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TRACEABILITY OF POSITIVE INTEGRAL OPERATORS IN THE ABSENCE OF A METRIC

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Abstract. We investigate the traceability of positive integral operators on $L^2(X,\mu)$ when X is a Hausdorff locally compact second countable space and μ is a non-degenerate, σ-finite and locally finite Borel measure. This setting includes other cases proved in the literature, for instance the one in which X is a compact metric space and μ is a special finite measure. The results apply to spheres, tori and other relevant subsets of the usual space $\mathbb{R}^m$.

1. Introduction and preliminaries

Let X be a Hausdorff locally compact and second countable topological space endowed with a non-degenerate, σ-finite and locally finite Borel measure μ. In this paper, we shall investigate the traceability of integral operators $\mathcal{K} : L^2(X,\mu) \to L^2(X,\mu)$ generated by a suitable kernel $K : X \times X \to \mathbb{C}$ from $L^2(X \times X, \mu \times \mu)$. The title of the paper refers to the fact that the space X carries a topological structure rather than a metric one. The setting just described allows the space $L^2(X,\mu)$ to have a countable complete orthonormal subset ([7, p.92]) while the operator $\mathcal{K}$, which is given by the formula

$$\mathcal{K}(f) := \int_X K(\cdot, y) f(y)\, d\mu(y), \quad f \in L^2(X,\mu),$$

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becomes compact. As so, the spectral theorem for compact operators is applicable and $\mathcal{K}$ can be represented in the form

$$\mathcal{K}(f)=\sum_{n=1}^{\infty}\lambda_n\langle f,f_n\rangle f_n,\quad f\in L^2(X,\mu),$$

in which $\{\lambda_n\}$ is a sequence of real numbers (possibly finite) converging to 0 and $\{f_n\}$ is a complete orthonormal sequence in $L^2(X,\mu)$. The symbol $\langle\cdot,\cdot\rangle$ will stand for the usual inner product of $L^2(X,\mu)$.

The basic requirement on the kernel K will be its positive definiteness. A kernel K from $L^2(X\times X,\mu\times\mu)$ is $L^2(X,\mu)$-*positive definite* when the corresponding integral operator $\mathcal{K}$, is positive:

$$\langle\mathcal{K}(f),f\rangle\geq 0,\quad f\in L^2(X,\mu).$$

Fubini's theorem is all that is need in order to show that a $L^2(X,\mu)$-positive definite kernel is hermitian $\mu\times\mu$-a.e.. As so, the integral operator $\mathcal{K}$ is automatically self-adjoint with respect to $\langle\cdot,\cdot\rangle$. In particular, the sequence $\{\lambda_n\}$ mentioned in the previous paragraph needs to be entirely composed of nonnegative numbers. In the present paper, we shall assume they are listed in a decreasing order, with repetitions to account for multiplicities.

Under the conditions established above, the specific aim of this paper is to establish additional conditions on K in order that $\mathcal{K}$ be *trace-class*, that is,

$$\sum_{f\in\mathfrak{B}}\langle\mathcal{K}^*\mathcal{K}(f),f\rangle^{1/2}<\infty$$

for every orthonormal basis $\mathfrak{B}$ of $L^2(X,\mu)$. In the formula above, $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$. We refer the reader to [5, 10, 11] for more information on trace-class operators.

The main result in this paper can be seen as a generalization of another one proved in [11] for the case $X=[a,b]$. The proof there used in a key manner the so-called Steklov's smoothing operator to construct an averaging process to generate a convenient approximation to $\mathcal{K}$. The upgrade to the case in which X is a subspace of $\mathbb{R}^n$ was discussed in [8] and references therein. By assuming that the Lebesgue measure of nonempty intersections of X with open balls of $\mathbb{R}^n$ was positive and using auxiliary approximation integral operators generated by an averaging process constructed via the Hardy-Littlewood theory, the main result in [8] described necessary and sufficient conditions for the traceability of the integral operator, under the assumption of positive definiteness of the kernel. The process used in [8] and other references as well provides a way to deal with the generating kernel on the diagonal of $X\times X$ and it is convenient when the kernel is not continuous. Despite using a similar average process, another achievement in the present paper is the inclusion of a setting in which the measure does not need to be finite.

Since our spaces are no longer metric, the Hardy-Littlewood theory in the average arguments needs to be replaced or adapted. We will use techniques involving the construction of auxiliary integral operators based on martingales constructed from special partitions of X, following very closely the development of Brislawn in [2]. A similar construction have appeared in [6] in an attempt to generalize Brislawn results to L^p spaces. The main difference between the construction to be delineated here and those in [2] and [6] is that, in the present one, we need to guarantee that the elements in the partitions belong to the topology of X. This is the exact point where the assumption of local compactness will play an important role.

For the sake of completeness we mention references [1, 14] where other characterizations for traceability were obtained.

An outline of the paper is as follows. Section 2 contains the basic information on martingales used in the paper, along with the key construction we will need in order to introduce approximating auxiliary operators in Section 3. There, the main technical results are established and proved. Section 4 contains the main results of the paper, including a convenient equivalence for traceability.

2. A special martingale

This section contains several results involving a special martingale on X. Some of them are just refined versions of results described in Section 2 of [2]. However, the reader is advised that the basic references we used for the concepts and results either quoted or used here are [4, 15].

