)\left[{}_2F_1\left(1, 1; p+2; \frac{1}{2}\right) - 1\right],\ z \in \mathrm{U}.$$

The result is the best possible.

For a function $f \in A(p)$, the integral operator $F_{\mu,p} : A(p) \to A(p)$ is defined by (cf., e.g. [1])

$$\begin{aligned} F_{\mu,p}(f)(z) &= \frac{\mu+p}{z^\mu}\int_0^z t^{\mu-1} f(t)\,dt = \left(z^p + \sum_{k=p+1}^{\infty}\frac{\mu+p}{\mu+k} z^{p+k}\right) * f(z) \\ &= z^p\,{}_2F_1(1, \mu+p; \mu+p+1; z) * f(z),\ z \in \mathrm{U} \quad (\mu > -p). \end{aligned} \tag{3.11}$$

Theorem 3.9. *Let $\lambda > 0$, $\alpha \in \mathbb{R}$, $-1 \le B \le 1$, $B < A$ and $\mu > -p$. Suppose that $f \in A(p)$ and $F_{\mu,p}(f)$ is given by (3.11). If*

$$(1-\lambda)\frac{\mathrm{I}_p^{\alpha} F_{\mu,p}(f)(z)}{z^p} + \lambda\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \tag{3.12}$$

then

$$\operatorname{Re}\frac{\mathrm{I}_p^{\alpha} F_{\mu,p}(f)(z)}{z^p} > \rho_0,\ z \in \mathrm{U},$$

where

$$\rho_0 = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1, 1; \dfrac{\mu+p}{\lambda} + 1; \dfrac{B}{B-1}\right), & \textit{if} \quad B \neq 0 \\ 1 - \dfrac{\mu+p}{\mu+p+\lambda}A, & \textit{if} \quad B = 0. \end{cases}$$

The result is the best possible.

Proof. It follows from the definition (3.11) that

$$z\left(\mathrm{I}_p^{\alpha} F_{\mu,p}(f)(z)\right)' = (\mu+p)\,\mathrm{I}_p^{\alpha} f(z) - \mu\,\mathrm{I}_p^{\alpha} F_{\mu,p}(f)(z). \tag{3.13}$$

If we let

$$G(z) = \frac{\mathrm{I}_p^{\alpha} F_{\mu,p}(f)(z)}{z^p}, \tag{3.14}$$

then the hypothesis (3.12) in conjunction with (3.13) and (3.14) would yield

$$(1-\lambda)\frac{\mathrm{I}_p^{\alpha} F_{\mu,p}(f)(z)}{z^p} + \lambda\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p} = G(z) + \frac{\lambda}{\mu+p} zG'(z) \prec \frac{1+Az}{1+Bz}.$$

The remaining part of the proof of Theorem 3.9 is similar to that of Theorem 3.5 and we omit the details. □

Theorem 3.10. *Let $\lambda > 0$, $\alpha \in \mathbb{R}$, $0 \le \beta < p$ and $p \in \mathbb{N}$, and let a function $f \in A(p)$ satisfy the following subordination condition:*

$$(1-\lambda)\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p} + \lambda \frac{\mathrm{I}_p^{\alpha-1} f(z)}{z^p} \prec 1 + M_1 z, \tag{3.15}$$

where

$$M_1 = \frac{\eta\xi}{|\eta-1| + \sqrt{1+\xi^2}}, \tag{3.16}$$

with $\eta = \dfrac{\lambda(p-\beta)}{p+1}$ *and* $\xi = \dfrac{p+1+\lambda}{p+1} = 1 + \dfrac{\eta}{p-\beta}$. *Then* $f \in S_p^{\alpha}(\beta)$.

