COMPREHENSIVE SURVEY ON AN ORDER PRESERVING OPERATOR INEQUALITY

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Dedicated to Professor Masatoshi Fujii and Professor Eizaburo Kamei on their retirements with respect and affection.

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ABSTRACT. In 1987, we established an operator inequality as follows; $A \geq B \geq 0 \implies (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$ holds for (*) $p \geq 0$, $q \geq 1$, $r \geq 0$ with $(1+r)q \geq p+r$. It is an extension of Löwner-Heinz inequality. The purpose of this paper is to explain geometrical background of the domain by (*), and to give brief survey of recent results of its applications.

1. Introduction

A capital letter means a bounded linear operator on a Hilbert space H. An operator T is called positive (simply $A > 0$) if T is positive semidefinite (simply $A \geq 0$) and invertble.

Theorem 1.1 (LH(Löwner-Heinz inequality)). *$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.*

Although Theorem LH is very useful, but the condition " $\alpha \in [0, 1]$ " is too restrictive. In fact Theorem LH does not always hold for $\alpha \notin [0, 1]$. The following result [25] has been obtained from this point of view.

Theorem 1.2 (F (Furuta inequality)). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^{\frac{r}{2}}A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

    and

(ii) $(A^{\frac{r}{2}}A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

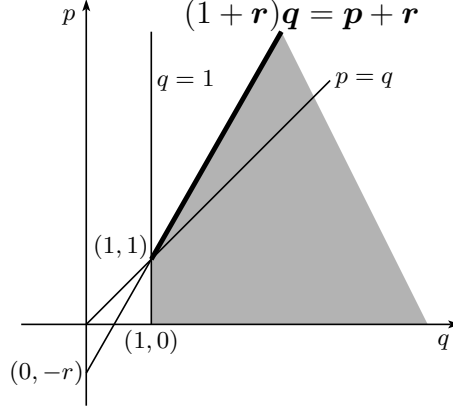*hold for* $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p + r$.

FIGURE 1.

The domain drawn for p,q and r in Figure (1) is the best possible one K. Tanahashi [64].

Theorem F yields Löwner-Heinz inequality asserting that $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$, when we put $r = 0$ in (i) or (ii). Consider two magic boxes

$$f(\square) = (B^{\frac{r}{2}}\square B^{\frac{r}{2}})^{\frac{1}{q}} \text{ and } g(\square) = (A^{\frac{r}{2}}\square A^{\frac{r}{2}})^{\frac{1}{q}}$$

.

Although $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for $p > 1$, Theorem F asserts the following " two order preserving operator inequalities"

$$f(A^p) \geq f(B^p) \text{ and } g(A^p) \geq g(B^p)$$

hold whenever $A \geq B \geq 0$ under the condition p, q and r in Figure (1).

We have been finding a lot of applications of Theorem F in the following three branches (A) operator inequalities, (B) norm inequalities, and (C) operator equations. We would like to concentrate ourselves to state typical examples of recent applications of Theorem F without their proofs.

(A) OPERATOR INEQUALITIES

(A-1) Several characterizations of operators $\log A \geq \log B$ and its applications.

(A-2) Applications to the relative operator entropy.

(A-3) Applications to Ando–Hiai log majorization.

(A-4) Generalized Aluthge transformation on p-hyponormal operators.

(A-5) Several classes associated with log-hyponormal and paranormal operators.

(A-6) Order preserving operator inequalities and operator functions implying them.

(A-7) Applications to Kantorovich type operator inequalities.

(A-8) Some variations of Choi inequality.

(A-9) Furuta inequalty of indefinite type on Krein space.

(B) NORM INEQUALITIES

(B-1) Several generalizations of Heinz–Kato theorem.

(B-2) Generalizations of some theorem on norms.

(B-3) An extension of Kosaki trace inequality and parallel results.

(C) OPERATOR EQUATIONS

(C-1) Generalizations of Pedersen-Takesaki theorem and related results.

(C-2) Positive semidefinite solutions of some operator equations.

Lemma 1.3 (Lemma A [28]**).** *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

Proof. Let $YX^{\frac{1}{2}} = UH$ be the polar decomposition of $YX^{\frac{1}{2}}$, where U is unitary and $H = |YX^{\frac{1}{2}}|$. Then we have

$$(YXY^*)^\lambda = (UH^2U^*)^\lambda = YX^{\frac{1}{2}}H^{-1}H^{2\lambda}H^{-1}X^{\frac{1}{2}}Y^* = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

$$\square$$

Proof. of Theorem F. At first we prove (ii). In the case $1 \geq p \geq 0$, the result is obvious by Theorem LH. We have only to consider $p \geq 1$ and $q = \frac{p+r}{1+r}$ since (ii) of Theorem F for values q larger than $\frac{p+r}{1+r}$ follows by Theorem LH, that is, we have only to prove the following

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \text{ for any } p \geq 1 \text{ and } r \geq 0. \tag{1.1}$$

We may assume that A and B are invertible without loss of generality. In the case $r \in [0, 1]$, $A \geq B \geq 0$ ensures $A^r \geq B^r$ holds by Theorem LH. Then we have

$$\begin{aligned}
(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} &= A^{\frac{r}{2}}B^{\frac{p}{2}}(B^{\frac{-p}{2}}A^{-r}B^{\frac{-p}{2}})^{\frac{p-1}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}} \text{ by Lemma 1.3} \\
&\leq A^{\frac{r}{2}}B^{\frac{p}{2}}(B^{\frac{-p}{2}}B^{-r}B^{\frac{-p}{2}})^{\frac{p-1}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}} \\
&= A^{\frac{r}{2}}BA^{\frac{r}{2}} \leq A^{1+r},
\end{aligned}$$

and the first inequality follows by $B^{-r} \geq A^{-r}$ and Theorem LH since $\frac{p-1}{p+r} \in [0, 1]$ holds, and the last inequality follows by $A \geq B \geq 0$, so we have the following (1.2)

$$A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \text{ for } p \geq 1 \text{ and } r \in [0, 1]. \tag{1.2}$$

Put $A_1 = A^{1+r}$ and $B_1 = (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ in 1.2. Repeating 1.2 again for $A_1 \geq B_1 \geq 0$, $r_1 \in [0, 1]$ and $p_1 \geq 1$,

$$A_1^{1+r_1} \geq (A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}$$

Put $p_1 = \frac{p+r}{1+r} \geq 1$ and $r_1 = 1$, then

$$A^{2(1+r)} \geq (A^{r+\frac{1}{2}}B^pA^{r+\frac{1}{2}})^{\frac{2(1+r)}{p+2r+1}} \text{ for } p \geq 1, \text{ and } r \in [0, 1] \tag{1.3}$$

Put $\frac{s}{2} = r + \frac{1}{2}$ in 1.3. Then $\frac{2(1+r)}{p+2r+1} = \frac{1+s}{p+s}$ since $2(1+r) = 1+s$, so that 1.3 can be rewritten as follows;

$$A^{1+s} \geq (A^{\frac{s}{2}} B^p A^{\frac{s}{2}})^{\frac{1+s}{p+s}} \text{ for } p \geq 1, \text{ and } s \in [1,3] \tag{1.4}$$

Consequently 1.2 and 1.4 ensure that 1.2 holds for any $r \in [0,3]$ since $r \in [0,1]$ and $s = 2r + 1 \in [1,3]$ and repeating this process, 1.1 holds for any $r \geq 0$, (ii) is shown.

If $A \geq B > 0$, then $B^{-1} \geq A^{-1} > 0$. Then by (ii), for each $r \geq 0$, $B^{\frac{-(p+r)}{q}} \geq (B^{\frac{-r}{2}} A^{-p} B^{\frac{-r}{2}})^{\frac{1}{q}}$ holds for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+r)q \geq p+r$. Taking inverses gives (i), so the proof of Theorem F is complete. $\qquad \square$

This one page proof of Theorem F in T. Furuta [26], T. Furuta [30] and the original one in T. Furuta [25]. Alternative proofs are in M. Fujii [14] and E. Kamei [53].

Remark 1.4. $A \geq B \geq 0 \Longleftrightarrow A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$.

Background of Theorem F

We would like to explain "how to conjecture the form of Theorem F" via Löwner-Heinz inequality by using "FIGURE" illustration.

$$A \geq B \geq 0$$

$\Longleftrightarrow$ (a) $A^\alpha \geq B^\alpha$ for $\alpha \in [0,1]$ (1-dimensional interval $\alpha \in [0,1]$)
    (Löwner-Heinz inequality).

FIGURE 2.

$\Longleftrightarrow$ (b) $A^{\frac{p}{q}} \geq B^{\frac{p}{q}}$ for $q \geq p \geq 0, q \geq 1$ (2-dimensional (q,p) domain).

FIGURE 3.