Let $(X, \mathcal{M}, \sigma)$ denote a σ-finite measure space and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{M}$ for which $(X, \mathcal{F}, \sigma)$ is a σ-finite measure space too. If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, Radon-Nikodyn's theorem asserts that we can find a unique $\mathcal{F}$-measurable function $g : X \to \mathbb{C}$ so that

$$\int_A f\, d\sigma = \int_A g\, d\sigma, \quad A \in \mathcal{F}.$$

The function g is called the *conditional expectation* of f relative to $\mathcal{F}$ and is written $g = E(f|\mathcal{F})$. If $\{\mathcal{F}_n\}$ is a family of sub-σ-algebras of $\mathcal{M}$, a sequence $\{f_n\}$ of $\mathcal{M}$-measurable functions on X is a *martingale* if every f_n is $\mathcal{F}_n$-measurable and $E(f_n|\mathcal{F}_m) = f_m$, $m < n$.

Next, we remind the reader about the basic setting we are assuming in the paper: X is a Hausdorff, locally compact and second countable topological space endowed with a non-degenerate, locally finite and σ-finite Borel measure μ. In addition to that, we will write $\mathcal{B}_X$ to denote the Borel σ-algebra of X.

Invoking the first countability axiom, we may infer that every point of X possesses an open neighborhood. Since X is Hausdorff and locally compact, these neighborhoods can be assumed to be the interior of a compact set. Thus, due to the local finiteness of (X, μ), we can assume, in addition, that the open neighborhoods of elements of X have finite measure.

We intend to construct a special sequence of partitions of X from an open covering $\{\mathcal{A}_x\}_{x\in X}$ of it, composed of neighborhoods of the type just described, and use them to define a particular martingale. If such a covering has been fixed, Lindelöff's theorem ([13, p.191]) implies that we can extract from it a countable sub-collection $\{\mathcal{A}_n\}$, still covering X. Such sub-collection can be used in the construction of a first stage partition $\mathcal{P}_0$ of X, following these steps: the first two elements in the partition are $\mathcal{A}_0$ and its frontier $\partial\mathcal{A}_0$. Observing that $\{\mathcal{A}_n \backslash \overline{\mathcal{A}_0}\}$ is an open and countable covering of $X\backslash\overline{\mathcal{A}_0}$, we pick $\mathcal{A}_1\backslash\overline{\mathcal{A}_0}$ and $\partial\mathcal{A}_1\backslash\overline{\mathcal{A}_0}$ to include in the partition. The family $\{\mathcal{A}_n\backslash\overline{\mathcal{A}_0\cup\mathcal{A}_1}\}$ is an open and countable covering of $X\backslash\overline{\mathcal{A}_0\cup\mathcal{A}_1}$. We proceed, including its elements $\mathcal{A}_2\backslash\overline{\mathcal{A}_0\cup\mathcal{A}_1}$ and $\partial\mathcal{A}_2\backslash\overline{\mathcal{A}_0\cup\mathcal{A}_1}$ in the partition. Proceeding inductively, we complete the construction of $\mathcal{P}_0$, which is countable and entirely composed of Borel sets of finite measure. Since Theorem 7.8 in [9] implies that μ is regular, all the sets of the form $\partial\mathcal{A}_n\backslash\overline{\mathcal{A}_0\cup\cdots\cup\mathcal{A}_{n-1}}$ in $\mathcal{P}_0$ have measure zero.

In the next step, we construct a sequence $\{\mathcal{P}_n\}$ of partitions of X from $\mathcal{P}_0$, using as we can, a countable basis $\{\mathcal{U}_n\}$ for the topology of X. For $n=0,1,\ldots$, we put

$$\mathcal{P}_{n+1}=\{\mathcal{U}_n\cap\mathcal{A}:\,\mathcal{A}\in\mathcal{P}_n\}\cup\{(X\setminus\overline{\mathcal{U}_n})\cap\mathcal{A}:\,\mathcal{A}\in\mathcal{P}_n\}\cup\{\partial\mathcal{U}_n\cap\mathcal{A}:\,\mathcal{A}\in\mathcal{P}_n\}.$$

Clearly, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ and the sequence $\{\mathcal{F}_n\}$ of the corresponding σ-algebras generated by those partitions increases to $\mathcal{B}_X$. In addition, every $(X,\mathcal{F}_n,\mu)$ is σ-finite.

It is easy to see that for each $x\in X$ and each positive n, there exists a unique set $O_n(x)\in\mathcal{P}_n$ such that $x\in O_n(x)$. We denote by $\mathfrak{N}$ the subset of X containing all $x\in X$ for which $\mu(O_m(x))=0$, for some $m\geq 0$. Since the sequence $\{O_n(x)\}$ is telescoping, the equality $\mu(O_m(x))=0$ implies $\mu(O_n(x))=0$, $n\geq m$. Being each $\mathcal{P}_n$ countable, it is easily seen that $\mu(\mathfrak{N})=0$.

The very same arguments used in [15, p.89] show that for every $x\in X\setminus\mathfrak{N}$ and every positive n, the conditional expectation $E_n(f)$ of f relative to $\mathcal{F}_n$ is given by the formula

$$E_n(f)(x)=\frac{1}{\mu(O_n(x))}\int_{O_n(x)}f\,d\mu.$$

The sequence $\{E_n(f)\}$ defines a martingale generated by just one (measurable) function, the *martingale associated with* f. Examples related to constructions similar to the one above can be found in [15, p.88].

The section will be completed with a list of results involving the previous formula and the *maximal function* Mf of the martingale associated with f, which is defined by the formula

$$Mf(x):=\sup\{|E_n(f)(x)|:n=1,2,\ldots\},\quad x\in X.$$

Since the results are quite general and are not attached to the particular setting introduced above, we will include sketches of the proofs for the convenience of the reader.

A classical result concerning the maximal function ([15, p.91]) implies that if $p\in(0,\infty)$ then

$$\|Mf\|_p\leq c_p\|f\|_p,\quad f\in L^p(X,\mu),$$

where c_p is a constant depending on p only and $\|\cdot\|_p$ denotes the usual norm of $L^p(X,\mu)$. As for the conditional expectation, it transforms convergence in the mean into convergence μ-a.e. Another basic result ([4, p.53] and [2, p.232]), commonly called Doob's martingale convergence theorem, states that $E_n(f)$ converges to f μ-a.e., as long as $f\in L^p(X,\mu)$ and $p\in[1,\infty]$.