Proof. Putting

$$\varphi(z) = \frac{\mathrm{I}_p^{\alpha} f(z)}{z^p}, \tag{3.17}$$

then φ is of the form (2.1) and is analytic in U. From Theorem 3.5 with $A = M_1$, $B = 0$ and $m = 1$, we have

$$\varphi(z) \prec 1 + \frac{p+1}{p+1+\lambda} M_1 z,$$

which is equivalent to

$$|\varphi(z) - 1| < \frac{M_1}{\xi} = N < 1, \ z \in \mathrm{U}. \tag{3.18}$$

If we set

$$P(z) = \frac{1}{p-\beta}\left(\frac{z\left(\mathrm{I}_p^{\alpha} f(z)\right)'}{\mathrm{I}_p^{\alpha} f(z)} - \beta\right) \quad (0 \le \beta < p), \tag{3.19}$$

then, by using the identity (1.2) followed by (3.17), we obtain

$$\frac{\mathrm{I}_p^{\alpha-1} f(z)}{z^p} = \left[1 - \frac{p-\beta}{p+1} + \frac{p-\beta}{p+1} P(z)\right] \varphi(z). \tag{3.20}$$

In view of (3.20), the assumption (3.15) can be written as follows:

$$|(1-\eta)\,\varphi(z) + \eta P(z)\varphi(z) - 1| < M_1 = \xi N, \ z \in \mathrm{U}. \tag{3.21}$$

We need to show that (3.21) yields

$$\operatorname{Re} P(z) > 0, \ z \in \mathrm{U}. \tag{3.22}$$

If we suppose that $\operatorname{Re} P(z) \not> 0$, $z \in \mathrm{U}$, then there exists a point $z_0 \in \mathrm{U}$ such that $P(z_0) = ix$ for some $x \in \mathbb{R}$. To prove (3.22), it is sufficient to obtain a contradiction from the following inequality:

$$W = |(1-\eta)\,\varphi(z_0) + \eta P(z_0)\varphi(z_0) - 1| \ge M_1.$$

Letting $\varphi(z_0) = u + iv$, then, by using (3.18) and the triangle inequality, we obtain that

$$\begin{aligned} W^2 &= |(1-\eta)\,\varphi(z_0) + \eta P(z_0)\varphi(z_0) - 1|^2 \\ &= (u^2+v^2)\eta^2 x^2 + 2\eta v x + |(1-\eta)\,\varphi(z_0) - 1|^2 \\ &\ge (u^2+v^2)\eta^2 x^2 + 2\eta v x + (\eta - |1-\eta|\, N)^2, \end{aligned}$$

and thus

$$W^2 - M_1^2 \geq \left(u^2 + v^2\right) \eta^2 x^2 + 2\eta v x + (\eta - |1 - \eta|\, N)^2 - \xi^2 N^2.$$

Setting

$$\Psi(x) = \left(u^2 + v^2\right) \eta^2 x^2 + 2\eta v x + (\eta - |1 - \eta|\, N)^2 - \xi^2 N^2,$$

we note that (3.21) holds true if $\Psi(x) \geq 0$ for any $x \in \mathbb{R}$. Since

$$\left(u^2 + v^2\right) \eta^2 > 0,$$

the inequality $\Psi(x) \geq 0$ holds true if the discriminant $\Delta \leq 0$, that is

$$\Delta = 4\left\{\eta^2 v^2 - \eta^2 \left(u^2 + v^2\right) \left[\left(\eta - |1 - \eta|\, N\right)^2 - \xi^2 N^2\right]\right\} \leq 0,$$

which is equivalent to

$$v^2 \left[1 - (\eta - |1 - \eta|\, N)^2 + \xi^2 N^2\right] \leq u^2 \left[(\eta - |1 - \eta|\, N)^2 - \xi^2 N^2\right].$$