$\Longleftrightarrow$ (c)
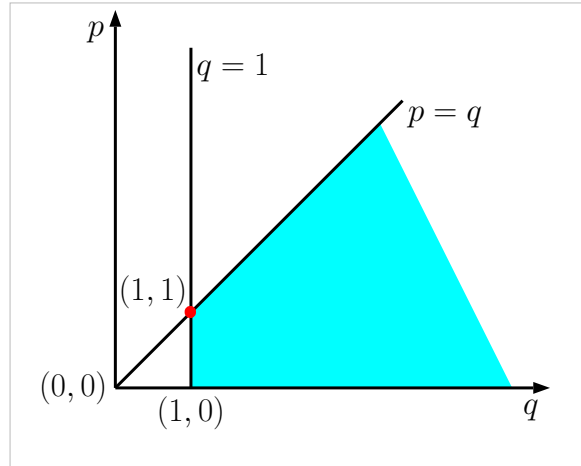
$$(A^{\frac{0}{2}}A^p A^{\frac{0}{2}})^{\frac{1}{q}} \geq (A^{\frac{0}{2}}B^p A^{\frac{0}{2}})^{\frac{1}{q}} \quad for$$
$$p \geq 0, \ q \geq 1 \ and \ (1+0)(q-1) \geq p-1.$$

$\Longleftrightarrow$ (d)

$$(A^{\frac{r}{2}}A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^{\frac{1}{q}} \quad for$$
$$(\star) \quad r \geq 0, \ p \geq 0, \ q \geq 1 \ and \ (1+r)(q-1) \geq p-1$$

$\Longleftrightarrow$ (e)

$$(A^{\frac{r}{2}}A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^{\frac{1}{q}} \quad for$$
$$(\star\star) \quad r \geq 0, \ p \geq 0, \ q \geq 1 \ and \ (1+r)q \geq p+r.$$

Recall that $(\star\star)$ in (e) is equivalent to $(\star)$ in (d). Since $(d) \Longrightarrow (c)$ is trivial and we prove the equivalence relation between (c) and (d) in the proof of Theorem F.

We would like to emphasize that the condition on $\alpha \in [0,1]$ in (a) could be converted to 2-dimensional domain $q \geq p \geq 0, \ q \geq 1$ in (b) and this idea is most important.
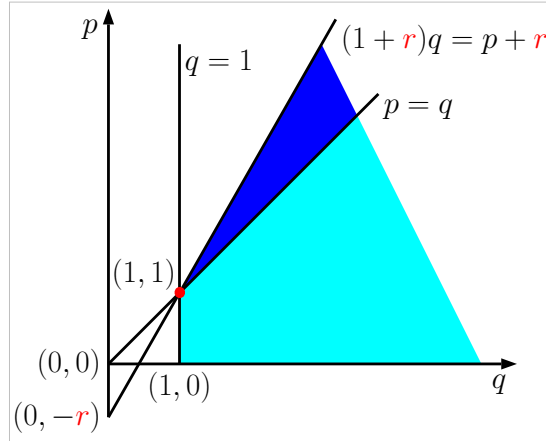
FIGURE 4.

An excellent and tough proof of the best possibility of Theorem F is obtained in K. Tanahashi [64], that is, the domain drawn for p,q and r in FIGURE 1 is the best possible one.

Some of closely related papers in this chapter: [13, 14, 15, 16, 25, 26, 30, 40, 53, 64].

2. (A-6) FURTHER EXTENSIONS OF FURUTA INEQUALITY AND OPERATOR FUNCTIONS IMPLYING THEM

We show the following Theorem G which interpolates Theorem F and the equality equivalent to log majorization in [8] (see §5 and §10).

Theorem 2.1 (Theorem G [28]). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \ for \ s \geq 1 \ and \ r \geq t. \qquad (2.1)$$

Proof. We may assume that B is invertible. First of all, we prove that if $A \geq B \geq 0$ with $A > 0$, then

$$A \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}} \text{ for } t \in [0,1], p \geq 1 \text{ and } s \geq 1. \qquad (2.2)$$

In case the $2 \geq s \geq 1$, as $s - 1, \frac{1}{(p-t)s+t} \in [0,1]$ and $A^t \geq B^t$ by Theorem LH, so by Lemma A and Theorem LH we have

$$\begin{aligned}
B_1 &= \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}} \\
&= \{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{-t}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{1}{(p-t)s+t}} \\
&\leq \{B^{\frac{p}{2}}(B^{\frac{p}{2}}B^{-t}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{1}{(p-t)s+t}} \\
&= B \leq A = A_1 \qquad (2.3)
\end{aligned}$$

for $t \in [0,1]$, $p \geq 1$ and $2 \geq s \geq 1$. Repeating (2.3) for $A_1 \geq B_1 \geq 0$, then we have

$$A_1 \geq \{A_1^{\frac{t_1}{2}}(A_1^{\frac{-t_1}{2}}B_1^{p_1}A_1^{\frac{-t_1}{2}})^{s_1}A_1^{\frac{t_1}{2}}\}^{\frac{1}{(p_1-t_1)s_1+t_1}} \text{ for } t_1 \in [0,1], p_1 \geq 1 \text{ and } 2 \geq s_1 \geq 1 \qquad (2.4)$$

Put $t_1 = t$ and $p_1 = (p-t)s + t \geq 1$ in (2.4). Then we obtain

$$\begin{aligned}
A &\geq \{A^{\frac{t}{2}}[A^{\frac{-t}{2}}A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}A^{\frac{-t}{2}}]^{s_1}A^{\frac{t}{2}}\}^{\frac{1}{(p-t)ss_1+t}} \qquad (2.5) \\
&= \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^{ss_1}A^{\frac{t}{2}}\}^{\frac{1}{(p-t)ss_1+t}} \text{ for } t \in [0,1], p \geq 1 \text{ and } 4 \geq ss_1 \geq 1
\end{aligned}$$

Repeating this process from (2.3) to (2.5), we obtain (2.2) for $t \in [0,1]$, $p \geq 1$ and any $s \geq 1$.

Put $A_2 = A$ and $B_2 = \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$ in (2.2).

Applying (ii) of Theorem F for $A_2 \geq B_2 \geq 0$ by (2.2) for $t \in [0,1]$, $p \geq 1$ and $s \geq 1$, so we have

$$A_2^{1+r_2} \geq (A_2^{\frac{r_2}{2}}B_2^{p_2}A_2^{\frac{r_2}{2}})^{\frac{1+r_2}{p_2+r_2}} \text{ holds} \quad \text{for } p_2 \geq 1 \text{ and} r_2 \geq 0 \qquad (2.6)$$

We have only to put $r_2 = r - t \geq 0$ and $p_2 = (p-t)s + t \geq 1$ in (2.6) to obtain the desired inequality (2.1) $\qquad \square$

Remark 2.2. Theorem G implies; $A \geq B \geq 0 \implies A^{1+r} \geq (A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$ so that Theorem G is an extension of Theorem F.

Remark 2.3 (Best possibility of Theorem G [66]). Let $p \geq 1$, $t \in [0,1]$, $r \geq t$ and $s \geq 1$. If $\dfrac{1-t+r}{(p-t)s+r} < \alpha$, then there exist positive invertible operators A and B such that $A \geq B > 0$ and $A^{\{(p-t)s+r\}\alpha} \ngeq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^\alpha$.

Theorem 2.4 ([30, 37]). *The following (i),(ii),(iii) and (iv) hold and follow from each other.*

(i) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \text{ holds for } r \geq t \text{ and } s \geq 1$$

(ii) *If $A \geq B \geq 0$ with $A > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,*

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \text{ holds for } r \geq t \text{ and } s \geq 1$$

(iii) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$.

(iv) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$, $q \geq 0$ and $p \geq t$,*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

Corollary 2.5 ([30, 37, 53]). *If $A \geq B > 0$, then the following inequalities* (i) *and* (ii) *hold.*

(i) $\{B^{\frac{t}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sB^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+r}} \geq A \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+r}}$

(ii)

$$B^{\frac{-(r-t)}{2}}(B^{\frac{r-t}{2}}A^pB^{\frac{r-t}{2}})^{\frac{1-t+r}{p-t+r}}B^{\frac{-(r-t)}{2}} \geq A \geq B \geq A^{\frac{-(r-t)}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{1-t+r}{p-t+r}}A^{\frac{-(r-t)}{2}}$$

for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$.

Corollary 2.6 ([30, 53]). *If $A \geq B > 0$, then the following inequality holds*

$$B^{\frac{-r}{2}}(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}}B^{\frac{-r}{2}} \geq A \geq B \geq A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}}$$

for $p \geq 1$ and $r \geq 0$

Some of closely related papers in this chapter: [21, 23, 24, 28, 30, 39, 40, 41, 50, 51, 52, 53, 66, 70, 72].

3. (A-1) SEVERAL CHARACTERIZATION OF OPERATORS $\log A \geq \log B$ AND ITS APPLICATIONS

A function f is said to be operator monotone if $f(A) \geq f(B)$ whenever $A \geq B \geq 0$.

$f(t) = t^\alpha$ is a famous typical example of operator monotone for $\alpha \in [0,1]$ by Theorem LH. Another typical example of operator monotone is $\log t$. In fact, then

If $A \geq B > 0$, $A^\alpha \geq B^\alpha > 0$ for any $\alpha \in [0,1]$ by Theorem L-H, so

$$\frac{A^\alpha - I}{\alpha} \geq \frac{B^\alpha - I}{\alpha}.$$

Hence we have the desired result $\log A \geq \log B$ by tending $\alpha \to +0$.

Theorem 3.1 ([68]). *Let A and B be positive invertible operators. Then the following* (i) *and* (ii) *are equivalent7:*

(i) $\log A \geq \log B$.