Moving forward, the inequalities

$$|E_n(f)(x)| \le Mf(x), \quad x \in X \setminus \mathfrak{N}, \quad n \ge 1, \tag{2.1}$$

and

$$|f(x)| \le |f(x) - E_n(f)(x)| + Mf(x), \quad x \in X \setminus \mathfrak{N}, \quad n \ge 1,$$

are easily deducted. Combining the last one with Doob's martingale convergence theorem, we are led to the inequality $|f| \le Mf$, μ-a.e.. As for the conditional expectation, we have the following result found in [15, p.90]: if $p \in [1,\infty]$ and $f \in L^p(X,\mu)$ then $\|E_n(f)\|_p \le \|f\|_p$, $n \ge 1$. As a consequence, the following theorem holds.

Theorem 2.1. *If $p \in [1,\infty]$ then the linear map $E_n : L^p(X,\mu) \to L^p(X,\mu)$ is bounded. If $p = 2$, then the previous map is a self-adjoint operator.*

We close the section with a result for convergence in the mean of the conditional expectation.

Theorem 2.2. *If $f \in L^2(X,\mu)$ then $E_n f$ converges to f in the mean.*

Proof. If $g_n := |f - E_n(f)|^2$, $n \ge 1$, the previous theorem yields that $\{g_n\} \subset L^1(X,\mu)$. Now, inequality (2.1) leads to

$$|g_n(x)| \le 2(|f(x)|^2 + |E_n(f)(x)|^2) \le 4|Mf(x)|^2, \quad x \in X \setminus \mathfrak{N}, \quad n \ge 1.$$

Clearly, $Mf \in L^2(X,\mu)$ while Doob's convergence theorem gives us $g_n \to 0$ μ-a.e.. The dominated convergence theorem connects the final arguments. □

3. Approximating kernels

This section is entirely composed of technical results involving a family of operators constructed from the martingale defined in Section 2.

Under the notation in Section 2, Theorem 7.20 in [9] informs that the product measure $\mu \times \mu$ is a regular Borel measure on $X \times X$ and the sequence $\{\mathcal{P}_n \times \mathcal{P}_n\}$ of partitions of $X \times X$ increases to the Borel σ-algebra $\mathcal{B}_{X\times X}$ of $(X \times X, \mu \times \mu)$. In particular, if $K \in L^1_{loc}(X \times X, \mu \times \mu)$, the conditional expectation with respect to the σ-algebra generated by the partition $\mathcal{P}_n \times \mathcal{P}_n$ of $X \times X$ can be defined by the formula

$$E_n(K)(u,v) := \frac{1}{\sigma(O_n(u))\sigma(O_n(v))} \int_{O_n(u)} \int_{O_n(v)} K(x,y)\, d\mu(y) d\mu(x).$$

Lemma 3.1 below provides information about a limit property regarding the open sets $O_n(x)$ previously defined. We will use the symbol χ_A to denote the characteristic function of the subset A of X. We remind the reader that given $x \in X$ and $n \ge 1$, the construction introduced in the previous section shows that there exists a unique $O_n(x) \subset P_n$ so that $x \in O_n(x)$.

Lemma 3.1. *If $x \in X$ and $n \ge 1$ then*

$$\lim_{u \to u_0} \chi_{O_n(u)}(x) = \chi_{O_n(u_0)}(x), \quad u_0 \in X \setminus \mathfrak{N}.$$

Proof. Fix $x \in X$ and $n \geq 1$. If $u \in X$ then $x \in O_n(u)$ if and only if $u \in O_n(x)$. Since $\chi_{O_n(x)}(u) = \chi_{O_n(u)}(x)$, we can write

$$|\chi_{O_n(u)}(x) - \chi_{O_n(u_0)}(x)| = |\chi_{O_n(x)}(u) - \chi_{O_n(x)}(u_0)|, \quad u_0 \in X.$$

Next, if $u_0 \in X \setminus \mathfrak{N}$, the fact that $O_n(u_0)$ is open, leaves us with two cases: if $x \in O_n(u_0)$ then $u_0 \in O_n(x)$ and, at the limit, we can assume $u \in O_n(u_0) = O_n(x)$ so that

$$\lim_{u \to u_0} |\chi_{O_n(x)}(u) - \chi_{O_n(x)}(u_0)| = |1 - 1| = 0.$$

If $x \notin O_n(u_0)$ then $u_0 \notin O_n(x)$, and assuming $u \in O_n(u_0)$ as we can, we conclude that

$$\lim_{u \to u_0} |\chi_{O_n(x)}(u) - \chi_{O_n(x)}(u_0)| = |0 - 0| = 0.$$

The proof is complete. □

It is now reasonable that the following result holds.

Lemma 3.2. *If* $u_0 \in X \setminus \mathfrak{N}$ *then* $\lim_{u \to u_0} \mu(O_n(u)) = \mu(O_n(u_0))$, $n = 1, 2, \ldots$.

