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GEOMETRY OF THE LEFT ACTION OF THE p-SCHATTEN GROUPS

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Abstract. Let $\mathcal{H}$ be an infinite dimensional Hilbert space, $\mathcal{B}_p(\mathcal{H})$ the p-Schatten class of $\mathcal{H}$ and $U_p(\mathcal{H})$ be the Banach-Lie group of unitary operators which are p-Schatten perturbations of the identity. Let A be a bounded self-adjoint operator in $\mathcal{H}$. We show that

$$\mathcal{O}_A := \{UA : U \in U_p(\mathcal{H})\}$$

is a smooth submanifold of the affine space $A + \mathcal{B}_p(\mathcal{H})$ if only if A has closed range. Furthermore, it is a homogeneous reductive space of $U_p(\mathcal{H})$. We introduce two metrics: one via the ambient Finsler metric induced as a submanifold of $A + \mathcal{B}_p(\mathcal{H})$, the other, by means of the quotient Finsler metric provided by the homogeneous space structure. We show that $\mathcal{O}_A$ is a complete metric space with the rectifiable distance of these metrics.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space; denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators acting in $\mathcal{H}$, by $\mathcal{U}(\mathcal{H})$ the set of unitary operators in $\mathcal{H}$ and by $\mathcal{B}_p(\mathcal{H})$ the p-Schatten class ($1 \leq p < +\infty$),

that is

$$\mathcal{B}_p(\mathcal{H}) := \{A \in \mathcal{B}(\mathcal{H}) : \operatorname{tr}(|A|^p) < \infty\}$$

where tr is the usual trace in $\mathcal{B}(\mathcal{H})$. If $A \in \mathcal{B}_p(\mathcal{H})$, define $\|A\|_p^p = \operatorname{tr}(|A|^p)$.

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Throughout this paper, $\|\cdot\|$ denotes the usual operator norm. Consider the following group of operators

$$U_p(\mathcal{H}) := \{U \in \mathcal{U}(\mathcal{H}) : U - I \in \mathcal{B}_p(\mathcal{H})\}$$

where $I \in \mathcal{B}(\mathcal{H})$ denotes the identity operator. This group is an example of what in the literature [8] is called a classical Banach-Lie group. It has differentiable structure when it is endowed with the metric $\|U_1 - U_2\|_p$ (note that $U_1 - U_2 \in \mathcal{B}_p(\mathcal{H})$). For instance, the Banach-Lie algebra of $U_p(\mathcal{H})$ is the (real) Banach space $\mathcal{B}_p(\mathcal{H})_{ah} := \{X \in \mathcal{B}_p(\mathcal{H}) : X^* = -X\}$. Hereafter, the subscript h (respectively ah) will be used to denote the sets of hermitic (respectively antihermitic) operators.

Fix a selfadjoint operator $A \in \mathcal{B}(\mathcal{H})$. If $A \notin \mathcal{B}_p(\mathcal{H})$, then A and $\mathcal{B}_p(\mathcal{H})$ are linearly independent. Denote by

$$A + \mathcal{B}_p(\mathcal{H}) := \{A + X : X \in \mathcal{B}_p(\mathcal{H})\}$$

Note that every element $S \in A + \mathcal{B}_p(\mathcal{H})$ has a unique decomposition $S = A + X, X \in \mathcal{B}_p(\mathcal{H})$. We shall endow this affine space with the metric induced by the p-norm: if $S_1 = A + X_1$ y $S_2 = A + X_2$, then $\|S_1 - S_2\| := \|X_1 - X_2\|_p$.

The main goal of this paper is the study of the set

$$\mathcal{O}_A := \{UA : U \in U_p(\mathcal{H})\}.$$

It is shown that it is a differential submanifold if only if A has closed range. Therefore it is regarded as a metric space with two natural metrics, obtained as length metrics (i.e., computed as infimum of length of curves inside $\mathcal{O}_A$).

Since any operator $U \in U_p(\mathcal{H})$ can be written as $U = I + X$, with $X \in \mathcal{B}_p(\mathcal{H})$, it can be seen that $\mathcal{O}_A$ is contained in $A + \mathcal{B}_p(\mathcal{H})$. Therefore, $\mathcal{O}_A$ will be considered with the topology induced by the p-metric of $A + \mathcal{B}_p(\mathcal{H})$.

Unitary orbits of operators have been studied before. For instance, Andruchow and Stojanoff [2] studied the geometry of the orbit $\{ubu^* : u \in U\}$ where b is a fixed element of a complex unital C^*-algebra and U is the unitary group of this C^*-algebra. In [7] can be found a thorough study of the geometric structure of the space Q of idempotent elements of a C^*-algebra. In this paper, Corach, Porta and Recht studied, among other things, the unitary orbits induced by the inner automorphisms of the group of elements which are unitary for the non-degenerate conjugate-bilinear symmetric form determined by a selfadjoint element of Q, on the space of selfadjoints elements of Larotonda in [12] studied the geometry of unitary orbits in a Riemannian, infinite dimensional manifold modeled on the full-matrix algebra of Hilbert-Schmidt operators. In this particular framework, restricting the action to classical groups, certain results can be found in [6] by Carey, [5] by Bóna and [4] by Beltiţă.

Let us describe the contents of the sections. Section 2 contains some preliminary results. In section 3 we establish that the necessary and sufficient condition for $\mathcal{O}_A$ to be a submanifold of $A + \mathcal{B}_p(\mathcal{H})$ is that the selfadjoint operator A has closed range. In section 4, we will endow $\mathcal{O}_A$ with a Finsler metric given by the Banach quotient norm of the Lie algebra of $U_p(\mathcal{H})$ by the Lie algebra of the isotropy group of the action and study the case $p = 2$, in particular, we show that for this p, is a Riemannian metric. In section 5, we endow $\mathcal{O}_A$ with the Finsler metric provided by the ambient norm of $\mathcal{B}_p(\mathcal{H})$ and show that it is complete metric space. In section 6, we will consider de Finsler quotient metric introduced in section 4 and characterize the rectifiable distance induced by this metric. We show this rectifiable distance coincides with the metric for the quotient topology of $U_p(\mathcal{H})/G_A$, where G_A is the isotropy group. We also show that $\mathcal{O}_A$ is a complete metric space with the rectifiable distance given by the quotient metric.