(ii) $A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all $p \geq 0$ and $r \geq 0$.*

Proof. (i) $\implies$ (ii). We recall the following obvious and crucial formula

$$\lim_{n\to\infty}(I + \frac{1}{n}\log X)^n = X \text{ for any } X > 0. \tag{3.1}$$

The hypothesis $\log A \geq \log B$ ensures

$$A_1 = I + \frac{\log A}{n} \geq I + \frac{\log B}{n} = B_1$$

for sufficiently large natural number n. Applying (ii) of Theorem F to A_1 and B_1, we have

$$A_1^{nr} \geq (A_1^{\frac{nr}{2}} B_1^{np} A_1^{\frac{nr}{2}})^{\frac{nr}{np+nr}} \text{ for all } p \geq 0 \text{ and } r \geq 0 \tag{3.2}$$

since $(1 + nr)(\frac{np+nr}{nr}) \geq np + nr$ holds and this condition satisfies the required condition of Theorem F. When $n \to \infty$, (3.2) ensures (ii) by (3.1).

(ii) $\Longrightarrow$ (i). Taking logarithm of both sides of (ii) since $\log t$ is operator monotone function , we have

$$r(p+r)\log A \geq r\log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}) \text{ for all } p \geq 0 \text{ and } r \geq 0$$

and tending $r \to +0$, hence we obtain $\log A \geq \log B$. $\qquad\square$

Theorem 3.2 ([17, 27]). *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(i) $A \gg B$ (i.e., $\log A \geq \log B$).
(ii) *For any fixed $t \geq 0$, $F(p,r) = B^{\frac{-r}{2}}(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{t+r}{p+r}} B^{\frac{-r}{2}}$ is an increasing function of both $p \geq t$ and $r \geq 0$.*
(iii) *For any fixed $t \geq 0$, $G(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{t+r}{p+r}} A^{\frac{-r}{2}}$ is a decreasing function of both $p \geq t$ and $r \geq 0$.*

Some of closely related papers in this chapter: [5, 16, 17, 27, 30, 39, 68].

4. (A-4) Generalized Aluthge transformation on p-hyponormal operators

An operator T on a Hilbert space H is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for positive number p.

Define $\widetilde{T}$ as follows:

$$\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$

which is called " Aluthge transformation".

Theorem 4.1 ([1]). *Let $T = U|T|$ be p-hyponormal for $p > 0$ and U be unitary. Then*

(i) $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ *is $(p + \frac{1}{2})$-hyponormal if $0 < p < \frac{1}{2}$*
(ii) $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ *is hyponormal if $\frac{1}{2} \leq p < 1$*

Proof. (i) Firstly we recall that if T is p-hyponormal for $p > 0$, the following (4.1) holds obviously

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^* \text{ for any } p > 0. \tag{4.1}$$

Let $A = U^*|T|^{2p}U$, $B = |T|^{2p}$ and $C = U|T|^{2p}U^*$ in (4.1). Then (4.1) means

$$A \geq B \geq C \geq 0. \tag{4.2}$$

As $(1 + \frac{1}{2p})\frac{2}{2p+1} = \frac{1}{p} = \frac{1}{2p} + \frac{1}{2p}$ holds, we can apply Theorem F, that is,

$$(\widetilde{T}^*\widetilde{T})^{p+\frac{1}{2}} = (|T|^{\frac{1}{2}}U^*|T|U|T|^{(\frac{1}{2})})^{p+\frac{1}{2}} = (B^{\frac{1}{4p}}A^{\frac{1}{2p}}B^{\frac{1}{4p}})^{p+\frac{1}{2}}$$

$$\geq (B^{\frac{1}{4p}}B^{\frac{1}{2p}}B^{\frac{1}{4p}})^{p+\frac{1}{2}} \geq (B^{\frac{1}{4p}}C^{\frac{1}{2p}}B^{\frac{1}{4p}})^{p+\frac{1}{2}}$$

$$= (|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}})^{p+\frac{1}{2}} = (\widetilde{T}\widetilde{T}^*)^{p+\frac{1}{2}} \tag{4.3}$$

Hence (4.3) ensure $(\widetilde{T}^*\widetilde{T})^{p+\frac{1}{2}} \geq B^{1+\frac{1}{2p}} \geq (\widetilde{T}\widetilde{T}^*)^{p+\frac{1}{2}}$ that is, $\widetilde{T}$ is $p + \frac{1}{2}$ -hyponormal.

(ii) As $|T|^{2p} \geq |T^*|^{2p}$, we have $|T| \geq |T^*|$ by Theorem L-H since $\frac{1}{2p} \in [\frac{1}{2}, 1]$, or equivalently

$$U^*|T|U \geq |T| \geq U|T|U^* \tag{4.4}$$

Then we have

$$\widetilde{T}^*\widetilde{T} - \widetilde{T}\widetilde{T}^* = |T|^{\frac{1}{2}}(U^*|T|U - U|T|U^*)|T|^{\frac{1}{2}} \geq 0 \text{ by (4.4)} \tag{4.5}$$

(4.5) implies $\widetilde{T}^*\widetilde{T} \geq \widetilde{T}\widetilde{T}^*$, that is, $\widetilde{T}$ is hyponormal.

$\square$

Some of closely related papers in this chapter: [1, 2, 3, 29, 30, 42, 43, 45, 46].

5. (A-3) Applications to Ando–Hiai log majorization

Let us write $A \underset{(\log)}{\succ} B$ in [8] for positive semidefinite matrices $A, B \geq 0$ and call the log-majorization if $\prod_{i=1}^{k} \lambda_i(A) \geq \prod_{i=1}^{k} \lambda_i(B)$, and $k = 1, 2, \cdots, n - 1$, and

$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B)$, i.e. $\det A = \det B$. where $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B)$ are the eigenvalues of A and B respectively arranged in decreasing order.

The α-power mean of $A, B > 0$ is defined by $A\#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$. for $0 \leq \alpha \leq 1$. Similarly define $A \natural_s B$ by for any $s \geq 0$ and for $A > 0$ and $B \geq 0$

$$A \natural_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}.$$

Using Theorem G and the same way as in the proof of [8, Theorem 2.1], we can transform Theorem G into the following log-majorization inequality.

Theorem 5.1 ([28]). *For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and each $t \in [0, 1]$*

$$(A\#_\alpha B)^h \underset{(\log)}{\succ} A^{1-t+r}\#_\beta (A^{1-t} \natural_s B)$$

holds for $s \geq 1$, and $r \geq t \geq 0$, where $\beta = \dfrac{\alpha(1 - t + r)}{(1 - \alpha t)s + \alpha r}$ and $h = \dfrac{(1 - t + r)s}{(1 - \alpha t)s + \alpha r}$.

Corollary 5.2 ([28]). *For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$, $(A\#_\alpha B)^h \underset{(\log)}{\succ} A^r\#_{\frac{h\alpha}{s}} B^s$ for $r \geq 1$ and $s \geq 1$, where $h = [\alpha s^{-1} + (1 - \alpha)r^{-1}]^{-1}$.*

Corollary 5.2 yields the following result of T. Ando and F. Hiai [8, Theorem 2.1].

Theorem 5.3 (AH [8])**.** *For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$, $(A\#_\alpha B)^r \underset{(\log)}{\succ} A^r \#_\alpha B^r$ for $r \geq 1$.*

Remark 5.4. The following result is pointed out in [8].

   (i) $(A\#_\alpha B)^r \underset{(\log)}{\succ} A^r \#_\alpha B^r$ for $r \geq 1$ and $0 \leq \alpha \leq 1$ in Theorem AH $\iff$

   (ii) if $A \geq B > 0$ with $A > 0$ ensures $A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^p A^{\frac{-1}{2}})^r A^{\frac{r}{2}}\}^{\frac{1}{p}}$ for $p \geq 1$ and $r \geq 1$.

(ii) follows by Theorem G since we have only to put $t = 1$ and $r = s$ in Theorem G.

Theorem G can be transformed into Theorem 5.1 as an extension of Corollary 5.2 containing Theorem AH. Theorem G interpolates both Theorem F and Theorem AH as follows,

<div align="center">

Theorem G

$t = 0$ ↙ ↘ $t = 1$ and $r = s$

Theorem F   **Theorem AH**

</div>

Some of closely related papers in this chapter: [6, 8, 15, 16, 28, 30, 34, 44, 69].

6. (A-3) Operator inequalities and log majorization

As stated in section §5, $A \natural_s B$ in the case $0 \leq s \leq 1$ just coincides with the usual α-power mean. We shall show a log majorization equivalent to an order preserving operator inequality.

Using Theorem G and the same way as in the proof of [8, Theorem 2.1], we can transform Theorem G into the following log majorization inequality different from Theorem 5.1.

Theorem 6.1 ([31])**.** *The following* (i) *and* (ii) *hold and are equivalent:*

   (i) *If $A, B \geq 0$, then for each $t \in [0, 1]$ and $r \geq t$*

$$A^{\frac{1}{2}}(A^{\frac{r-t}{2}}B^p A^{\frac{r-t}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{(p-tq)s+rq}{2ps}}\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^r B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{q}{ps}}A^{\frac{(p-tq)s+rq}{2ps}}$$

   holds for any $s \geq 1$ and $p \geq q > 0$.

   ii) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $r \geq t$*

$$A^{\frac{(p-tq)s+rq}{ps}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{\frac{p}{q}}A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q}{ps}}$$

   holds for any $s \geq 1$ and $p \geq q > 0$

Corollary 6.2 ([31])**.** *The following* (i) *and* (ii) *hold and are equivalent:*

   (i) *If $A, B \geq 0$, then for each $r \geq 0$*

$$A^{\frac{1}{2}}(A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1}{2}(1+\frac{r}{p}q)}B^q A^{\frac{1}{2}(1+\frac{r}{p}q)}$$

   holds for any $p \geq q > 0$.