Proof. Since

$$\mu(O_n(u)) = \int_X \chi_{O_n(u)}(x)\, d\mu(x), \quad u \in X,$$

it follows that

$$|\mu(O_n(u)) - \mu(O_n(u_0))| \leq \int_X |\chi_{O_n(u)}(x) - \chi_{O_n(u_0)}(x)|\, d\mu(x), \quad u \in X.$$

As so, the assertion of the lemma will be proved if we can show that

$$\lim_{u \to u_0} \int_X |\chi_{O_n(u)}(x) - \chi_{O_n(u_0)}(x)|\, d\mu(x) = 0, \quad u_0 \in X \setminus \mathfrak{N}.$$

Hence, in view of the previous lemma, it suffices to show that the integral and the limit in the previous equation commute. The family $\{g_u\}$ defined by

$$g_u(x) = |\chi_{O_n(u)}(x) - \chi_{O_n(u_0)}(x)|, \quad u, x \in X,$$

and the function $g = \chi_{O_n(u_0)}$ belong to $L^1(X, \mu)$. Since $|g_u| \leq g$, μ-a.e., when $u \to u_0$, the desired commuting property follows from the dominated convergence theorem. □

We now turn to kernels of the form

$$D_n(u, x) = \frac{1}{\mu(O_n(u))} \chi_{O_n(u)}(x), \quad u, x \in X, \quad n = 1, 2, \ldots.$$

and the corresponding integral operators $\mathcal{D}_n$ generated by D_n. For use ahead, we mention the immediate formula

$$E_n(\chi_{O_n(u)}\, f) = \mathcal{D}_n(f), \quad u \in X \setminus \mathfrak{N}, \quad f \in L^2(X, \mu). \tag{3.1}$$

Initially, we will use the above kernels to prove the following result.

Theorem 3.3. *If* $K \in L^2(X \times X, \mu \times \mu)$ *and* $n \geq 1$ *then* $E_n(K)$ *is continuous* $\mu \times \mu$*-a.e..*

Proof. It suffices to show that $E_n(K)$ is continuous in the set $(X \setminus \mathfrak{N}) \times (X \setminus \mathfrak{N})$. Let $u_0, v_0 \in X \setminus \mathfrak{N}$. It is not hard to see that

$$E_n(K)(u,v) = \int_X \int_X D_n(u,x)K(x,y)D_n(v,y)\, d\mu(y) d\mu(x), \quad u, v \in X,$$

and that we can use Lemma 3.1 and Lemma 3.2 to deduce that

$$\lim_{(u,v)\to(u_0,v_0)} D_n(u,x)K(x,y)D_n(v,y) = D_n(u_0,x)K(x,y)D_n(v_0,y),$$

for $x, y \in X$ a.e.. If $(u,v) \in O_n(u_0) \times O_n(v_0)$, we have

$$|D_n(u,x)K(x,y)D_n(v,y)| \leq \frac{1}{\mu(O_n(u_0))\mu(O_n(v_0))}|K(x,y)|,$$

for $x, y \in X$ a.e.. So, the continuity at (u_0, v_0) now follows from the dominated convergence theorem. □

Next, we will state and prove a list of technical results that will lead to the following conclusion: $\mathcal{D}_n \mathcal{K} \mathcal{D}_n$ coincides with the integral operator generated by $E_n(K)$.

The sequence of partitions $\{\mathcal{P}_n\}$ was constructed in such a way that each one of them has the following feature: every element of $\{\mathcal{A}_i\}$ is a subset of at most finitely many $O_n(x)$. That been said, if n and i are fixed, we can write

$$\mathcal{A}_i \subset \left(\bigcup_{j=1}^{m(n,i)} O_n(x_j) \right) \bigcup \mathfrak{N}(n,i),$$

in which $\mu(\mathfrak{N}(n,i)) = 0$ and $0 < \mu(O_n(x_j)) < \infty$, $j = 1, 2, \ldots, m(n,i)$. The set $\mathfrak{N}(n,i)$ is nothing but the union of all elements of $\mathcal{P}_n$ for which the intersection with $\mathcal{A}_i$ has measure zero.

In the next results, we will deal with a continuous function $f : X \to \mathbb{C}$ with compact support X_f. Since X_f can be covered by finitely many $\mathcal{A}_i$, after reordering if necessary, we can find an index l so that

$$X_f \subset \left(\bigcup_{k=1}^{l} \bigcup_{j=1}^{m(n,k)} O_n(x_j) \right) \bigcup \left(\bigcup_{k=1}^{l} \mathfrak{N}(n,k) \right),$$

with $\mu(\cup_{k=1}^{l} \mathfrak{N}(n,k)) = 0$. In that case, we will write

$$Y_f = \bigcup_{k=1}^{l} \bigcup_{j=1}^{m(n,k)} O_n(x_j). \tag{3.2}$$

Lemma 3.4. *Let $f : X \to \mathbb{C}$ be a continuous function with compact support X_f and K an element of $L^1_{loc}(X \times X, \mu \times \mu)$. Then*

$$\int_{X\times X} \int_X D_n(u,x)K(x,y)D_n(y,z)f(z)\, d\mu(z)\, d(\mu\times\mu)(x,y)$$
$$= \int_X \int_{X\times X} D_n(u,x)K(x,y)D_n(y,z)f(z)\, d(\mu\times\mu)(x,y)\, d\mu(z).$$

Proof. Pick $M > 0$ so that $|f(x)| \leq M$, $x \in X$. We have

$$\int_{X\times X}\int_X |D_n(u,x)K(x,y)D_n(y,z)f(z)|\,d\mu(z)\,d(\mu\times\mu)(x,y)$$
$$\leq \frac{M}{\mu(O_n(u))}\int_{O_n(u)\times X}\frac{|K(x,y)|}{\mu(O_n(y))}\int_{X_f}\chi_{O_n(y)}(z)\,d\mu(z)\,d(\mu\times\mu)(x,y).$$