2. Preliminaries

Let $\mathcal{B}\left(\mathcal{H}, \mathcal{H}'\right)$ be the algebra of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}'$. From now on, $N(T)$ and $\mathrm{R}(T)$ will denote the nullspace and range of T, respectively, for every operator $T \in \mathcal{B}\left(\mathcal{H}, \mathcal{H}'\right)$.

Let $T \in \mathcal{B}\left(\mathcal{H}, \mathcal{H}'\right)$ be an operator with closed range. $T^\dagger$ will denote the Moore–Penrose pseudoinverse of T, i.e., the unique bounded linear operator from $\mathcal{H}'$ to $\mathcal{H}$ such that:

(1) $TT^\dagger = \left(TT^\dagger\right)^*$,
(2) $T^\dagger T = \left(T^\dagger T\right)^*$,
(3) $TT^\dagger T = T$,
(4) $T^\dagger T T^\dagger = T^\dagger$.

This is an operator defined by $T^\dagger(Tx) = x$ if $x \in N(T)^\perp$ and $T^\dagger|_{\mathrm{R}(T)^\perp} = 0$. Note that $T^\dagger T$ coincides with the orthogonal projection onto $N(T)^\perp$ and $TT^\dagger$ with the orthogonal projection onto $\mathrm{R}(T)$. For further information about Moore–Penrose pseudoinverse, the reader is referred to Penrose's paper [14] or the book by Groetsch [10].

If $\mathcal{H}_0$ is a closed linear subspace of $\mathcal{H}$, then $H = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$. If $X \in \mathcal{B}(\mathcal{H})$, then X can be written as a 2×2 matrix with operators entries,

$$X = \begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix}$$

where $X^{11} \in \mathcal{B}(\mathcal{H}_0)$, $X^{12} \in \mathcal{B}\left(\mathcal{H}_0^\perp, \mathcal{H}_0\right)$, $X^{21} \in \mathcal{B}\left(\mathcal{H}_0, \mathcal{H}_0^\perp\right)$, $X^{22} \in \mathcal{B}\left(\mathcal{H}_0^\perp\right)$.

Given any $x, y \in \mathcal{H}$, the operator $x \otimes y$ on $\mathcal{H}$ will be defined by

$$(x \otimes y)(z) = \langle z, y \rangle\, x$$

for all $z \in \mathcal{H}$. If $x, y \in \mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$, then the following equalities are readily verified:

$$\begin{aligned} (x \otimes y)^* &= y \otimes x \\ T(x \otimes y) &= T(x) \otimes y \\ (x \otimes y)T &= x \otimes T^*(y). \end{aligned}$$

Given $A \in \mathcal{B}(\mathcal{H})$ and $X \in \mathcal{B}_p(\mathcal{H})$, denote by R_A the map given by:

$$R_A(X) = XA.$$

Since $\mathcal{B}_p(\mathcal{H})$ is two-sided ideal, $R_A : \mathcal{B}_p(\mathcal{H}) \to \mathcal{B}_p(\mathcal{H})$.

3. Differentiable structure of $\mathcal{O}_A$

In this section we study under what conditions the set

$$\mathcal{O}_A = \{UA : U \in U_p(\mathcal{H})\} \subseteq A + \mathcal{B}_p(\mathcal{H})$$

is a differentiable (real analytic) submanifold of the affine space $A+\mathcal{B}_p(\mathcal{H})$, where A is any fixed operator in $\mathcal{B}(\mathcal{H})$ and $A \notin \mathcal{B}_p(\mathcal{H})$. The Banach-Lie group $U_p(\mathcal{H})$ acts on $\mathcal{O}_A$ on the left, by means of

$$U \times VA \mapsto UVA$$

for any $U \in U_p(\mathcal{H})$ and $VA \in \mathcal{O}_A$. This action induces the map

$$\begin{array}{ccc} U_p(\mathcal{H}) & \xrightarrow{\pi_A} & \mathcal{O}_A \\ U & \longmapsto & UA \end{array}$$

This map, regarded as a map on $A + \mathcal{B}_p(\mathcal{H})$, is real analytic. Its differential at the identity is the linear map

$$\begin{array}{ccc} \mathcal{B}_p(\mathcal{H})_{ah} & \xrightarrow{\delta_A} & \mathcal{B}_p(\mathcal{H}) \\ X & \longmapsto & XA \end{array}$$

Here we have identified the Banach-Lie algebra of $U_p(\mathcal{H})$ with the space $\mathcal{B}_p(\mathcal{H})_{ah}$ of antihermitian elements in $\mathcal{B}_p(\mathcal{H})$. We start with a lemma.

Lemma 3.1. *For any $A \in \mathcal{B}(\mathcal{H})$ selfadjoint , the following assertions are equivalent:*

(1) $\mathrm{R}(A)$ *is closed in* $\mathcal{H}$.
(2) $\mathrm{R}(R_A)$ *is closed in* $\mathcal{B}_p(\mathcal{H})$.
(3) $\mathrm{R}(\delta_A)$ *is closed in* $\mathcal{B}_p(\mathcal{H})$.

Proof. $1 \to 2$: Let $(X_n) \in \mathcal{B}_p(\mathcal{H})$ such that $R_A(X_n) \to Y$ in $\mathcal{B}_p(\mathcal{H})$. Since $\mathrm{R}(A)$ is closed, $A^\dagger$ is well defined and since $X_nA \to Y$, $Y|_{N(A)} = 0$. Then if $X := YA^\dagger$, we have that $R_A(X) = YA^\dagger A = Y$.