(ii) *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$A^{1+\frac{r}{p}q} \geq (A^{\frac{r}{2}}B^{\frac{p}{q}}A^{\frac{r}{2}})^{\frac{q}{p}}$$

holds for any $p \geq q > 0$.

Theorem 6.3 ([31]). *If $A, B \geq 0$, then, for every $t \in [0,1]$ and $p \geq 0$,*

$$\mathrm{Tr}[A\log(A^{\frac{r-t}{2}}B^p A^{\frac{r-t}{2}})^s] \geq (r - ts)\mathrm{Tr}[A\log A] + \mathrm{Tr}[A\log\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^r B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}]$$

holds for any $r \geq t$ and $s \geq 1$.

Sketch of the proof of Theorem 6.3. Since log majorization yields weak mojorization, (ii) of Theorem 6.1 ensures the following

$$p\mathrm{Tr}[A(A^{\frac{r-t}{2}}B^p A^{\frac{r-t}{2}})^{\frac{q}{p}}] \geq \mathrm{Tr}[A^{\frac{(p-tq)s+rq}{2}}\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^p B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{q}{ps}}]$$

holds for $t \in [0,1]$, $r \geq t$, $s \geq 1$ and $p \geq q > 0$. Since both sides of the inequality stated above are equal to $\mathrm{Tr}[A]$ when $q = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}q}\mathrm{Tr}\left[A(A^{\frac{r-t}{2}}B^p A^{\frac{r-t}{2}})^{\frac{q}{p}}\right]\bigg|_{q=0} \geq \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{Tr}\left[A^{\frac{(p-tq)s+rq}{2}}\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^p B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{q}{ps}}\right]\bigg|_{q=0}$$

and the desired result follows by simple calculation of q derivation.

Theorem 6.3 easily implies the following result.

Corollary 6.4 ([31]). *If $A, B \geq 0$, then, for every $p \geq 0$ and $r \geq 0$,*

$$\mathrm{Tr}[A\log(A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^s] \geq \mathrm{Tr}[A\log A^r] + \mathrm{Tr}[A\log\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^r B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}]$$

holds for any $s \geq 1$. In particular,

$$\mathrm{Tr}[A\log(A^{\frac{r}{2}}B^p A^{\frac{r}{2}})] \geq \mathrm{Tr}[A\log A^r + A\log B^p]$$

and

$$\mathrm{Tr}[A\log(A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^2] \geq \mathrm{Tr}[A\log A^r] + \mathrm{Tr}[A\log(B^p A^r B^p)].$$

We need the following useful lemma to prove Theorem 6.7 and Theorem 6.9

Lemma 6.5 ([31]). *If A, B, C and D are Hermitian, then for any positive numbers α and β*

$$e^{A+\alpha B+\alpha\beta(C+D)} = \lim_{p\downarrow 0}\{e^{\frac{pA}{2}}(e^{\frac{pB}{2}}(e^{\frac{pC}{2}}e^{pD}e^{\frac{pC}{2}})^{\beta}e^{\frac{pB}{2}})^{\alpha}e^{\frac{pA}{2}}\}^{\frac{1}{p}}$$

in particular,

$$e^{A+\alpha(B+C)} = \lim_{p\downarrow 0}\{e^{pA}(e^{\frac{pB}{2}}e^{pC}e^{\frac{pB}{2}})^{\alpha}e^{pA}\}^{\frac{1}{p}}$$

Remark 6.6. When $C = 0$ and $\alpha = 1$, Lemma 6.5 implies the famous Lie-Trotter formula

$$e^{A+B} = \lim_{p\downarrow 0}(e^{\frac{pA}{2}}e^{pB}e^{\frac{A}{2}})^{\frac{1}{p}}.$$

When $B = -A$ and $C = B$, Lemma 6.5 implies the well known $\alpha-$ mean version of the Lie-Trotter formula by Hiai and Petz

$$e^{(1-\alpha)A+\alpha B} = \lim_{p\downarrow 0}(e^{pA}\sharp_\alpha e^{pB})^{\frac{1}{p}}.$$

We remark that by using Theorem 6.1 and Lemma 6.5, we have Theorem 6.7 and Theorem 6.9.

Theorem 6.7 ([31]). *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$\frac{s}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] - \frac{1}{p}\mathrm{Tr}[A\log\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}] \geq \mathrm{Tr}[A\log A]$$

holds for any $p \geq 0$ and $s \geq 1$, and the left hand side converges to the right hand side as $p \downarrow 0$.

Corollary 6.8 ([31]).
  (i) *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] \geq \mathrm{Tr}[A\log A + A\log B]$$

  holds and the left hand side converges to the right hand side as $p \downarrow 0$.
  (ii) *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$\frac{2}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] - \frac{1}{p}\mathrm{Tr}[A\log(B^pA^pB^p)] \geq \mathrm{Tr}[A\log A]$$

  holds and the left hand side converges to the right hand side as $p \downarrow 0$.

Theorem 6.9 ([31]). *If $A > 0$ and $B \geq 0$, then, for every positive number β,*

$$\frac{s}{p}\mathrm{Tr}[A\log(A^p\natural_\beta B^p)] - \frac{1}{p}\mathrm{Tr}[A\log\{A^{\frac{-p}{2}}(A^p\natural_\beta B^p)^sA^{\frac{-p}{2}}\}] \geq \mathrm{Tr}[A\log A]$$

holds for any $p \geq 0$, $s \geq 1$, and the left hand side converges to the right hand side as $p \downarrow 0$.

Closely related papers in this chapter: [6, 8, 31, 44]

7. (A-5) LOG-HYPONORMAL $\Longrightarrow$ CLASS A OPERATOR $\Longrightarrow$ PARANORMAL

An operator T is said to be paranormal if $||T^2x|| \geq ||Tx||^2$ for $||x|| = 1$ and T is said to be lass A operator if $|T^2| \geq |T|^2$ and also T is said to be log-hyponormal if T is invertible and $\log|T| \geq \log|T^*|$

We recall that $\log|T| \geq \log|T^*|$ implies $|T|^{2p} \geq (|T|^p|T^*|^{2p}|T|^p)^{\frac{1}{2}}$ for all $p \geq 0$ by Theorem 3.1, so that we have easily the following Theorem 7.1

Theorem 7.1 ([30, 38]). $\log|T| \geq \log|T^*| => |T^2| \geq |T|^2 => ||T^2x|| \geq ||Tx||^2$ *for $||x|| = 1$, that is,*

$$\text{log-hyponormal} \Longrightarrow \text{class A operator} \Longrightarrow \text{paranormal.}$$

We show the following interesting parallelism between Theorem 7.2 on paranormal operators and Theorem 7.3 on class A operators.

Theorem 7.2 ([47]).
  (1) *If T is a paranormal, then $||T^nx||^{\frac{1}{n}} \geq ||Tx||$ holds for every unit vector x and for all positive integer n.*
  (2) *If T is a paranormal, then T^n is also a paranormal operator for all positive integer n.*
  (3) *If T is invertible and paranormal, then T^{-1} is also a paranormal operator.*

(4) *If T is a paranormal, then*

$$||Tx|| \leq ||T^2x||^{\frac{1}{2}} \leq \cdots \leq ||T^nx||^{\frac{1}{n}}$$

holds for every unit vector x and for all positive integer n.

Theorem 7.3 ([47]).
   (1) *If T is an invertible class A operator, then $|T^n|^{\frac{2}{n}} \geq |T|^2$ holds for all positive integer n.*
   (2) *If T is an invertible class A operator, then T^n is also a class A operator for all positive integer n.*
   (3) *If T is an invertible class A operator, then T^{-1} is also a class A operator.*
   (4) *If T is an invertible class A operator , then $|T|^2 \leq |T^2| \cdots \leq |T^n|^{\frac{2}{n}}$ holds for all positive integer n.*

Some of closely related papers in this chapter: [30, 38, 46, 47, 65, 71, 73].

8. (A-9) Furuta inequality of indefinite type on Krein space

Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices. For a selfadjoint involution, $J = J^*$ and $J^2 = I$, we consider an indefinite inner product $[,]$ on $\mathbb{C}^n$ given by

$$[x, y] = \langle Jx, y \rangle \qquad (x, y \in \mathbb{C}^n)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^n$.

The J- adjoint matrix $A^\sharp$ of A is defined by

$$[Ax, y] = [x, A^\sharp y] \qquad (x, y \in \mathbb{C}^n)$$

equivalently, $A^\sharp = JA^*J$.

A matrix A is said to be J-selfadjoit if $A^\sharp = A$ or JA is selfadjoint: $JA = A^*J$.

For a pair of J-selfadjoint matrices A, B, the J-order, denoted as $A \geq^J B$, is defined by

$$[Ax, x] \geq [Bx, x] \qquad (x \in \mathbb{C}^n),$$

that is, $JA \geq JB$.

A matrix A is called J-positive if $[Ax, x] \geq 0$ for $x \in \mathbb{C}^n$, that is, $JA \geq 0$.