Putting $\varphi(z_0) - 1 = \rho e^{i\theta}$ for some real $\theta \in \mathbb{R}$, we get

$$\frac{v^2}{u^2} = \frac{\rho^2 \sin^2 \theta}{(1 + \rho \cos \theta)^2}.$$

Since the above expression attains its maximum value at $\cos\theta = -\rho$, by using (3.18), we obtain

$$\frac{v^2}{u^2} \leq \frac{\rho^2}{1 - \rho^2} \leq \frac{N^2}{1 - N^2} = \frac{(\eta - |1 - \eta|\, N)^2 - \xi^2 N^2}{1 - (\eta - |1 - \eta|\, N)^2 + \xi^2 N^2},$$

which yields $\Delta \leq 0$. Therefore, $W \geq M_1$, which contradicts (3.12), hence $\operatorname{Re} P(z) > 0$, $z \in \mathrm{U}$. This proves that $f \in S_p^{\alpha}(\beta)$, which completes the proof of Theorem 3.10. □

Taking $\alpha = 1$ in Theorem 3.10, we obtain:

Corollary 3.11. *Let $\lambda > 0$ and suppose that $f \in A(p)$ satisfies the following differential subordination*

$$(1 - \lambda)\frac{p + 1}{z^{p+1}} \int_0^z f(t)\, dt + \lambda \frac{f(z)}{z^p} \prec 1 + M_1 z,$$

where M_1 is given by (3.16). Then $f \in S_p^{*}(\beta)$, $0 \leq \beta < p$.

Theorem 3.12. *Let $\lambda > 0$, $0 \leq \beta < p$, $p \in \mathbb{N}$ and $\mu \geq 0$. If $f \in A(p)$ such that*

$$\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p} \neq 0,\ z \in \mathrm{U},$$

and satisfies the following differential subordination:

$$(1 - \lambda) \left(\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p}\right)^{\mu} + \lambda \frac{\left(\mathrm{I}_p^{\alpha} f(z)\right)'}{p z^{p-1}} \left(\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p}\right)^{\mu - 1} \prec 1 + M_2 z, \tag{3.23}$$

where the powers are understood as the principal value, and

$$M_2=\begin{cases}\dfrac{(p-\beta)\lambda\left(1+\dfrac{\lambda}{\mu p}\right)}{|p-(p-\beta)\lambda|+\sqrt{p^2+\left(p+\dfrac{\lambda}{\mu}\right)^2}}, & \text{if} \quad \mu>0\\ \dfrac{p-\beta}{p}\lambda, & \text{if} \quad \mu=0,\end{cases}$$

then $f\in S_p^{\alpha}(\beta)$.

Proof. If $\mu=0$, then the condition (3.23) is equivalent to

$$\left|\frac{z\left(\mathrm{I}_p^{\alpha}f(z)\right)'}{\mathrm{I}_p^{\alpha}f(z)}-p\right|<p-\beta,\ z\in\mathrm{U},$$

which, in turn, implies that $f\in S_p^{\alpha}(\beta)$.

If we consider $\mu>0$, let denotes

$$\varphi(z)=\left(\frac{\mathrm{I}_p^{\alpha}f(z)}{z^p}\right)^{\mu}. \tag{3.24}$$

Choosing the principal value in (3.24), we note that φ is of the form (2.1) and is analytic in U. Differentiating (3.24) with respect to z, we obtain

$$(1-\lambda)\left(\frac{\mathrm{I}_p^{\alpha}f(z)}{z^p}\right)^{\mu}+\lambda\frac{\left(\mathrm{I}_p^{\alpha}f(z)\right)'}{pz^{p-1}}\left(\frac{\mathrm{I}_p^{\alpha}f(z)}{z^p}\right)^{\mu-1}=\varphi(z)+\frac{\lambda}{\mu p}z\varphi'(z),$$

which, in view of Lemma 2.1 (with $c=\dfrac{\mu p}{\lambda}$), yields

$$\varphi(z)\prec 1+\frac{\mu p}{\mu p+\lambda}M_2 z.$$

Also, with the aid of (3.24), the subordination (3.23) can be written as follows:

$$\varphi(z)\left\{1-\lambda+\lambda\left[\left(1-\frac{\beta}{p}\right)P(z)+\frac{\beta}{p}\right]\right\}\prec 1+M_2 z,$$