If $y \notin Y_f$ then $y \notin X_f$ and, consequently, $O_n(y)\cap X_f = \emptyset$. Thus $\chi_{O_n(y)} = 0$ in X_f and we can take the above integral on $O_n(u)\times Y_f$. Now, if $u \in X \setminus \mathfrak{N}$, the local integrability of K implies that

$$\int_{O_n(u)\times Y_f}\frac{|K(x,y)|}{\mu(O_n(y))}\int_{X_f}\chi_{O_n(y)}(z)\,d\mu(z)\,d(\mu\times\mu)(x,y)$$
$$= \int_{O_n(u)\times Y_f}\frac{|K(x,y)|}{\mu(O_n(y))}\,\mu(O_n(y)\cap X_f)d(\mu\times\mu)(x,y)$$
$$\leq \int_{O_n(u)\times Y_f}|K(x,y)|\,d(\mu\times\mu)(x,y) < \infty.$$

Fubini's theorem ([12, p.386]) completes the proof. □

Lemma 3.5. *Let $f : X \to \mathbb{C}$ be a continuous function with compact support X_f and K an element of $L^1_{loc}(X\times X, \mu\times\mu)$. If $u \in X \setminus \mathfrak{N}$ then*

$$\int_{O_n(u)\times Y_f} K(x,y)D_n(y,z)d(\mu\times\mu)(x,y) = \int_{O_n(u)}\int_{Y_f}K(x,y)D_n(y,z)d\mu(y)d\mu(x)$$
$$= \int_{Y_f}\int_{O_n(u)}K(x,y)D_n(y,z)d\mu(x)d\mu(y),$$

with Y_f as defined in (3.2).

Proof. If $u \in X \setminus \mathfrak{N}$ and $z \in X$ then

$$\int_{O_n(u)}\int_{Y_f}|K(x,y)D_n(y,z)|\,d\mu(y)\,d\mu(x) \leq \int_{O_n(u)}\int_{Y_f}\frac{|K(x,y)|}{\mu(O_n(y))}\,d\mu(y)\,d\mu(x).$$

Introducing the decomposition (3.2) in the last expression above and recalling the uniqueness property of the $O_n(x)$, we deduce that

$$\int_{O_n(u)}\int_{Y_f}\frac{|K(x,y)|}{\mu(O_n(y))}\,d\mu(y)\,d\mu(x) = \int_{O_n(u)}\sum_{k=1}^{l}\sum_{j=1}^{m(n,k)}\int_{O_n(y_j)}\frac{|K(x,y)|}{\mu(O_n(y))}\,d\mu(y)\,d\mu(x)$$
$$= \int_{O_n(u)}\sum_{k=1}^{l}\sum_{j=1}^{m(n,k)}\frac{1}{\mu(O_n(y_j))}\int_{O_n(y_j)}|K(x,y)|\,d\mu(y)\,d\mu(x)$$
$$\leq \max_{\substack{1\leq k\leq l\\ 1\leq j\leq m(n,k)}}\left\{\frac{1}{\mu(O_n(y_j))}\right\}\|K\|_{L^1(O_n(u)\times Y_f)} < \infty.$$

Once again, Fubini's theorem leads to the concluding statement. □

Lemma 3.6. *Let $f : X \to \mathbb{C}$ be a continuous function with compact support X_f and K an element of $L^1_{loc}(X \times X, \mu \times \mu)$. Then*

$$\int_X \int_X \int_X D_n(u,x)K(x,y)D_n(y,z)f(z)\,d\mu(z)\,d\mu(y)\,d\mu(x)$$
$$= \int_{X\times X} \int_X D_n(u,x)K(x,y)D_n(y,z)f(z)\,d\mu(z)\,d(\mu\times\mu)(y,x).$$

Proof. If $u \in X \setminus \mathfrak{N}$ and $M > 0$ is a bound for f in X then it is easily seen that

$$\int_X \int_X \left| \int_X D_n(u,x)K(x,y)D_n(y,z)f(z)\,d\mu(z)\right| d\mu(y)\,d\mu(x)$$
$$\le \frac{M}{\mu(O_n(u))} \int_{O_n(u)} \int_{Y_f} \frac{|K(x,y)|}{\mu(O_n(y))}\, \mu(O_n(y)\cap X_f)\,d\mu(x)\,d\mu(y)$$
$$\le \frac{M}{\mu(O_n(u))} \int_{O_n(u)} \int_{Y_f} |K(x,y)|\,d\mu(x)\,d\mu(y) < \infty.$$

So, the result follows from Fubini's theorem once again. $\square$

The proof of the next lemma is analogous and will be omitted.

Lemma 3.7. *Let $f : X \to \mathbb{C}$ be a continuous function with compact support and K and element in $L^1_{loc}(X \times X, \mu \times \mu)$. Then*

$$\int_X \int_X \int_X D_n(u,x)K(x,y)D_n(y,z)f(z)\,d\mu(z)\,d\mu(y)\,d\mu(x)$$
$$= \int_X \int_X \int_X D_n(u,x)K(x,y)D_n(y,z)f(z)\,d\mu(x)\,d\mu(y)\,d\mu(z).$$

Recalling that if $y, z \in X$ then $z \in O_n(y)$ if and only if $y \in O_n(z)$, the following lemma becomes obvious.

Lemma 3.8. *If $n \ge 1$ then $D_n(y,z) = D_n(z,y)$, $y, z \in X \setminus \mathfrak{N}$.*

Below, $\mathcal{E}^n_K$ will denote the integral operator generated by $E_n(K)$.