A matrix A is said to be a J-contraction if $I \geq^J A^\sharp A$ or $[x, x] \geq [Ax, Ax]$ for $x \in \mathbb{C}^n$.

Theorem 8.1 ([62]). *Let A, B be J-selfadjoint matrices with non-negative eigenvalues and $I \geq^J A \geq^J B$. Then for each $r \geq 0$,*

$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for $p \geq 0$, $q \geq 1$ with $(1 + r)q \geq p + r$.

As an application of Theorem 8.1, the following characterization of the J-chaotic order has been obtained.

Theorem 8.2 ([63]). *If A, B are J-selfadjoint matrices with positive eigenvalues and $I \geq^J A$ and $I \geq^J B$. Then the following statements are equivalent:*
   (i) $\text{Log}(A) \geq^J \text{Log}(B)$
   (ii) $A^r \geq^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all $p \geq 0$ and $r \geq 0$.*

Remark 8.3. Theorem 8.1 is regarded as Theorem F of indefinite type (compare Theorem 8.1 with Theorem F). Also Theorem 8.2 is regarded as Theorem 3.1 of indefinite type (compare Theorem 8.2 with Theorem 3.1)

Some of closely related papers in this chapter: [7, 9, 10, 62, 63].

9. (C-2) POSITIVE SEMIDEFINITE SOLUTIONS OF THE OPERATOR EQUATION

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$$

In [13], the following result is shown; let A be positive definite matrix and B is positive semidefinite matrix. The solution X of the following matrix equation is always positive semidefinite

$$A^2 X + X B^2 = AB + BA.$$

In [13], the following question was posed. How can one characterize all the functions f such that the solution of the matrix equation

$$f(A)X + Xf(B) = AB + BA$$

is positive semidefinite?

Although Theorem F in §1 itself is *operator inequality*, we show that Theorem F is useful to discuss positive semidefinite solutions of the following operator equation:

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$$

where B is of special type.

We need the following lemma to prove Theorem 9.2 which is the main result.

Lemma 9.1 ([35]). *Let A be a positive definite matrix and B be a positive semi-definite matrix. Let m be a natural number and $t \geq 0$. Let the following equation be the polynomial expansion of $(A + tB)^m$ with respect to t:*

$$(A+tB)^m = A^m + tF_1(A,B,m) + t^2 F_2(A,B,m) + \cdots + t^j F_j(A,B,m) + \cdots t^m B^m$$

Then $F_1(A,B,m)$ can be expressed as

$$F_1(A,B,m) = A^{m-1}B + A^{m-2}BA + \cdots + A^{m-j}BA^{j-1} + \cdots + BA^{m-1}.$$

Theorem 9.2 ([35]).

Let A be a positive definite operator and B be a positive semidefinite operator. Let m and n be natural numbers. There exists positive semidefinite operator solution X of the following operator equation:

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{nr}{2(m+r)}} \left(\sum_{j=1}^{m} A^{\frac{n(m-j)}{m+r}} B A^{\frac{n(j-1)}{m+r}} \right) A^{\frac{nr}{2(m+r)}}$$

m for r such that
$$\begin{cases} r \geq 0 & \text{if } n \geq m & \text{(i)} \\ r \geq \dfrac{m-n}{n-1} & \text{if } m \geq n \geq 2 & \text{(ii)}. \end{cases}$$

Sketch of the proof of Theorem (9.2)**.** The inequality (i) of Theorem F and Theorem LH ensure

$$A \geq B \geq 0 \text{ ensures } (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}\alpha} \geq B^{(1+r)\alpha} \text{ for } p \geq 1, r \geq 0 \text{ and } \alpha \in [0,1] \quad (9.1)$$

Since $A + tB \geq B$ holds for $t \geq 0$, so that we replace A by $A + tB$ and B by A in (9.1) and we have

$$(A^{\frac{r}{2}} (A + tB)^m A^{\frac{r}{2}})^{\frac{1+r}{m+r}\alpha} \geq A^{(1+r)\alpha} \text{ for } m \geq 1, t \geq 0, r \geq 0 \text{ and } \alpha \in [0,1] \quad (9.2)$$

For $\frac{1+r}{m+r}\alpha = \frac{1}{n}$ in (9.2), we take α as follows: $\alpha = \frac{m+r}{n(1+r)} \in [0,1]$ *for r such that*

$$\begin{cases} r \geq 0 & \text{if } n \geq m \quad \text{(i)} \\ r \geq \dfrac{m-n}{n-1} & \text{if } m \geq n \geq 2 \quad \text{(ii)}. \end{cases}$$

Then (9.2) implies

$$Y(t) = [A^{\frac{r}{2}}(A+tB)^m A^{\frac{r}{2}}]^{\frac{1}{n}} \geq A^{\frac{m+r}{n}} \text{ for } r \text{ under the condition (i) or (ii).} \quad (9.3)$$

Then (9.3) ensures $Y(t) \geq Y(0) = A^{\frac{m+r}{n}}$ for any $t \geq 0$. Therefore

$$X = Y'(0) \geq 0. \quad (9.4)$$

Differentiating the equation $Y^n(t) = A^{\frac{r}{2}}(A+tB)^m A^{\frac{r}{2}}$ and then letting $t = 0$,

$$Y(0)^{n-1} Y'(0) + \cdots + Y(0)^{n-j} Y'(0) Y(0)^{j-1} + \cdots + Y'(0) Y(0)^{n-1}$$

$$= \frac{\mathrm{d}}{\mathrm{dt}} [A^{\frac{r}{2}}(A+tB)^m A^{\frac{r}{2}}]_{t=0}$$

$$= A^{\frac{r}{2}} (A^{m-1} B + A^{m-2} BA + \cdots + A^{m-j} BA^{j-1} + \cdots + BA^{m-1}) A^{\frac{r}{2}} \text{ by Lemma } 9.1$$

and we have the following operator equation for $X = Y'(0)$ since $Y(0) = A^{\frac{m+r}{n}}$ holds:

$$A^{\frac{(m+r)(n-1)}{n}} X + A^{\frac{(m+r)(n-2)}{n}} XA^{\frac{(m+r)}{n}} + \cdots + A^{\frac{(m+r)(n-j)}{n}} XA^{\frac{(m+r)(j-1)}{n}} + \cdots$$

$$+ XA^{\frac{(m+r)(n-1)}{2}} A^{\frac{r}{2}} (A^{m-1} B + A^{m-2} BA + \cdots + A^{m-j} BA^{j-1} + \cdots$$

$$+ A^{m-2} BA + \cdots + A^{m-j} BA^{j-1} + \cdots + BA^{m-1}) A^{\frac{r}{2}} \quad (9.5)$$

and we can replace A by $A^{\frac{n}{m+n}}$ in (9.5) and (9.5) can be rewritten as

$$\sum_{j=1}^{n} A^{n-j} XA^{j-1} = A^{\frac{nr}{2(m+r)}} \left(\sum_{j=1}^{m} A^{\frac{n(m-j)}{m+r}} BA^{\frac{n(j-1)}{m+r}} \right) A^{\frac{nr}{2(m+r)}}$$

for r such that $\begin{cases} r \geq 0 & \text{if } n \geq m \quad \text{(i)} \\ r \geq \dfrac{m-n}{n-1} & \text{if } m \geq n \geq 2 \quad \text{(ii)}. \end{cases}$

Corollary 9.3. [35] *Let A be a positive definite operator and B be a positive semidefinite operator. There exists positive semidefinite operator solution X of the following operator equation* (i),(ii), (iii), (iv) *and* (v) *respectively:*

(i) $A^{\frac{2+r}{2}} X + XA^{\frac{2+r}{2}} = A^{\frac{r}{2}}(AB + BA)A^{\frac{r}{2}}$ *for $r \geq 0$.*

(ii) $A^{\frac{(2+r)2}{3}}X + A^{\frac{2+r}{3}}XA^{\frac{2+r}{3}} + XA^{\frac{(2+r)2}{3}} = A^{\frac{r}{2}}(AB + BA)A^{\frac{r}{2}}$ for $r \geq 0$.

(iii) $A^{\frac{(3+r)2}{3}}X + A^{\frac{3+r}{3}}XA^{\frac{3+r}{3}} + XA^{\frac{(3+r)2}{3}} = A^{\frac{r}{2}}(A^2B + ABA + BA^2)A^{\frac{r}{2}}$ for $r \geq 0$

(iv) $A^{\frac{3+r}{2}}X + XA^{\frac{3+r}{2}} = A^{\frac{r}{2}}(A^2B + ABA + BA^2)A^{\frac{r}{2}}$ for $r \geq 1$.

(v) $A^{\frac{5+r}{2}}X + XA^{\frac{5+r}{2}} = A^{\frac{r}{2}}(A^4B + A^3BA + A^2BA^2 + ABA^3 + BA^4)A^{\frac{r}{2}}$ for $r \geq 3$.