where P is given by (3.19). Therefore, by Lemma 2.3, we find that

$$\operatorname{Re}P(z)>0,\ z\in\mathrm{U},$$

that is

$$\operatorname{Re}\frac{z\left(\mathrm{I}_p^{\alpha}f(z)\right)'}{\mathrm{I}_p^{\alpha}f(z)}>\beta,\ z\in\mathrm{U}\quad(0\le\beta<p),$$

which completes the proof of Theorem 3.12. □

Putting $\alpha=0$ in Theorem 3.12, we obtain the following result.

Corollary 3.13. *Let* $\lambda>0$, $0\le\beta<p$, $p\in\mathbb{N}$ *and* $\mu\ge 0$. *If* $f\in A(p)$ *such that*

$$\frac{f(z)}{z^p}\neq 0,\ z\in\mathrm{U},$$

and satisfies the following differential subordination:

$$(1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\mu}+\lambda\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\mu}\prec 1+M_2z,$$

where the powers are understood as the principal value and M_2 is given as in Theorem 3.12, then $f\in S_p^*(\beta)$.

Remark 3.14. (i) We note that this result was also obtained by Patel et al. [9, Corollary 4];

(ii) Putting p=1 in Corollary 3.13, we obtain the result of Liu [3, Theorem 2.2 with $n=1$].

Putting $\alpha=0$ and $\lambda=1$ in Theorem 3.12, we obtain the following result.

Corollary 3.15. *Let $\mu\geq 0$, $0\leq\beta<p$ and $p\in\mathbb{N}$. If $f\in A(p)$ such that*

$$\frac{f(z)}{z^p}\neq 0,\ z\in \mathrm{U},$$

and satisfies the inequality:

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z^p}\right)^{\mu}-p\right|<\frac{(p-\beta)\,(p\mu+1)}{\mu\beta+\sqrt{p^2\mu^2+(p\mu+1)^2}},\ z\in \mathrm{U},$$

where the powers are understood as the principal value, then $f\in S_p^*(\beta)$.

Remark 3.16. (i) Putting $p=1$ in Corollary 3.15, we obtain the result of Liu [3, Corollary 2.1, with $n=1$];

(ii) Putting $p=\mu=1$ in Corollary 3.15, we obtain the result of Mocanu and Oros [7, Corollary 2.2 with $n=1$].

Taking $\alpha=0$ and $\lambda=\dfrac{1}{p-\beta}$, $0\leq\beta<p$ in Theorem 3.12, we obtain the following result.

Corollary 3.17. *Let $\mu\geq 0$, $0\leq\beta<p$ and $p\in\mathbb{N}$. If $f\in A(p)$ such that*

$$\frac{f(z)}{z^p}\neq 0,\ z\in \mathrm{U},$$

and satisfies the inequality:

$$\left|(p-\beta-1)\left(\frac{f(z)}{z^p}\right)^{\mu}+\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\mu}+\beta-p\right|$$
$$<\frac{(p-\beta)\,[p\mu(p-\beta)+1]}{(p-1)+\sqrt{p^2\mu^2(p-\beta)^2+[p\mu(p-\beta)+1]^2}},\ z\in \mathrm{U},$$

where the powers are understood as the principal value, then $f\in S_p^*(\beta)$.

Remark 3.18. (i) Putting $p=1$ in Corollary 3.17, we get the result of Liu [3, Corollary 2.2, with $n=1$];

(ii) Putting $p=\mu=1$ in Corollary 3.17, we obtain the result of Mocanu and Oros [7, Corollary 2.4 with $n=1$].