$2 \to 1$: Let $(x_n) \subseteq H$ such that $Ax_n \to y$. Let $X_n = y \otimes x_n \in \mathcal{B}_p(\mathcal{H})$. It is easy to see that $R_A(X_n) = X_nA = y \otimes Ax_n \to y \otimes y$ in $\mathcal{B}_p(\mathcal{H})$.

Now, using the fact that $\mathrm{R}(R_A)$ is closed in $\mathcal{B}_p(\mathcal{H})$ and that A is selfadjoint, there exists $X \in \mathcal{B}_p(\mathcal{H})$ such that $y \otimes y = AX^*$. Evaluating the last equality in y, we conclude that $y \in \mathrm{R}(A)$.

$2 \to 3$: Let $(X_n) \in \mathcal{B}_p(\mathcal{H})_{ah}$ such that $\delta_A(X_n) = X_nA \to Y$ in $\mathcal{B}_p(\mathcal{H})$. Since $\mathrm{R}(R_A)$ is closed in $\mathcal{B}_p(\mathcal{H})$, there is a $Z \in \mathcal{B}_p(\mathcal{H})$ such that $Y = ZA$. (Note that Z is not necessarily anti-hermitic.).

If Y is represented as a 2×2 operator matrix relative to $H = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, then

$$Y = \begin{pmatrix} Z^{11}A_0 & 0 \\ Z^{21}A_0 & 0 \end{pmatrix}$$

where $\mathcal{H}_0 = (N(A))^\perp = \overline{\mathrm{R}(A)}$ and $A_0 = A|_{\mathcal{H}_0} : \mathcal{H}_0 \to \mathcal{H}_0$. Note that $\mathrm{R}(A)$ is closed because $\mathrm{R}(R_A)$ is closed in $\mathcal{B}_p(\mathcal{H})$ and it was proved that assertion 2 implies assertion 1. Also note that Z^{11} is anti-hermitic, since $\mathrm{R}(A)$ is closed and Z^{11} is the limit of anti-hermitics operators.

Then, if we consider

$$X = \begin{pmatrix} Z^{11} & -\left(Z^{21}\right)^* \\ Z^{21} & 0 \end{pmatrix}$$

we have that $X \in \mathcal{B}_p(\mathcal{H})_{ah}$ and $Y = XA = \delta_A(X)$, then $\mathrm{R}(\delta_A)$ is closed in $\mathcal{B}_p(\mathcal{H})$.

$3 \to 2$: Let us consider again the decomposition $H = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, where $\mathcal{H}_0 = (N(A))^\perp = \overline{\mathrm{R}(A)}$ and $A_0 = A|_{\mathcal{H}_0} : \mathcal{H}_0 \to \mathcal{H}_0$. Since $\mathrm{R}(\delta_A)$ is closed in $\mathcal{B}_p(\mathcal{H})$, the set

$$\{X_0A_0 : X_0 \in \mathcal{B}_p(\mathcal{H}_0)_h\}$$

is closed in $\mathcal{B}_p(\mathcal{H}_0)$.

Now, let

$$\begin{array}{ccc} \mathcal{B}_p(\mathcal{H}_0)_h & \xrightarrow{\delta_0} & \{X_0A_0 : X_0 \in \mathcal{B}_p(\mathcal{H}_0)_h\} \\ X_0 & \longmapsto & X_0A_0 \end{array}$$

Since $\mathrm{R}(\delta_0) = \{X_0A_0 : X_0 \in \mathcal{B}_p(\mathcal{H}_0)_h\}$ is closed in $\mathcal{B}_p(\mathcal{H}_0)$ and A_0 is injective, by the inverse mapping theorem, there is a constant C such that

$$\|X_0\|_{\mathcal{B}_p(\mathcal{H}_0)} \leq C \, \|X_0A_0\|_{\mathcal{B}_p(\mathcal{H}_0)} \tag{3.1}$$

for all $X_0 \in \mathcal{B}_p(\mathcal{H}_0)_h$.

Let $Y_0 \in \mathcal{B}_p(\mathcal{H}_0)$. Replacing X_0 by $|Y_0| \in \mathcal{B}_p(\mathcal{H}_0)_h$ in (3.1), and using the fact that $\||Y_0|\|^p_{\mathcal{B}_p(\mathcal{H}_0)} = \|Y_0\|^p_{\mathcal{B}_p(\mathcal{H}_0)}$ and $\||Y_0|A_0\|^p_{\mathcal{B}_p(\mathcal{H}_0)} = \|Y_0A_0\|^p_{\mathcal{B}_p(\mathcal{H}_0)}$, we see that

$$\|Y_0\|_{\mathcal{B}_p(\mathcal{H}_0)} \leq C \, \|Y_0A_0\|_{\mathcal{B}_p(\mathcal{H}_0)} \, .$$

Then $\mathrm{R}(R_{A_0})$ is closed in $\mathcal{B}_p(\mathcal{H}_0)$. It follows, using the fact that assertion 1 is equivalent to assertion 2, that $\mathrm{R}(R_A)$ is closed in $\mathcal{B}_p(\mathcal{H})$.

□

For the next proposition, the following notation will be used. Let $x = U_0A$, π_x will denote

$$\begin{array}{ccc} U_p(\mathcal{H}) & \xrightarrow{\pi_x} & \mathcal{O}_A \\ U & \longmapsto & Ux \end{array}$$

Note that π_x is real analytic and it's differential at the identity is the linear map

$$\begin{array}{ccc} \mathcal{B}_p(\mathcal{H})_{ah} & \xrightarrow{\delta_x} & \mathcal{B}_p(\mathcal{H}) \\ X & \longmapsto & Xx \end{array}$$

Proposition 3.2. *Let A be a selfadjoint operator with closed range. Then π_x has continuous local cross sections with uniform radius. That is, for any $x = U_0A \in \mathcal{O}_A$ there is a continuous map*

$$\sigma_x : V_x = \left\{ UA \in \mathcal{O}_A : \|UA - U_0A\|_p < \frac{1}{2\,\|A^\dagger\|} \right\} \longrightarrow U_p(\mathcal{H})$$

such that $\pi_x \circ \sigma_x = id_{V_x}$.