Proposition 9.4 ([35])**.** *Let the diagonal matrix* $A = \mathrm{diag}(a_1, a_2, \cdots, a_l)$ *with each* $a_j > 0$ *and* B *be the* $l \times l$ *matrix all of whose entries are 1. Let* m *and* n *be natural numbers. There exists positive semidefinite matrix solution* X *of the following matrix equation:*

$$\sum_{j=1}^{n} A^{\frac{(m+r)(n-j)}{n}} X A^{\frac{(m+r)(j-1)}{n}} = A^{\frac{r}{2}}\left(\sum_{j=1}^{m} A^{m-j}BA^{j-1}\right)A^{\frac{r}{2}}$$

for r *such that* $\begin{cases} r \geq 0 & \text{if } n \geq m & (i) \\ r \geq \dfrac{m-n}{n-1} & \text{if } m \geq n \geq 2 & (ii). \end{cases}$

The positive semidefinite matrix solution X *can be expressed as:*

$$X = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} \left(\sum_{k=1}^{m} a_i^{m-k} a_j^{k-1}\right)}{\sum_{k=1}^{n} a_i^{\frac{(m+r)(n-k)}{n}} a_j^{\frac{(m+r)(k-1)}{n}}} \right)_{i,j=1,2,\ldots,l}.$$

Examples of positive semidefinite matrices. Let the diagonal matrix $A = (a_1, a_2, \cdots, a_n)$ with each $a_j > 0$ and B be $n \times n$ matrix all of whose entries are 1. Then the positive semidefinite solutions X_i of (i),(ii),(iii),(iv) and (v) of Corollary 2 are given by:

$$X_1 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i + a_j)}{a_i^{\frac{2+r}{2}} + a_j^{\frac{2+r}{2}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 0.$$

$$X_2 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i + a_j)}{a_i^{\frac{2(2+r)}{3}} + a_i^{\frac{2+r}{3}} a_j^{\frac{2+r}{3}} + a_j^{\frac{2(2+r)}{3}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 0.$$

$$X_3 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i^2 + a_i a_j + a_j^2)}{a_i^{\frac{2(3+r)}{3}} + a_i^{\frac{3+r}{3}} a_j^{\frac{3+r}{3}} + a_j^{\frac{2(3+r)}{3}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 0.$$

$$X_4 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i^2 + a_i a_j + a_j^2)}{a_i^{\frac{3+r}{2}} + a_j^{\frac{3+r}{2}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 1.$$

$$X_5 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i^4 + a_i^3 a_j + a_i^2 a_j^2 + a_i a_j^3 + a_j^4)}{a_i^{\frac{5+r}{2}} + a_j^{\frac{5+r}{2}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 3.$$

Some of closely related papers in this chapter: [4, 11, 12, 13, 35, 60, 61, 80, 81].

10. (A-6) Further extensions of order preserving operator inequalities

We recall the following order preserving operator inequalities:

(i) $A \geq B \geq 0 \implies A^{1+r} \geq (A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$.

(ii) $A \geq B \geq 0$ with $A > 0 \implies A^{1-t+r} \geq \{(A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}})\}^{\frac{1-t+r}{(p-t)s+r}}$ for $t \in [0,1]$, $r \geq t$, $p \geq 1$ and $s \geq 1$.

In fact (i) is the essential part of Theorem F in §1 (see Remark 1.4) and also (ii) is (G-1) itself of Theorem G in §2 which is an extension of (i).

In this chapter we study further extensions of order preserving operator inequalities including (i) and (ii) by applications of Theorem F and Lemma A.

Theorem 10.1 ([32]). *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \cdots, p_{2n} \geq 1$ for natural number n. Then the following inequality holds for $r \geq t$:*

$$A^{1-t+r}$$

$$\geq \left\{ A^{\frac{r}{2}} \Big[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\cdots\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n\ times\ and\ A^{\frac{t}{2}}\ n-1\ times\ by\ turns} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\cdots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n\ times\ and\ A^{\frac{t}{2}}\ n-1\ times\ by\ turns} \Big]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{\varphi[2n;r,t]}}$$

where $\varphi[2n; r, t]$ is defined by

$$\varphi[2n; r, t] = \underbrace{\left\{\ldots\ldots[\{[(p_1 - t)p_2 + t]p_3 - t\}p_4 + t]p_5 - \cdots - t\right\}p_{2n} + r}_{\text{-t appears n times and t appears n-1 times by turns}}$$

$$= r + \prod_{i=1}^{2n} p_i + \underbrace{\left(\prod_{i=3}^{2n} p_i + \prod_{i=5}^{2n} p_i + \cdots + \prod_{i=7}^{2n} p_i + \cdots + p_{2n-1}p_{2n}\right)t}_{n\text{-1 terms}}$$

$$- \underbrace{\left(\prod_{i=2}^{2n} p_i + \prod_{i=4}^{2n} p_i + \prod_{i=6}^{2n} p_i + \cdots + p_{2(n-1)}p_{2n-1}p_{2n} + p_{2n}\right)t}_{n\ terms}.$$

Theorem 10.1 easily yields the following result.

Corollary 10.2 ([32]). *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geq 1$,*

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}}\Big[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\Big]^{p_4}A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $r \geq t$, where $[\{(p_1 - t)p_2 + t\}p_3 - t]p_4 + r = p_1 p_2 p_3 p_4 + (p_3 p_4 - p_2 p_3 p_4 - p_4)t + r$.

Theorem 10.3 ([33]). *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \cdots, p_{2n} \geq 1$ for natural number n. Then*

$$G_{A,B}[r, p_{2n}] = A^{\frac{-r}{2}}$$

$$\cdot \left\{ A^{\frac{r}{2}} \Big[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\cdots\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n\ times\ and\ A^{\frac{t}{2}}\ n-1\ times\ by\ turns} \quad B^{p_1} \quad \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\cdots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n\ times\ and\ A^{\frac{t}{2}}\ n-1\ times\ by\ turns} \Big]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1+r-t}{\varphi[2n;r,t]}} A^{\frac{-r}{2}}$$

is a decreasing function of $p_{2n} \geq 1$ and $r \geq t$, and the following inequality holds $G_{A,A}[r, p_{2n}] \geq G_{A,B}[r, p_{2n}]$, that is,

$$A^{1-t+r}$$

$$\geq \left\{ A^{\frac{r}{2}} \Big[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n\ times\ and\ A^{\frac{t}{2}}\ n-1\ times\ by\ turns} \quad B^{p_1} \quad \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\}A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n\ times\ and\ A^{\frac{t}{2}}\ n-1\ times\ by\ turns} \Big]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{\varphi[2n;r,t]}}.$$

Corollary 10.4 ([32]). *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geq 1$,*

$$G_{A,B}[r, p_4] = A^{\frac{-r}{2}} \left\{ A^{\frac{r}{2}} \Big[A^{\frac{-t}{2}} \{ A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3} A^{\frac{-t}{2}} \Big]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}} A^{\frac{-r}{2}}$$

is a decreasing function of $p_4 \geq 1$ and $r \geq t$, and the following inequality holds $G_{A,A}[r, p_4] \geq G_{A,B}[r, p_4]$, that is,

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \Big[A^{\frac{-t}{2}} \{ A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3} A^{\frac{-t}{2}} \Big]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $t \in [0,1]$, $r \geq t$ and $p_1, p_2, p_3, p_4 \geq 1$.

Theorem 10.5 ([33]). *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \cdots, p_{2n} \geq 1$, then*

$$A \geq B$$

$$\geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{\frac{1}{(p_1-t)p_2+t}}$$

$$\geq \left\{ A^{\frac{t}{2}}\Big[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\Big]^{p_4}A^{\frac{t}{2}} \right\}^{\frac{1}{\{[(p_1-t)p_2+t]p_3-t\}p_4+t}}$$

$$\cdots$$

$$\geq \left[\underbrace{A^{\frac{t}{2}}\{A^{\frac{-t}{2}}\Big[A^{\frac{t}{2}}\cdots\Big[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ and\ A^{\frac{t}{2}}\ alternately\ n\ times} \quad B^{p_1} \quad \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\Big]^{p_4}\cdots A^{\frac{t}{2}}\Big]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ and\ A^{\frac{t}{2}}\ alternately\ n\ times} \right]^{\frac{1}{q[2n]}}$$

$$(10.1)$$

where $q[2n]$ is defined by

$$q[2n] = \underbrace{\left\{\ldots..[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\cdots-t\right\}p_{2n}+t}_{-t\ and\ t\ alternatively\ n\ times\ appear}.$$

Although Corollary 10.6 is nothing but a simple corollary of Theorem 10.1, we shall show an interesting relation among Corollary 10.6, Theorem G, Theorem F, and log majorization (Theorem AH under below) by Ando–Hiai [8].

In fact, we recall in Remark 5.4 that Theorem G interpolates Theorem F and an inequality equivalent to this log majorization.

Corollary 10.6 ([33]). *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$, $p_1, p_2, p_3, p_4 \geq 1$ and $r \geq t$, then*

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[A^{\frac{-t}{2}} \left\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \right\}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

$p_2 = p_3 = 1 \qquad \downarrow$

Theorem 10.7 (G). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*
$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}}$$
holds for $r \geq t$ and $s \geq 1$.

$t = 0$ and $s = 1$ $\swarrow$ $\qquad\qquad$ $t = 1$ and $r = s$ $\searrow$

Theorem 10.8 (F).

$A \geq B \geq 0 \Longrightarrow A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$

for $p \geq 1$ and $r \geq 0$.

If $A \geq B \geq 0$ with $A > 0$, then

$$A^r \geq \{ A^{\frac{r}{2}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r A^{\frac{r}{2}} \}^{\frac{1}{p}}$$

for $r, p \geq 1$.