Theorem 3.19. *If the function $g \in A(p)$ satisfies the condition*

$$\operatorname{Re} \frac{\mathrm{I}_p^{\alpha-1} g(z)}{z^p} > \widetilde{\gamma}(p),\ z \in \mathrm{U} \quad (\alpha \in \mathbb{R}),$$

then

$$\operatorname{Re} \frac{\mathrm{I}_p^{\alpha} g(z)}{\mathrm{I}_p^{\alpha+1} g(z)} > 0,\ z \in \mathrm{U},$$

where

$$\widetilde{\gamma}(p) := \frac{1-\widetilde{F}(p)}{2p+4-\widetilde{F}(p)} \quad \text{and} \quad \widetilde{F}(p) := {}_2F_1\left(1,1;p+2;\frac{1}{2}\right). \tag{3.25}$$

Proof. According to the Remark 3.4 (ii), it is easy to check the inequalities

$$\frac{1-\widetilde{F}(p)}{2-\widetilde{F}(p)} < \widetilde{\gamma}(p) < 0, \tag{3.26}$$

while a simple computation shows that

$$\operatorname{Re}\left[\frac{\mathrm{I}_p^{\alpha} g(z)}{z^p} + \frac{z}{p+1}\left(\frac{\mathrm{I}_p^{\alpha} g(z)}{z^p}\right)'\right] = \operatorname{Re} \frac{\mathrm{I}_p^{\alpha-1} g(z)}{z^p} > \widetilde{\gamma}(p),\ z \in \mathrm{U}. \tag{3.27}$$

If we denote

$$\varphi(z) = \frac{\mathrm{I}_p^{\alpha} g(z)}{z^p},$$

then φ is analytic in U with $\varphi(0) = 1$, and the above inequality is equivalent to

$$\varphi(z) + \frac{1}{p+1} z\varphi'(z) \prec \frac{1+(1-2\widetilde{\gamma}(p))z}{1-z}.$$

Now, according to Lemma 2.1, this subordination implies

$$\varphi(z) \prec \frac{1+(1-2\widetilde{\gamma}(p))z}{1-z}, \quad \text{i.e.} \quad \operatorname{Re} \frac{\mathrm{I}_p^{\alpha} g(z)}{z^p} > \widetilde{\gamma}(p),\ z \in \mathrm{U}.$$

By applying Corollary 3.8 together with the relation (3.26), this last inequality yields

$$\operatorname{Re} \frac{\mathrm{I}_p^{\alpha+1} g(z)}{z^p} > \widetilde{\gamma}(p) + (1-\widetilde{\gamma}(p))\left[{}_2F_1\left(1,1;p+2;\frac{1}{2}\right) - 1\right] > 0,\ z \in \mathrm{U}, \tag{3.28}$$

hence, if we let

$$\phi(z) = \frac{\mathrm{I}_p^{\alpha} g(z)}{\mathrm{I}_p^{\alpha+1} g(z)}, \tag{3.29}$$

then ϕ is of the form (2.1), and from (3.28) and (3.29) we have that ϕ is analytic in U.

It is easy to show that

$$\frac{\mathrm{I}_p^{\alpha} g(z)}{z^p} + \frac{z}{p+1}\left(\frac{\mathrm{I}_p^{\alpha} g(z)}{z^p}\right)' = \frac{\mathrm{I}_p^{\alpha+1} g(z)}{z^p}\left(\phi^2(z) + \frac{z\phi'(z)}{p+1}\right) = \Psi(\phi(z), z\phi'(z); z),$$

where

$$\Psi(u,v;z) = \frac{\mathrm{I}_p^{\alpha+1} g(z)}{z^p}\left(u^2 + \frac{v}{p+1}\right),$$

then the inequality (3.27) can be written as

$$\operatorname{Re}\Psi(\phi(z), z\phi'(z); z) > \widetilde{\gamma}(p),\ z \in \mathrm{U}.$$