In particular, π_x is a locally trivial fibre bundle.

Proof. Let's denote $P_A = AA^{\dagger}$. Since A is selfadjoint and $\mathrm{R}(A)$ is closed, P_A is the orthogonal projection onto $\mathrm{R}(A)$.

Let $\phi : \mathcal{O}_A \to \{UP_A : U \in U_p(\mathcal{H})\}$ be the map given by $\phi(UA) = UP_A$. Note that ϕ is well defined and that, since $\|U_nP_A - UP_A\|_p \leq \|(U_n - U)\, A\|_p \left\|A^{\dagger}\right\|$, $\|\phi(U_nA) - \phi(UA)\|_p \to 0$ when $\|U_nA - UA\|_p \to 0$. Also note that given $R > 0$, $\phi\left(B_{\mathcal{O}_A}\left(A, \frac{R}{\|A^{\dagger}\|}\right)\right)$ lies in

$$V_{P_A} := \left\{UP_A : U \in U_p(\mathcal{H}), \|UP_A - P_A\|_p < R\right\}.$$

Indeed,

$$\|\phi(UA) - P_A\|_p = \left\|(U - I)AA^{\dagger}\right\|_p \leq \|UA - A\|_p \left\|A^{\dagger}\right\| < R.$$

Now, let $U \in V_{P_A}$ and call Q_0 and Q the projections $Q_0 = P_A$ y $Q = UP_AU^*$. Since

$$\begin{gathered}\|Q - Q_0\| = \|UP_AU^* - UP_A + UP_A - P_A\| = \\ \|UP_A(P_AU^* - P_A) + UP_A - P_A\| \leq \\ \|UP_A\|\,\|P_AU^* - P_A\| + \|UP_A - P_A\| < 2R,\end{gathered}$$

if $R \leq \frac{1}{2}$, there is $Z \in U_p(\mathcal{H})$ such that $ZQ_0Z^* = Q$, namely, $ZP_AZ^* = UP_AU^*$. (See, for instance, [11]).

Let $W = (U - Z)\, P_A + Z$. Then W is unitary and $W - I \in \mathcal{B}_p(\mathcal{H})$, namely, $W \in U_p(\mathcal{H})$. Also note that $WP_A = UP_A$.

Let $\sigma_{P_A}(UP_A) = W \in U_p(\mathcal{H})$ for any $UP_A \in V_{P_A}$. Then σ_{P_A} is well defined on V_{P_A}, is continuous and $\sigma_{P_A}(UP_A)P_A = UP_A$.

Let $\sigma = \sigma_{P_A} \circ \phi$. Then σ is a continuous map (since it is the composition of continuous maps) defined in the set $\left\{UA \in \mathcal{O}_A : \|UA - A\|_p < \frac{R}{\|A^{\dagger}\|}\right\}$ and $\sigma(UA) \in U_p(\mathcal{H})$. More over,

$$\pi_A(\sigma(UA)) = \sigma_{P_A}(UP_A)\, A = \sigma_{P_A}(UP_A)\, P_AA = UP_AA = UA.$$

Thus σ is a continuous local cross section for π_A on a neighbourhood of A.

Finally, note that since σ can be translated with the action to any $x = U_0A \in \mathcal{O}_A$, one obtains cross sections defined on translated balls with the same radius. Namely, calling $V_x := \left\{UA \in \mathcal{O}_A : \|UA - U_0A\|_p < \frac{R}{\|A^{\dagger}\|}\right\}$ one can define σ_x in any $UA \in V_x$ by $\sigma_x(UA) := U_0\sigma\left(U_0^{-1}UA\right)U_0^{-1}$.

□

To establish the equivalence between the existence of the submanifold structure for $\mathcal{O}_A \subseteq A + \mathcal{B}_p(\mathcal{H})$ and the closed $\mathrm{R}(A)$ condition, the following general result on homogeneous spaces is useful. It is contained in the appendix of the paper [15] by I. Raeburn, and it is a consequence of the implicit function theorem in Banach spaces.

Lemma 3.3. *Let G be a Banach-Lie group acting smoothly on a Banach space X. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \to X$ the smooth map $\pi_{x_0}(g) := g \cdot x_0$. Suppose that*

(1) *π_{x_0} is an open mapping, when regarded as a map from G onto the orbit $\{g \cdot x_0 : g \in G\}$ of x_0 (with the relative topology of X).*
(2) *The differential $d\,(\pi_{x_0})_1 : (TG)_1 \to X$ splits: it's kernel and range are closed complemented subspaces.*

Then the orbit $\{g \cdot x_0 : g \in G\}$ is a smooth submanifold of X, and the map $\pi_{x_0} : G \to \{g \cdot x_0 : g \in G\}$ is a smooth submersion.

Theorem 3.4. *The orbit $\mathcal{O}_A$ is a real analytic submanifold of $A + \mathcal{B}_p(\mathcal{H})$ if and only if* $\mathrm{R}(A)$ *is closed in $\mathcal{H}$.*

In this case, the map $\pi_x : U_p(\mathcal{H}) \to \mathcal{O}_A$ is a real analytic submersion and $\mathcal{O}_A$ is a homogeneous space of $U_p(\mathcal{H})$.