Theorem 3.9. *If $K \in L^2(X\times X, \mu\times\mu)$ then $\mathcal{D}_n\mathcal{K}\mathcal{D}_n(f) = \mathcal{E}^n_K(f)$, $f \in L^2(X,\mu)$.*

Proof. Clearly $L^2(X \times X, \mu \times \mu) \subset L^1_{loc}(X \times X, \mu \times \mu)$. If $K \in L^2(X \times X, \mu \times \mu)$ and $f\colon X \to \mathbb{C}$ is continuous with compact support then the previous lemmas imply that

$$\begin{aligned}
\mathcal{D}_n\mathcal{K}\mathcal{D}_n(f)(u) &= \int_X \int_X \int_X D_n(u,x)K(x,y)D_n(y,z)f(z)\,d\mu(z)\,d\mu(y)\,d\mu(x)\\
&= \int_{O_n(u)} \int_{Y_f} \int_{X_f} D_n(u,x)K(x,y)D_n(y,z)f(z)d\mu(z)d\mu(y)d\mu(x)\\
&= \int_X \int_X \int_X D_n(u,x)K(x,y)D_n(z,y)f(z)\,d\mu(x)\,d\mu(y)\,d\mu(z)\\
&= \mathcal{E}^n_K(f)(u), \quad u \in X \setminus \mathfrak{N}.
\end{aligned}$$

Hence, the result in the statement of the theorem follows from the equality $\mu(X \setminus \mathfrak{N}) = 0$ and from a basic approximation theorem from measure theory ([12, p.197]). □

The last result of the section refers to the positive definiteness of $E_n(K)$.

Theorem 3.10. *If K is $L^2(X,\mu)$-positive definite then so is $E_n(K)$.*

Proof. If K is $L^2(X,\mu)$-positive definite then both, K and $E_n(K)$, belong to the space $L^2(X \times X, \mu \times \mu)$. On the other hand, Theorem 2.1 and (3.1) imply that $\mathcal{D}_n(L^2(X,\mu)) \subset L^2(X,\mu)$. Thus, an application of Theorem 3.9 leads to

$$\langle \mathcal{E}_K^n(f), f\rangle_2 = \langle \mathcal{K}\mathcal{D}_n(f), \mathcal{D}_n(f)\rangle_2 \geq 0, \quad f \in L^2(X,\mu).$$

The proof is complete. □

4. Traceability

This section contains the main results of the paper. They can be interpreted as generalizations of results obtained in [8] and other references quoted here. The traceability results described here will be obtained via several known results on trace-class operators and singular values of operators. We will quote some of them and just mention others. The construction developed in Section 2 reveals that the diagonal of X is, up to a set of measure zero, a subset of $(X \setminus \mathfrak{N}) \times (X \setminus \mathfrak{N})$. This remark justify why some of the integrals appearing below are not identically zero. Given $K \in L^2(X \times X, \mu \times \mu)$, we will consider $\mathcal{E}_K^n$ acting like an operator on $L^2(X,\mu)$. All other operators mentioned here are to be understood acting in the same way.

The following lemma is a known consequence of Mercer's theorem ([3]).

Lemma 4.1. *If K is a continuous ($\mu \times \mu$-a.e.) $L^2(X,\mu)$-positive definite kernel and $x \in X \to K(x,x)$ is integrable then $\mathcal{K}$ is trace-class and*

$$tr(\mathcal{K}) = \int_X K(x,x)\, d\mu(x).$$

Lemma 4.2. *Let K be $L^2(X,\mu)$-positive definite. If*

$$\limsup_{n\to\infty} \int_X E_n(K)(x,x)\, d\mu(x) < \infty, \tag{4.1}$$

then $\limsup_{n\to\infty} tr(\mathcal{E}_K^n) < \infty$.

Proof. If (4.1) holds then there exists $n_0 \in \mathbb{N}$ such that $x \in X \to E_n(K)(x,x)$ is integrable for $n \geq n_0$. Theorem 3.10 implies that $E_n(K)$ is $L^2(X,\mu)$-positive definite while Theorem 3.3 shows that $E_n(K)$ is continuous $\mu \times \mu$-a.e.. Applying Lemma 4.1 we see that

$$\mathrm{tr}(\mathcal{E}_K^n) = \int_X E_n(K)(x,x)\, d\mu(x), \quad n \geq n_0.$$

The result follows. □

Next, we recall some facts involving singular values of an operator. If T is a compact operator on a Hilbert space, a *singular value of* T is an eigenvalue of $(T^*T)^{1/2}$. We shall enumerate the nonzero singular values of T in decreasing order, taking multiplicities into account: $s_1(T) \geq s_2(T) \geq \ldots$. If the rank ρ of $(T^*T)^{1/2}$ is finite, obviously $s_j(T) = 0$, $j \geq \rho + 1$. If the eigenvalues of T are ordered like $|l_1(T)| \geq |l_2(T)| \geq \ldots$, then a classical result from operator theory states that $s_j(T) = |l_j(T)|$, $j = 1, 2, \ldots$, as long as T is either hermitian or normal. If S is another compact operator of same type as T, and assuming the same ordering on the singular values of S, the following inequality holds: $|s_n(T) - s_n(S)| \leq \|T - S\|$, $n = 1, 2 \ldots$. All of these results can be found with proofs in [10, 11].

In Theorem 4.4, a complement of Lemma 4.2, we also use the following non-trivial result on convergence of operators ([8]).

Lemma 4.3. *Let $\{T_n\}$ be a countable set of bounded linear operators on a Hilbert space $\mathcal{H}$ such that $\lim_{n\to\infty} \|T_n(f) - f\|_{\mathcal{H}} = 0$, $f \in \mathcal{H}$. If every T_n is self-adjoint and T is a bounded compact operator on $\mathcal{H}$ then $\lim_{n\to\infty} \|T_n T T_n - T\| = 0$.*

Theorem 4.4. *Let K be $L^2(X, \mu)$-positive definite. If*

$$\limsup_{n\to\infty} \int_X E_n(K)(x, x)\, d\mu(x) < \infty,$$

then $\mathcal{K}$ is trace-class.