$\Updownarrow$

Theorem 10.9 (AH). *For every $A, B \geq 0$, $0 \leq \alpha \leq 1$ and $r \geq 1$*
$$(A \#_\alpha B)^r \underset{(\log)}{\succ} A^r \#_\alpha B^r$$

Some of closely related papers in this chapter: [15, 16, 18, 19, 20, 21, 24, 25, 30, 32, 34, 39, 40, 41, 44, 48, 49, 52, 54, 55, 56, 57, 58, 59, 67, 72, 74, 75, 76, 77, 78, 79].

11. (A-6) Operator functions on chaotic order involving order preserving operator inequalities

The purpose of this paper is to emphasize that the chaotic order $A \gg B$ is sometimes more convenient and more useful than the usual order $A \geq B \geq 0$.

Definitions of $\mathbb{C}_{A,B}\big[n; p_1, p_2, \cdots, p_{n-1}, p_n \big| r_1, r_2, \cdots, r_{n-1}, r_n\big]$, (*denoted by $\mathbb{C}_{A,B}[n]$ or $\mathbb{C}_{[n]}$ briefly*) *and* $\mathfrak{q}\big[n; p_1, p_2, \cdots, p_{n-1}, p_n \big| r_1, r_2, \cdots, r_{n-1}, r_n\big]$ (*denoted by $\mathfrak{q}[n]$ briefly*):

Let $A, B \geq 0$, $p_1, p_2, \cdots, p_n \geq 0$ and $r_1, r_2, \cdots, r_n \geq 0$ for a natural number n. Let $\mathbb{C}_{A,B}\big[n; p_1, p_2, \cdots, p_{n-1}, p_n \big| r_1, r_2, \cdots, r_{n-1}, r_n\big]$ be defined by

$$\mathbb{C}_{A,B}\big[n; p_1, p_2, \cdots, p_{n-1}, p_n | r_1, r_2, \cdots, r_{n-1}, r_n\big]$$
$$= A^{\frac{r_n}{2}}\Big\{A^{\frac{r_{n-1}}{2}}[\cdots A^{\frac{r_3}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{p_3}A^{\frac{r_3}{2}}\cdots]^{p_{n-1}}A^{\frac{r_{n-1}}{2}}\Big\}^{p_n}A^{\frac{r_n}{2}}$$

$$(11.1)$$

For examples,

$$\mathbb{C}_{A,B}[1] = A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}} \text{ and } \mathbb{C}_{A,B}[2] = A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}$$

Particularly put $A = B$ in $\mathbb{C}_{A,B}\big[n\big]$ in (11.1). Then

$$\mathbb{C}_{A,A}\big[n; p_1, p_2, \cdots, p_{n-1}, p_n | r_1, r_2, \cdots, r_{n-1}, r_n\big]$$
$$= A^{\frac{r_n}{2}}\Big\{A^{\frac{r_{n-1}}{2}}[\cdots A^{\frac{r_3}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}A^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{p_3}A^{\frac{r_3}{2}}\cdots]^{p_{n-1}}A^{\frac{r_{n-1}}{2}}\Big\}^{p_n}A^{\frac{r_n}{2}}$$
$$= A^{[\cdots\{(p_1+r_1)p_2+r_2\}p_3+\cdots r_{n-1}]p_n+r_n.}$$

$$(11.2)$$

Next let $\mathfrak{q}\big[n; p_1, p_2, \cdots, p_{n-1}, p_n | r_1, r_2, \cdots, r_{n-1}, r_n\big]$ be defined by
$\mathfrak{q}\big[n; p_1, p_2, \cdots, p_{n-1}, p_n | r_1, r_2, \cdots, r_{n-1}, r_n\big] = $ the exponential power of A in (11.2)

$$= [\cdots\{(p_1+r_1)p_2+r_2\}p_3+\cdots r_{n-1}]p_n+r_n. \qquad (11.3)$$

$\mathfrak{q}\big[n; p_1, p_2, \cdots, p_{n-1}, p_n | r_1, r_2, \cdots, r_{n-1}, r_n\big]$ denoted by $\mathfrak{q}\big[p_1, p_2, \cdots, p_{n-1}, p_n\big]$ or denoted by $\mathfrak{q}\big[r_1, r_2, \cdots, r_{n-1}, r_n\big]$ or by $\mathfrak{q}\big[n\big]$ briefly.

For examples, $\mathfrak{q}[1] = p_1 + r_1$ and $\mathfrak{q}[2] = (p_1 + r_1)p_2 + r_2$

For the sake of convenience, we define

$$\mathbb{C}_{A,B}[0] = B \text{ and } \mathfrak{q}[0] = 1. \qquad (11.4)$$

We have the following basic and fundamental relations.

$$\mathbb{C}_{A,B}[n] = A^{\frac{r_n}{2}}\mathbb{C}_{A,B}[n-1]^{p_n}A^{\frac{r_n}{2}} \qquad (11.5)$$

$$\mathfrak{q}[n] = \mathfrak{q}[n-1]p_n + r_n \qquad (11.6)$$

In this chapter §11, we shall state further extensions of the results in §2 and §3.

By using Theorem 3.1 in §3 and Mathematical Induction we can easily show the following result.

Theorem 11.1 ([36]). *Let $A \gg B$ and $r_1, r_2, \cdots, r_n \geq 0$ for a natural number n. Then the following inequality holds,*

$$A^{r_1+r_2\cdots+r_n} = \mathbb{C}_{A,A}[n]^{\frac{r_1+r_2\cdots+r_n}{\mathfrak{q}[n]}} \geq \mathbb{C}_{A,B}[n]^{\frac{r_1+r_2\cdots+r_n}{\mathfrak{q}[n]}} \qquad (11.7)$$

for $p_1, p_2, \cdots, p_n$ satisfying

$$p_j \geq \frac{r_1 + r_2 + \cdots + r_{j-1}}{\mathfrak{q}[j-1]} \text{ for } j = 1, 2, \cdots, n \quad (r_0 = 0 \text{ and } \mathfrak{q}[0] = 1), \qquad (11.8)$$

that is,

$$p_1 \geq 0, p_2 \geq \frac{r_1}{p_1+r_1}, p_3 \geq \frac{r_1+r_2}{(p_1+r_1)p_2+r_2}, \cdots, p_n \geq \frac{r_1+r_2+\cdots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

Corollary 11.2 ([36]). *Let $A \gg B$ and $r_1, r_2, r_3 \geq 0$. Then*

(i) $A^{r_1+r_2+r_3} \geq \{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{r_1+r_2+r_3}{[(p_1+r_1)p_2+r_2]p_3+r_3}}$.
   holds for $p_2 \geq \frac{r_1}{p_1+r_1}$ and $p_3 \geq \frac{r_1+r_2}{(p_1+r_1)p_2+r_2}$.

(ii) $A^{r_1+r_2} \geq \{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{r_1+r_2}{(p_1+r_1)p_2+r_2}}$
   holds for $p_1 \geq 0$ and $p_2 \geq \frac{r_1}{p_1+r_1}$.

Similarly we have the following two results on usual order which are contained in Corollary 11.7 and also this Corollary 11.7 is a simple corollary of the forthcoming Theorem 11.6 on chaotic order.

Theorem 11.3 ([36]). *Let $A \geq B \geq 0$ and $r_1, r_2, \cdots, r_n \geq 0$ for a natural number n. Then the following inequality holds,*

$$A^{1+r_1+r_2\cdots+r_n} = \mathbb{C}_{A,A}[n]^{\frac{1+r_1+r_2\cdots+r_n}{\mathfrak{q}[n]}} \geq \mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2\cdots+r_n}{\mathfrak{q}[n]}} \qquad (11.9)$$

for $p_1, p_2, \cdots, p_n$ satisfying

$$p_1 \geq 1, p_2 \geq \frac{1+r_1}{p_1+r_1}, p_3 \geq \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}, \cdots, p_n \geq \frac{1+r_1+r_2+\cdots+r_{n-1}}{\mathfrak{q}[n-1]}. \qquad (11.10)$$

Corollary 11.4 ([46]). *Let $A \geq B \geq 0$ and $r_1, r_2, r_3 \geq 0$. Then*

(i) $A^{1+r_1+r_2+r_3} \geq \{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{1+r_1+r_2+r_3}{[(p_1+r_1)p_2+r_2]p_3+r_3}}$
   holds for $p_1 \geq 1$, $p_2 \geq \frac{1+r_1}{p_1+r_1}$ and $p_3 \geq \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}$.

(ii) $A^{1+r_1+r_2} \geq \{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}}$
   holds for $p_1 \geq 1$ and $p_2 \geq \frac{1+r_1}{p_1+r_1}$.

Theorem 11.5 ([36]). *Let $A \gg B$ and $r_1, r_2, \cdots, r_n \geq 0$ for a natural number n. For any fixed $\delta \geq 0$, let $p_1, p_2, \cdots, p_n$ be satisfied by*

$$p_1 \geq \delta, p_2 \geq \frac{\delta+r_1}{p_1+r_1}, \cdots, p_k \geq \frac{\delta+r_1+r_2+\cdots+r_{k-1}}{\mathfrak{q}[k-1]}, \cdots, p_n \geq \frac{\delta+r_1+r_2+\cdots+r_{n-1}}{\mathfrak{q}[n-1]}. \qquad (11.11)$$

The operator function $\mathfrak{F}_k(p_k, r_k)$ for any natural number k such that $1 \leq k \leq n$ is defined by

$$\mathfrak{F}_k(p_k, r_k) = A^{\frac{-r_k}{2}}\mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+\cdots+r_k}{\mathfrak{q}[k]}}A^{\frac{-r_k}{2}}. \qquad (11.12)$$

Then the following inequality holds:

$$A^{\frac{r_{k-1}}{2}}\mathfrak{F}_{k-1}(p_{k-1}, r_{k-1})A^{\frac{r_{k-1}}{2}} \geq \mathfrak{F}_k(p_k, r_k) \quad (\mathfrak{F}_0(p_0, r_0) = B^{\delta}) \qquad (11.13)$$

for every natural number k such that $1 \leq k \leq n$.