Proof. Suppose first that $\mathrm{R}(A)$ is closed in $\mathcal{H}$ and use lemma 3.3. In this case, $G = U_p(\mathcal{H})$ and $X = A + \mathcal{B}_p(\mathcal{H})$. Let $x = U_0 A \in \mathcal{O}_A$. Since $\mathrm{R}(A)$ is closed in $\mathcal{H}$, π_x is open because, by proposition 3.2, π_x has continuous local cross sections. In order to see that δ_x has closed complemented range and kernel, let's consider the map E_x given by:

$$E_x(Z) = \frac{1}{2} x x^\dagger Z x^\dagger - \frac{1}{2} (x^*)^\dagger Z^* x x^\dagger + \left(1 - x x^\dagger\right) Z x^\dagger - (x^*)^\dagger Z^* \left(1 - x x^\dagger\right)$$

for any $Z \in \mathcal{B}_p(\mathcal{H})$. Note that since $\mathrm{R}(A)$ is closed, $x^\dagger$ is well defined and also note that $E_x(Z)$ is anti-hermitic, then $E_x : \mathcal{B}_p(\mathcal{H}) \to \mathcal{B}_p(\mathcal{H})_{ah}$.

Note that $\delta_x \circ E_x \circ \delta_x = \delta_x$. This implies that $\delta_x \circ E_x$ and $E_x \circ \delta_x$ are both idempotent operators in the Banach Space $\mathcal{B}_p(\mathcal{H})$, therefore their ranges and kernels are complemented.

Since the range of $\delta_x \circ E_x$ equals the range of δ_x and the kernel of $E_x \circ \delta_x$ equals the kernel of δ_x, δ_x splits.

Then, by lemma 3.3 and using the fact that, in this context, smooth means real analytic (the group and the action are real analytic), $\mathcal{O}_A$ is a real analytic submanifold of $A + \mathcal{B}_p(\mathcal{H})$ and π_x is a real analytic submersion.

Conversely, if $\mathcal{O}_A$ is a submanifold of $A + \mathcal{B}_p(\mathcal{H})$, then the tangent space $T_A(\mathcal{O}_A) = \{XA : X^* = -X \in \mathcal{B}_p(\mathcal{H})\}$ is closed in $\mathcal{B}_p(\mathcal{H})$. Then, by lemma 3.1, the range of A is closed in $\mathcal{H}$. □

4. Finsler and Riemannian metrics in $\mathcal{O}_A$

Since $\mathcal{O}_A$ is a homogeneous space, it is natural to endow each the tangent space with the quotient metric. We will show that, in the case $p = 2$, this metric is a Riemannian metric.

Fix $x = U_0 A \in \mathcal{O}_A$. Since π_x is a submersion, the tangent space of $\mathcal{O}_A$ at x is:

$$T_x(\mathcal{O}_A) = \mathrm{R}(\delta_x) = \left\{ Zx : Z \in \mathcal{B}_p(\mathcal{H})_{ah} \right\}.$$

As noted before, it is a closed linear subspace of $\mathcal{B}_2(\mathcal{H})_{ah}$.

Denote by G_x the isotropy group and by $\mathcal{G}_x$ the Banach-Lie algebra of G_x. Let $X \in T_x(\mathcal{O}_A)$, the **Finsler quotient metric** is usually defined (c.f. [1]) as:

$$\begin{aligned} \|X\|_x &= \inf\left\{ \|Y\|_p : \ Y \in \mathcal{B}_p(\mathcal{H})_{ah}, \delta_x(Y) = X \right\} \\ &= \inf\left\{ \|Z+W\|_p : \ Z \in \mathcal{G}_x \right\}, \end{aligned} \tag{4.1}$$

where $W \in \mathcal{B}_p(\mathcal{H})_{ah}, \delta_x(W) = X$. Note that since π_x is a surjective map, such W exists.

Also note that this metric is invariant under the left action of the group. Namely, for each fixed $U \in U_p(\mathcal{H})$, $x \in \mathcal{O}_A$ and $X \in T_x(\mathcal{O}_A)$, we have that $\|X\|_x = \|UX\|_{Ux}$.

Let us now study the case $p = 2$. $\mathcal{B}_2(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$ and a Hilbert space with the inner product $\langle Z, W\rangle_2 = \operatorname{tr}(W^*Z)$. Note that the norm induced by this inner product coincides with the $\|\cdot\|_2$ previously defined.

In the previous section, it was noted that $E_x \circ \delta_x$ is an idempotent real linear operator on $\mathcal{B}_p(\mathcal{H})_{ah}$ (in particular, on $\mathcal{B}_2(\mathcal{H})_{ah}$). Let's call this operator Q_x. Explicitly, $Q_x : \mathcal{B}_2(\mathcal{H})_{ah} \to \mathcal{B}_2(\mathcal{H})_{ah}$

$$Q_x(Z) = Zxx^\dagger + xx^\dagger Z - xx^\dagger Z xx^\dagger.$$

Then, Q_x is a real linear idempotent operator of $\mathcal{B}_2(\mathcal{H})_{ah}$ with $N(Q_x) = N(\delta_x)$ and $\mathrm{R}(Q_x) = \left\{Z \in \mathcal{B}_2(\mathcal{H})_{ah} : (1 - xx^\dagger)Z(1 - xx^\dagger) = 0\right\}$. Moreover, since Q_x is symmetric for the inner product $\langle , \rangle_2$,

Q_x is a tr-orthogonal projection.