Proof. Since $\{s_j(\mathcal{E}_K^n)\} \subset (0, \infty)$, it is quite clear that

$$\sum_{j=1}^{k} s_j(\mathcal{E}_K^n) \leq \operatorname{tr}(\mathcal{E}_K^n), \quad k = 1, 2, \ldots.$$

Theorem 3.9 and the inequality mentioned before Lemma 4.3 imply that

$$|s_j(\mathcal{E}_K^n) - s_j(\mathcal{K})| \leq \|\mathcal{E}_K^n - \mathcal{K}\| = \|\mathcal{D}_n \mathcal{K} \mathcal{D}_n - \mathcal{K}\|, \quad j = 1, 2, \ldots. \tag{4.2}$$

Since each $\mathcal{D}_n$ is self-adjoint, $\mathcal{K}$ is compact and

$$\lim_{n\to\infty} \|\mathcal{D}_n(f) - f\|_2 = 0, \quad f \in L^2(X, \mu),$$

we are authorized to apply Lemma 4.3 to conclude, from (4.2), that

$$\lim_{n\to\infty} s_j(\mathcal{E}_K^n) = s_j(\mathcal{K}), \quad j = 1, 2, \ldots.$$

It is now clear that

$$\sum_{j=1}^{k} s_j(\mathcal{K}) = \limsup_{n\to\infty} \sum_{j=1}^{k} s_j(\mathcal{E}_K^n) \leq \limsup_{n\to\infty} \operatorname{tr}(\mathcal{E}_K^n), \quad k = 1, 2, \ldots,$$

and that concludes the proof. ∎

In order to deal with the converse of the previous result, we will need the following result ([10, p.51]): if S_1, S_2 and T are bounded linear operators on a Hilbert space and T is compact then so is the composition $S_1 T S_2$ and $s_j(S_1 T S_2) \leq \|S_1\| s_j(T) \|S_2\|$, $j = 1, 2, \ldots$.

Lemma 4.5. *Let $p \in [1,\infty)$ and $K \in L^p(X \times X, \mu \times \mu)$. If $x \in X \to K(x,x)$ is integrable and $\mu(X) < \infty$ then there is a positive integer n_0 for which $x \in X \to E_n(K)(x,x)$ is integrable when $n \geq n_0$.*

Proof. Since

$$|E_n(K)(u,u)| \leq |E_n(K)(u,u) - K(u,u)| + |K(u,u)|, \quad u \in X \setminus \mathfrak{N},$$

we can use Doob's convergence theorem to select a positive integer n_0 so that

$$|E_n(K)(u,u)| \leq 1 + |K(u,u)|, \quad u \in X \setminus \mathfrak{N}, \quad n \geq n_0.$$

Our assumptions on X and $x \in X \to K(x,x)$ imply the result. $\square$

Theorem 4.6. *Let K be $L^2(X,\mu)$-positive definite. If $x \in X \to K(x,x)$ is integrable and $\mu(X) < \infty$ then there is $n_0 \in \mathbb{N}$ so that $\mathcal{E}_K^n \in \mathcal{B}_1(L^2(X))$ and*

$$tr(\mathcal{E}_K^n) = \int_X E_n(K)(x,x)\, d\mu(x), \quad n \geq n_0.$$

Proof. The previous lemma reveals that $x \in X \to E_n(K)(x,x)$ is integrable for n large. As so, the result follows from Theorem 3.3 and Lemma 4.1. $\square$

Theorem 4.7. *Let $K \in L^2(X \times X, \mu \times \mu)$. If $\mathcal{K}$ is trace-class then so is every $\mathcal{E}_K^n$. The number $tr(\mathcal{K})$ is an upper bound for the sequence $\{tr(\mathcal{E}_K^n)\}$.*

Proof. Assume $\mathcal{K}$ is trace-class. Since each $\mathcal{D}_n$ is bounded, Theorem 3.10 and the comments preceding Lemma 4.5 imply that

$$s_j(\mathcal{E}_K^n) = s_j(\mathcal{D}_n\mathcal{K}\mathcal{D}_n) \leq \|\mathcal{D}_n\| s_j(\mathcal{K}) \|\mathcal{D}_n\|, \quad n = 1, 2, \ldots.$$

Hence,

$$\sum_{j=1}^{\infty} s_j(\mathcal{E}_K^n) \leq \|\mathcal{D}_n\|^2 \sum_{j=1}^{\infty} s_j(\mathcal{K}),$$

and the two assertions of the lemma follow. $\square$

The following result is very close to a converse of Theorem 4.4.

Theorem 4.8. *Let $K \in L^2(X \times X, \mu \times \mu)$. If $\mathcal{K}$ is trace-class then*

$$\lim_{n\to\infty} tr(\mathcal{E}_K^n) = tr(\mathcal{K}).$$

Proof. A basic inequality for the trace ([10, p.54]) implies that

$$|\mathrm{tr}\,(\mathcal{E}_K^n) - \mathrm{tr}\,(\mathcal{K})| \leq \sum_{j=1}^{\infty} s_j(\mathcal{E}_K^n - \mathcal{K}), \quad n = 1, 2, \ldots,$$

as long as $\mathcal{K}$ is trace-class. On the other hand, since (see [10, p.89])

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} s_j(\mathcal{D}_n\mathcal{K}\mathcal{D}_n - \mathcal{K}) = 0,$$

Theorem 3.9 completes the proof. $\square$

Next, we move to a proof of the converse of Theorem 4.4 in the case when $\mu(X) < \infty$.

Theorem 4.9. *Let K be $L^2(X,\mu)$- positive definite. If $\mathcal{K}$ is trace-class and $\mu(X)<\infty$ then*

$$\lim_{n\to\infty}\int_X E_n(K)(x,x)\,d\mu(x)<\infty.$$

Proof. Assume $\mathcal{K}$ is trace-class. Since the function $x\in X\to K(x,x)$ is integrable already, if $\mu(X)<\infty$, we can use Theorem 4.6 to find a positive integer n_0 such that

$$tr(\mathcal{E}_K^n)=\int_X E_n(K)(x,x)\,d\mu(x),\quad n\geq n_0.$$

An application of Theorem 4.8 finishes the proof. □

At this point, it is very important to remind the reader that the results we have obtained includes the case in which X is either a sphere or a torus.