Proof. Since $\mathbb{C}_{A,B}[0] = B$, $\mathfrak{q}[0] = 1$ in (11.4) and $p_0 = r_0 = 0$ in (11.8), we may define $\mathfrak{F}_0(p_0, r_0) = B^{\delta}$ in (11.13). Let $A \gg B$. Then for any fixed $\delta \geq 0$,

$$B^{\delta} \geq A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{\delta+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}} \text{ for } p_1 \geq \delta \text{ and } r_1 \geq 0 \qquad (11.14)$$

since $F_{A,B}(\delta, r_1) \geq F_{A,B}(p_1, r_1)$ holds by (iii) of Theorem 3.2 in §3. And (11.14) can be expressed as

$$B^{\delta} = A^{\frac{r_0}{2}}\mathfrak{F}_0(p_0, r_0)A^{\frac{r_0}{2}} \geq A^{\frac{-r_1}{2}}\mathbb{C}_{A,B}[1]^{\frac{\delta+r_1}{\mathfrak{q}[1]}}A^{\frac{-r_1}{2}} = \mathfrak{F}_1(p_1, r_1) \text{ by (11.12).} \qquad (11.15)$$

Since the condition (11.11) with $\delta \geq 0$ suffices (11.8) in Theorem 11.1, in fact, (11.8) is itself (11.11) without $\delta \geq 0$, we can apply Theorem 11.1 and we have the following (11.16) for natural number k such that $1 \leq k \leq n$

$$A^{r_1+r_2+\cdots+r_k} \geq \mathbb{C}_{A,B}[k]^{\frac{r_1+r_2+\cdots+r_k}{\mathfrak{q}[k]}}. \tag{11.16}$$

Since $X \geq Y > 0$ implies $X \gg Y$ and then $X^t \gg Y^t$ holds for any $t \geq 0$, (11.16) ensures

$$A^{\delta+r_1+r_2+\cdots+r_k} \gg \mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+\cdots+r_k}{\mathfrak{q}[k]}}.$$

Put $A_1 = A^{\delta+r_1+r_2+\cdots+r_k}$ and $B_1 = \mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+\cdots+r_k}{\mathfrak{q}[k]}}$ and applying (11.14) for $\delta = 1$ and $A_1 \gg B_1$, we have

$$B_1 \geq A_1^{\frac{-r}{2}}(A_1^{\frac{r}{2}}B_1^p A_1^{\frac{r}{2}})^{\frac{1+r}{p+r}} A_1^{\frac{-r}{2}} \text{ holds for } p \geq 1 \text{ and } r \geq 0. \tag{11.17}$$

Put $r_{k+1} = r(\delta + r_1 + r_2 + \cdots + r_k)$ in (11.17). Then (11.17) can be rewritten by

$$B_1 \geq A^{\frac{-r_{k+1}}{2}}(A^{\frac{r_{k+1}}{2}}\mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+\cdots+r_k}{\mathfrak{q}[k]}p}A^{\frac{r_{k+1}}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r_{k+1}}{2}}. \tag{11.18}$$

Put $p = \frac{\mathfrak{q}[k]}{\delta+r_1+\cdots+r_k}p_{k+1} \geq 1$, that is, $p_{k+1} \geq \frac{\delta+r_1+\cdots+r_k}{\mathfrak{q}[k]}$ in (11.18), then we have

$$A^{\frac{r_k}{2}}\mathfrak{F}_k(p_k, r_k)A^{\frac{r_k}{2}} = \mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+\cdots+r_k}{\mathfrak{q}[k]}} = B_1 \text{ by (11.12)}$$

$$\geq A^{\frac{-r_{k+1}}{2}}(A^{\frac{r_{k+1}}{2}}\mathbb{C}_{A,B}[k]^{p_{k+1}}A^{\frac{r_{k+1}}{2}})^{\frac{\delta+r_1+r_2+\cdots+r_k+r_{k+1}}{\mathfrak{q}[k]p_{k+1}+r_{k+1}}}A^{\frac{-r_{k+1}}{2}}$$

$$= A^{\frac{-r_{k+1}}{2}}(\mathbb{C}_{A,B}[k+1])^{\frac{\delta+r_1+r_2+\cdots+r_k+r_{k+1}}{\mathfrak{q}[k+1]}}A^{\frac{-r_{sk+1}}{2}}$$

$$\text{by (11.5) and (11.6)}$$

$$= \mathfrak{F}_{k+1}(p_{k+1}, r_{k+1}) \text{ by (11.12) for } k+1 \tag{11.19}$$

and we have (11.13) for k such that $1 \leq k \leq n$ by (11.19) and (11.15) since (11.15) means (11.13) for $k = 1$. $\square$

Theorem 11.5 easily implies Theorem 11.6 and Corollary 11.7.

Theorem 11.6 ([36]). *Let $A \gg B$ and $r_1, r_2, \cdots, r_n \geq 0$ for a natural number n. Then the following inequalities hold for any fixed $\delta \geq 0$:*

$$B^\delta \geq A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{\delta+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}}$$

$$\geq A^{\frac{-(r_1+r_2)}{2}}\left\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\right\}^{\frac{\delta+r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}}$$

$$\geq A^{\frac{-(r_1+r_2+r_3)}{2}}\left\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\right\}^{\frac{\delta+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}}.$$

$$\cdots$$

$$\geq A^{\frac{-(r_1+r_2+\cdots+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2+\cdots+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+\cdots+r_n)}{2}}$$

for $p_1, p_2, \cdots, p_n$ satisfying

$$p_1 \geq \delta, p_2 \geq \frac{\delta+r_1}{p_1+r_1}, \cdots, p_k \geq \frac{\delta+r_1+r_2+\cdots+r_{k-1}}{\mathfrak{q}[k-1]}, \cdots, p_n \geq \frac{\delta+r_1+r_2+\cdots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

Corollary 11.7 ([36]). *Let $A \geq B \geq 0$ and $r_1, r_2, \cdots, r_n \geq 0$ for a natural number n. Then*

$$A \geq B \geq A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}}$$

$$\geq A^{\frac{-(r_1+r_2)}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}}$$

$$\geq A^{\frac{-(r_1+r_2+r_3)}{2}}\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{1+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}}$$

$$\cdots$$

$$\geq A^{\frac{-(r_1+r_2+\cdots+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2+\cdots+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+\cdots+r_n)}{2}}$$

holds for $p_1, p_2, \cdots, p_n$ satisfying

$$p_1 \geq 1, p_2 \geq \frac{1+r_1}{p_1+r_1}, p_3 \geq \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}, \cdots, p_n \geq \frac{1+r_1+r_2+\cdots+r_{n-1}}{q[n-1]}.$$

By using Theorem F and Lemma A in §1, we have the following Theorem, which is further extension of both Theorem 2.4 in §2 and Theorem 3.2 in §3

Theorem 11.8 ([36]). *Let $A \gg B$ and $r_1, r_2, \cdots, r_n \geq 0$ for a natural number n. For any fixed $\delta \geq 0$, let $p_1, p_2, \cdots, p_n$ be satisfied by*

$$p_1 \geq \delta, p_2 \geq \frac{\delta+r_1}{p_1+r_1}, \cdots, p_k \geq \frac{\delta+r_1+r_2+\cdots+r_{k-1}}{q[k-1]}, \cdots, p_n \geq \frac{\delta+r_1+r_2+\cdots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

Then

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}}\mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2+\cdots+r_n}{\mathfrak{q}[n]}}A^{\frac{-r_n}{2}}$$

is a decreasing function of both $r_n \geq 0$ and p_n which satisfies

$$p_n \geq \frac{\delta+r_1+r_2+\cdots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

Corollary 11.9 ([36]). *Let $A \gg B$ and $r_1, r_2, \cdots, r_n \geq 0$ and also $p_1, p_2, \cdots, p_n \geq 1$ for a natural number n. Then*

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}}\mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2+\cdots+r_n}{\mathfrak{q}[n]}}A^{\frac{-r_n}{2}}$$

is a decreasing function of both $r_n \geq 0$ and $p_n \geq 1$.

We remark that we can give an alternative proof of Theorem 11.5 via Theorem 11.8.

Theorem 11.6 can be considered as further extension of the following result.

Theorem 11.10 (FKN. [22]). *If $A \gg B$ for $A, B > 0$, then*

$$A^{t-r}\natural_{\frac{1+r-t}{(p-t)s+r}}(A^t\natural_s B^p) \leq A^t\natural_{\frac{1-t}{p-t}}B^p \leq B$$

holds for $p \geq 1$, $s \geq 1$, $r \geq 0$ and $t < 0$.

Some of closely related papers in this chapter: [15, 15, 18, 20, 21, 22, 28, 30, 32, 33, 36, 49].

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