Note that if $x \in \mathcal{O}_A$, $X \in T_x(\mathcal{O}_A)$ and $Z \in \mathcal{B}_p(\mathcal{H})_{ah}$ is a lifting for X (i.e., $\delta_x(Z) = X$), then

$$\|X\|_x = \|Q_x(Z)\|_2 .$$

It is easy to see that

$$G_A = \left\{ U = \begin{pmatrix} I_0 & 0 \\ 0 & U_{22} \end{pmatrix} : I_0 \text{ is the identity of } \mathcal{B}(\mathcal{H}_0) \text{ and } U_{22} \in U_p\left(\mathcal{H}_0^\perp\right) \right\}$$

and since it is an algebraic subgroup, it is a Banach-Lie subgroup of $U_p(\mathcal{H})$ (see [3]). It's Lie algebra is given by

$$\mathcal{G}_A = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x_{22} \end{pmatrix} : x_{22} \in B_p\left(\mathcal{H}_0^\perp\right)_{ah} \right\}.$$

Thus G_x is exponential for any given $x \in \mathcal{O}_A$. Consider the decomposition of the Banach-Lie algebra $\mathcal{B}_2(\mathcal{H})_{ah}$ of $U_2(\mathcal{H})$:

$$\mathcal{B}_2(\mathcal{H})_{ah} = \mathcal{G}_x \oplus \mathrm{R}(Q_x)$$

explicitly,

$$\mathcal{B}_2(\mathcal{H})_{ah} = \{Z \in \mathcal{B}_2(\mathcal{H})_{ah} : ZP = 0\} \oplus \{Z \in \mathcal{B}_2(\mathcal{H})_{ah} : (1-P)Z(1-P) = 0\},$$

where $P = xx^\dagger$.

Note that $\mathrm{R}(Q_x)$ is invariant under the inner action of the isotropy group. Then, this decomposition is what in differential geometry is called a reductive structure of the homogeneous space [16].

Let's denote by η_x the inverse of the linear isomorphism

$$\overline{\delta_x} := \delta_x|_{\mathrm{R}(Q_x)} : \mathrm{R}(Q_x) \longrightarrow T_x\mathcal{O}_A$$

Note that $\delta_x \circ \eta_x = I_{T_x\mathcal{O}_A}$ and that $\eta_x \circ \delta_x = Q_x$.

There is a natural Riemann-Hilbert metric induced by the isomorphisms η_x, $x \in \mathcal{O}_A$. Given any $X, Y \in T_x\mathcal{O}_A$ define the inner product

$$\langle X, Y\rangle_x = \mathrm{tr}\left(\eta_x(Y)^*\eta_x(X)\right) = -\,\mathrm{tr}\left(\eta_x(Y)\eta_x(X)\right). \tag{4.2}$$

Note that

$$\langle X, X\rangle_x = \|\eta_x(X)\|_2^2 .$$

Then if z is a lifting for X, we have that

$$\langle X, X\rangle_x = \|\eta_x(X)\|_2^2 = \|\eta_x(\delta_x(z))\|_2^2 = \|Q_x(Z)\|_2^2 = \|X\|_x^2$$

so the metric (4.2) coincides with the quotient metric (4.1), and the quotient metric is Riemannian when $p = 2$.

Also note that if $\mathcal{X}, \mathcal{Y}$ are tangent vector fields of $\mathcal{O}_A$, then the map that sends any $x \in \mathcal{O}_A$ to $\langle \mathcal{X}_x, \mathcal{Y}_x\rangle_x$ is smooth, thus, this inner product defines a Riemann-Hilbert metric in $\mathcal{O}_A$.

Now, let us recall two natural linear connections for this type of homogeneous reductive spaces defined in [13], the **reductive connection** and the **classifying connection**.

First the **reductive connection** ∇^r. If $\mathcal{X}, \mathcal{Y}$ are a tangent vector fields of $\mathcal{O}_A$ then the field $\nabla^r_{\mathcal{X}}\mathcal{Y}$ in $x \in \mathcal{O}_A$ is given by

$$\eta_x(\nabla^r_{\mathcal{X}}\mathcal{Y}(x)) = \eta_x(\mathcal{X}_x)(\eta_x(\mathcal{Y}_x)) + \left[\eta_x(\mathcal{Y}_x), \eta_x(\mathcal{X}_x)\right],$$

where $X(Y)$ denotes the derivative of Y in the direction of X and $[,]$ the commutator of operators in $\mathcal{B}(\mathcal{H})$.

Note that the reductive connection is compatible with the metric defined since the quotient metric coincides with the metric (4.2).

The **clasifying connection** ∇^c is defined by:

$$\eta_x(\nabla^c_{\mathcal{X}}\mathcal{Y}(x)) = (\eta_x \circ \delta_x)(\eta_x(\mathcal{X}_x)(\eta_x(\mathcal{Y}_x))) = Q_x(\eta_x(\mathcal{X}_x)(\eta_x(\mathcal{Y}_x))),$$

where $\mathcal{X}, \mathcal{Y}$ are a tangent vector fields of $\mathcal{O}_A$

Remark 4.1. The Levi–Civita connection of the metric (4.2) is

$$\nabla = \frac{1}{2}\left(\nabla^r + \nabla^c\right).$$

The geodesics of these connection were computed in [13]: $\gamma(t) = e^{t\eta_x(X)}x$ $t \in \mathbb{R}$, is the geodesic with $\gamma(0) = x$ and $\gamma^{\cdot}(0) = X$.

Remark 4.2. If $\gamma(t),\ t \in [0,1]$ is a smooth curve in $\mathcal{O}_A$ such that $\gamma(0) = x$ then the *horizontal lifting* is a curve Γ in $U_p(\mathcal{H})$ that is the unique solution of the linear differential equation in $\mathcal{B}_2(\mathcal{H})$:

$$\begin{cases} \dot{\Gamma} = \eta_\gamma(\dot{\gamma})\Gamma, \\ \Gamma(0) = 1. \end{cases} \tag{4.3}$$

The existence and uniqueness is guaranteed because γ is smooth and then the map that sends $t \in [0,1]$ to $\eta_{\gamma(t)}(\dot{\gamma}(t)) \in \mathcal{B}_2(\mathcal{H})_{ah}$ is smooth.