Next, we intend to consider cases in which X has no finite measure. In order to handle that, we use the cover $\{\mathcal{A}_m\}$ of X constructed before to define a sequence of subsets of X that increases to X. Precisely, defining $X_j=\cup_{m=1}^{j}\mathcal{A}_m$, $j\geq 1$, we immediately have the following two properties: $X=\cup_{j=1}^{\infty}X_j$ and if $x\in X$ then there exists $j_0\geq 0$ such that $x\in X_j$, $j\geq j_0$. Using the sequence just defined, we now take linear operators $P_j: L^2(X,\mu)\to L^2(X,\mu)$ defined by the formula $P_j(f)=f\chi_{X_j}$, $f\in L^2(X,\mu)$. They are self-adjoint and the uniform boundedness principle shows that the sequence $\{P_j\}$ is bounded in the space of bounded linear operators on $L^2(X,\mu)$. Also, the dominated convergence theorem implies that $\{P_j\}$ converges pointwise to the identity operator on $L^2(X,\mu)$. The following technical lemma contains a critical information on the sequence $\{P_j\}$.

Lemma 4.10. *If $T: L^2(X,\mu)\to L^2(X,\mu)$ is trace-class then each P_jTP_j is so and the limit formula $\lim_{j\to\infty} tr(P_jTP_j)=tr(T)$ holds.*

Proof. The first assertion is a consequence of the remark preceding Lemma 4.5. As for the other, it follows from Theorem 11.3 in [10] . □

The converse of Theorem 4.4 reads as follows.

Theorem 4.11. *Let K be $L^2(X,\mu)$-positive definite. If $\mathcal{K}$ is trace-class then the limit*

$$\lim_{n\to\infty}\int_X E_n(K)(x,x)\,d\mu(x)$$

exists and is finite.

Proof. The proof requires the double-indexed operator $\mathcal{Q}_j^n$ given by the formula

$$\mathcal{Q}_j^n(f)(x)=\int_{X_j}E_n(K)(x,y)f(y)\,d\mu(y),\quad x\in X_j,\quad f\in L^2(X_j,\mu).$$

If $f \in L^2(X_j)$, let us write $\tilde{f}$ to denote a function on X that coincides with f on X_j and is zero in $X \setminus X_j$. It is now clear that

$$\begin{aligned}\int_{X_j} \mathcal{Q}_j^n(f)(x)\overline{f(x)}\,d\mu(x) &= \int_X \int_X E_n(K)(x,y)\tilde{f}(y)\overline{\tilde{f}(x)}\,d\mu(y)\,d\mu(x)\\ &= \int_X \int_X E_n(K)(x,y)\tilde{f}(y)\,d\mu(y)\overline{\tilde{f}(x)}\,d\mu(x)\\ &= \int_X \mathcal{E}_K^n(\tilde{f})(x)\overline{\tilde{f}(x)}\,d\mu(x), \quad f \in L^2(X_j,\mu).\end{aligned}$$

Since K is $L^2(X,\mu)$-positive definite, Theorem 3.10 implies that $\mathcal{Q}_j^n$ is $L^2(X_j,\mu)$-positive definite. Also, the fact that $\mathcal{K}$ is trace-class guarantees that $x \in X \to K(x,x)$ is integrable. Hence, due to Lemma 4.5, there exists $n_0 \geq 0$ such that $x \in X_j \to E_n(K)(x,x)$ whenever $n \geq n_0$. Recalling Theorem 3.3 and applying Lemma 4.1, we deduce that $\mathcal{Q}_j^n$ is trace-class and

$$\operatorname{tr}(\mathcal{Q}_j^n) = \int_{X_j} E_n(K)(x,x)\,d\mu(x),$$

as long as $n \geq n_0$. Let us keep the previous condition on n in force. If V_j is the closed subspace $L^2(X,\mu)$ encompassing the functions on X which are zero in $X \setminus X_j$ and $\mathcal{R}_j^n : V_j \to V_j$ is the operator given by

$$\mathcal{R}_j^n(f)(x) = \chi_{X_j}(x)\int_X E_n(K)(x,y)\chi_{X_j}(y)f(y)\,d\mu(y),$$

with $x \in X, f \in V_j$, then $\mathcal{R}_j^n$ and $\mathcal{Q}_j^n$ possess the same eigenvalues. Having in mind the previous lemma,

$$(\mathcal{P}_j\mathcal{E}_K^n\mathcal{P}_j)(f)(x) = \int_X E_n(K)(x,y)\chi_{X_j\times X_j}(x,y)f(y)\,d\mu(y),$$

for $x \in X, f \in L^2(X)$, and we can conclude now that $\mathcal{R}_j^n$ and $\mathcal{P}_j\mathcal{E}_K^n\mathcal{P}_j$ have the same eigenvalues. Therefore,

$$\operatorname{tr}(\mathcal{P}_j\mathcal{E}_K^n\mathcal{P}_j) = \operatorname{tr}(\mathcal{R}_j^n) = \operatorname{tr}(\mathcal{Q}_j^n) = \int_{X_j} E_n(K)(x,x)\,d\mu(x). \tag{4.3}$$

The monotone convergence theorem leads to

$$\operatorname{tr}(\mathcal{E}_K^n) = \int_X E_n(K)(x,x)\,d\mu(x).$$

Finally, (4.3) and the observation made before Lemma 4.3 lead to the assertion of the theorem. □

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