For all real x and $y \leq -\frac{1}{2}(1+x^2)$, we have

$$\operatorname{Re}\Psi(ix, y; z) = \left(\frac{y}{p+1} - x^2\right)\operatorname{Re}\frac{\mathrm{I}_p^{\alpha+1}g(z)}{z^p}$$

$$\leq -\frac{1}{2(p+1)}\left\{1 + [2(p+1)+1]x^2\right\}\operatorname{Re}\frac{\mathrm{I}_p^{\alpha+1}g(z)}{z^p} \leq -\frac{1}{2(p+1)}\operatorname{Re}\frac{\mathrm{I}_p^{\alpha+1}g(z)}{z^p}$$

$$\leq -\frac{1}{2(p+1)}\left[\widetilde{\gamma}(p) + (1-\widetilde{\gamma}(p))\left[{}_2F_1\left(1, 1; p+2; \frac{1}{2}\right) - 1\right]\right] = \widetilde{\gamma}(p),\ z \in \mathrm{U},$$

where we used (3.28) and the definition of $\widetilde{\gamma}(p)$ from the assumption. By Lemma 2.2 we get $\operatorname{Re}\phi(z) > 0$, $z \in \mathrm{U}$, which completes the proof of Theorem 3.19. $\square$

Theorem 3.20. *Let $f_j \in A(p)$ $(j = 1, 2)$. If the functions*

$$\frac{\mathrm{I}_p^{\alpha-1}f_j(z)}{z^p} \in P(\gamma_j), \quad (0 \leq \gamma_j < 1,\ \alpha \in \mathbb{R}),$$

then the function $g = \mathrm{I}_p^{\alpha}(f_1 * f_2)$ satisfies the following inequality:

$$\operatorname{Re}\frac{\mathrm{I}_p^{\alpha-1}g(z)}{\mathrm{I}_p^{\alpha}g(z)} > 0,\ z \in \mathrm{U},$$

provided that

$$1 - 2(1-\gamma_1)(1-\gamma_2) \geq \widetilde{\gamma}(p), \tag{3.30}$$

where $\widetilde{\gamma}(p)$ is defined by (3.25).

Proof. Denoting $g = \mathrm{I}_p^{\alpha}f_0$, where $f_0 = f_1 * f_2$, by using (1.2) it is easy to show that

$$\frac{\mathrm{I}_p^{\alpha-1}f_1(z)}{z^p} * \frac{\mathrm{I}_p^{\alpha-1}f_2(z)}{z^p} = \frac{1}{z^p}\mathrm{I}_p^{\alpha-1}\left(\mathrm{I}_p^{\alpha-1}f_0(z)\right)$$

$$= \frac{1}{z^p}\frac{1}{p+1}\left[\mathrm{I}_p^{\alpha}\left(\mathrm{I}_p^{\alpha-1}f_0(z)\right) + z\left(\mathrm{I}_p^{\alpha}\left(\mathrm{I}_p^{\alpha-1}f_0(z)\right)\right)'\right]$$

$$= \frac{1}{z^p}\frac{1}{p+1}\left(\mathrm{I}_p^{\alpha-1}g(z) + z\left(\mathrm{I}_p^{\alpha-1}g(z)\right)'\right).$$

Then, by the assumption of Theorem 3.20, it follows from Lemma 2.5 that

$$\operatorname{Re}\left(\frac{\mathrm{I}_p^{\alpha-1}f_1(z)}{z^p} * \frac{\mathrm{I}_p^{\alpha-1}f_2(z)}{z^p}\right) = \operatorname{Re}\left[\frac{\mathrm{I}_p^{\alpha-1}g(z)}{z^p} + \frac{z}{p+1}\left(\frac{\mathrm{I}_p^{\alpha-1}g(z)}{z^p}\right)'\right]$$

$$> \beta := 1 - 2(1-\gamma_1)(1-\gamma_2),\ z \in \mathrm{U},$$

which is equivalent to

$$\operatorname{Re}\frac{\mathrm{I}_p^{\alpha-2}g(z)}{z^p} > \beta,\ z \in \mathrm{U}.$$

Using the assumption (3.30) and applying Theorem 3.19, for the case when the parameter α is replaced by $\alpha - 1$, we obtain our result. $\square$

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