Similarly, as it was proved in [16] in the context of classical homogeneous reductive spaces, it can be proved that if $\gamma(t),\ t \in [0,1]$ is a smooth curve in $\mathcal{O}_A$, then the unique solution Γ of (4.3) verifies:

(1) $\Gamma(t) \in U_2(\mathcal{H})\ t \in [0,1]$.
(2) Γ lifts γ: $\pi_\gamma(\Gamma) = \gamma$.
(3) Γ is horizontal: $\Gamma^*\dot{\Gamma} \in \mathrm{R}(Q_x)$.

5. Completeness of the metric space $\mathcal{O}_A$ with the ambient Finsler metric.

In this section we will introduce the ambient Finsler metric induced as a submanifold of $A+\mathcal{B}_p(\mathcal{H})$ and prove that $\mathcal{O}_A$ is complete with the rectifiable distance given by this metric.

Let $x \in \mathcal{O}_A$ and $Zx \in T_x(\mathcal{O}_A) = \left\{Zx : Z \in \mathcal{B}_p(\mathcal{H})_{ah}\right\}$. The **Finsler ambient metric** is defined by

$$F_{amb}(Zx) := \|Zx\|_p .$$

If $\gamma(t) \in \mathcal{O}_A$, $t \in [0,1]$ is a piecewise smooth curve, we measure its length

$$L_{amb}(\gamma) = \int_0^1 F_{amb}(\dot{\gamma}(t))dt = \int_0^1 \|\dot{\gamma}(t)\|_p \, dt.$$

Then the rectifiable distance is given by:

$$d_{amb}(b_0, b_1) = \inf\left\{L_{amb}(\gamma) :\ \gamma \text{ is piecewise smooth curve and } \gamma(0) = b_0,\ \gamma(1) = b_1\right\}.$$

Since $\mathcal{O}_A$ is a homogeneous space of $U_p(\mathcal{H})$ (theorem 3.4), by [15], the next remark follows.

Remark 5.1. Let $x = U_0A \in \mathcal{O}_A$. There exist $r > 0$ (r does not depend on x), a neighborhood U_I of the identity in $U_p(\mathcal{H})$ and a smooth map σ_x such that

$$\sigma_x : V_x = \left\{UA \in \mathcal{O}_A : \|UA - x\|_p < r\right\} \to U_I \subset U_p(\mathcal{H})$$

and $\pi_x \circ \sigma_b|_{V_x} = id|_{V_x}$.

The next lemma will be used later on.

Lemma 5.2. *Let $(v_n)_n$ be a sequence in $\mathcal{O}_A$ such that $\|v_n - v_m\|_p \to 0$. Then there is $v \in \mathcal{O}_A$ such that $\|v_n - v\|_p \to 0$.*

Proof. Consider $r > 0$ given in the remark 5.1 and $n_0 = n(r) \in \mathbb{N}$ such that $\|v_n - v_{n_0}\|_p < r$ for all $n \geq n_0$.

If we call $x = v_{n_0}$ and $V_x = \left\{UA \in \mathcal{O}_A : \|UA - x\|_p < r\right\}$, we have that $v_n \in V_x$ for all $n \geq n_0$ and that $(\sigma_x(v_n))_{n \geq n_0} \subset U_p(\mathcal{H})$ is a Cauchy sequence because σ_x is locally Lipschitz.

Then, since $U_p(\mathcal{H})$ is complete, there is $u \in U_p(\mathcal{H})$ such that $\|\sigma_x(v_n) - u\|_p \to 0$. Thus, $\|v_n - ux\|_p \to 0$. □

We end this section showing that $(\mathcal{O}_A, d_{amb})$ is a complete metric space.

Proposition 5.3. *$\mathcal{O}_A$ is a complete metric space with the rectifiable distance d_{amb}.*

Proof. Let v_n a Cauchy sequence $\mathcal{O}_A$ for d_{amb}.

Since $\|v_n - v_m\|_p \leq d_{amb}(v_n, v_m)$, (v_n) is a Cauchy sequence also for $\|\cdot\|_p$. Then, by lemma 5.2, there is $x \in \mathcal{O}_A$ such that $\|v_n - x\|_p \to 0$.

By proposition 3.2, π_x has continuous local cross sections. Then, there is $n_0 \in \mathbb{N}$ such that v_n is in the domain of σ_x for all $n \geq n_0$ and also $\|\sigma_x(v_n) - I\|_p \to 0$.

Note that there is $Z_n \in \mathcal{B}_p(\mathcal{H})_{ah}$ such that $\sigma_x(v_n) = e^{Z_n}$ and $\|Z_n\|_p \to 0$. This happens because $U_p(\mathcal{H})$ is a Banach-Lie group and then the exponential map is a local diffeomorphism.

Let $\gamma_n(t) = e^{tZ_n}x \in \mathcal{O}_A$. Note that $\gamma_n(0) = x$ and $\gamma_n(1) = v_n$ for all $n \in \mathbb{N}$, then

$$d_{amb}(v_n, x) \leq L_{amb}(\gamma_n) \leq \|Z_n\|_p \|x\|.$$

□

6. A characterization of the rectifiable distance. Completeness of $\mathcal{O}_A$ with the quotient Finsler metric.

In this section we prove the completeness of $\mathcal{O}_A$ as a metric space with the rectifiable distance given by the Finsler quotient metric introduced in section 4. In order to prove this, the rectifiable distance induced by this metric will be characterized as a quotient distance of groups, furthermore, we show that it coincides with the metric for the quotient topology of $U_p(\mathcal{H})/G_A$.

Let $\Gamma(t)$ with $t \in [0,1]$ be a piecewise C^1 curve in $U_p(\mathcal{H})$. The length of Γ can be measured as

$$L_p(\Gamma) = \int_0^1 \left\|\dot{\Gamma}(t)\right\|_p dt.$$

Note that since, for any $U \in U_p(\mathcal{H})$, the tangent space of $U_p(\mathcal{H})$ can be identified with $U\mathcal{B}_p(\mathcal{H})_{ah}$, as well as with $\mathcal{B}_p(\mathcal{H})_{ah}U$, this length functional is well defined.

Then, the rectifiable distance in $U_p(\mathcal{H})$ is given by:

$$d_p(U_0, U_1) = \inf\left\{L_p(\Gamma) : \ \Gamma \subset U_p(\mathcal{H}), \ \Gamma(0) = U_0, \ \Gamma(1) = U_1\right\}.$$

The quotient metric (4.1), previously defined in $\mathcal{O}_A$, induces another length functional:

$$L(\gamma) = \int_0^1 \|\dot\gamma\|_\gamma \, dt,$$

where $\gamma(t)$ with $t \in [0,1]$ is a continuous and piecewise smooth curve in $\mathcal{O}_A$.

Analogously, the rectifiable distance in $\mathcal{O}_A$ is given by:

$$d(b_0, b_1) = \inf \{L(\gamma) : \ \gamma \subset \mathcal{O}_A, \ \gamma(0) = b_0, \ \gamma(1) = b_1\}$$

where the curves γ considered are continuous and piecewise smooth.

The following result (whose proof is adapted from [1]) shows that the rectifiable distance in $\mathcal{O}_A$ can be approximated by lifting curves to the group $U_p(\mathcal{H})$.

Lemma 6.1. *Let x_0 and $x_1 \in \mathcal{O}_A$. Then*

$$d(x_0, x_1) = \inf \{L_p(\Gamma) : \ \Gamma \subseteq U_p(\mathcal{H}), \ \pi_{x_0}(\Gamma(0)) = x_0, \ \pi_{x_0}(\Gamma(1)) = x_1\},$$

where the curves Γ considered are continuous and piecewise smooth.

Proof. Let $\Gamma(t)$ be a piecewise smooth curve in $U_p(\mathcal{H})$ such that $\pi_{x_0}(\Gamma(0)) = x_0$ and $\pi_{x_0}(\Gamma(1)) = x_1$. Note that the existence of curves that verify $\pi_{x_0}(\Gamma(t)) = \gamma(t)$ for $t \in [0,1]$ is assured because

$$\pi_{x_0} : U_p(\mathcal{H}) \to \mathcal{O}_A, \qquad \pi_{x_0}(U) = Ux_0$$

is a real analytic submersion (lemma 3.3).

Using the definition of the quotient metric (4.1), it can be seen that the differential map of π_{x_0} at the identity (denoted by δ_{x_0}) is contractive. More over, is contractive at any $U \in U_p(\mathcal{H})$. Then

$$d(x_0, x_1) \le L(\pi_{x_0}(\Gamma)) \le L_p(\Gamma).$$

Now let us show that $L(\gamma)$ can be approximated by lengths of curves in $U_p(\mathcal{H})$ joining the fibers of x_0 y x_1.

Fix $\epsilon > 0$ and consider $0 = t_0 < t_1 < \ldots < t_n = 1$ an uniform partition of $[0,1]$ satisfying:

(1) $\|\dot\gamma(s) - \dot\gamma(s)\|_p < \epsilon/4$, if $s, \ s' \in [t_{i-1}, t_i]$.

(2) $\left| L(\gamma) - \sum_{i=0}^{n-1} \|\dot\gamma(t_i)\|_{\gamma(t_i)} \Delta t_i \right| < \epsilon/2.$

For $i = 0, \ldots, n-1$, let $Z_i \in \mathcal{B}_p(\mathcal{H})_{ah}$ such that

$$\delta_{\gamma(t_i)}(Z_i) = \dot\gamma(t_i) \qquad \text{and} \qquad \|Z_i\|_p \le \|\dot\gamma(t_i)\|_{\gamma(t_i)} + \epsilon/2.$$

Consider the following curve:

$$\Gamma(t) = \begin{cases} e^{tZ_0} & t \in [0, t_1), \\ e^{(t-t_1)Z_1} e^{t_1 Z_0} & t \in [t_1, t_2), \\ e^{(t-t_2)Z_2} e^{(t_2-t_1)Z_1} e^{t_1 Z_0} & t \in [t_2, t_3), \\ \ldots & \ldots \\ e^{(t-t_{n-1})Z_{n-1}} \ldots e^{(t_2-t_1)Z_1} e^{t_1 Z_0} & t \in [t_{n-1}, 1]. \end{cases}$$

Γ is continuous and piecewise smooth curve in $U_p(\mathcal{H})$ satisfying $\Gamma(0)=1$ and

$$L_p(\Gamma)=\sum_{i=0}^{n-1}\|Z_i\|_p\,\Delta t_i\leq\left(\sum_{i=0}^{n-1}\|\dot\gamma(t_i)\|_{\gamma(t_i)}+n\epsilon/2\right)\Delta t_i\leq L(\gamma)+\epsilon.$$

and $\pi(\Gamma(1))$ lies close to x_1.

In order to prove the last statement, first apply the mean value theorem in Banach spaces [9] to the map $\alpha(t)=\pi_{x_0}\left(e^{tZ_0}\right)-\gamma(t)$. Then, for some $s_1\in[0,t_1]$, we see that

$$\begin{gathered}\left\|\pi_{x_0}\left(e^{t_1Z_0}\right)-\gamma(t_1)\right\|_p=\|\alpha(t_1)-\alpha(0)\|_p\leq\|\dot\alpha(s_1)\|_p\,\Delta t_1=\\ \left\|e^{s_1Z_0}\delta_{x_0}(Z_0)-\dot\gamma(s_1)\right\|_p\Delta t_1=\left\|e^{s_1Z_0}\dot\gamma(0)-\dot\gamma(s_1)\right\|_p\Delta t_1\leq\\ \left(\left\|e^{s_1Z_0}\dot\gamma(0)-\dot\gamma(0)\right\|_p+\|\dot\gamma(0)-\dot\gamma(s_1)\|_p\right)\Delta t_1<\\ \left(\left\|e^{s_1Z_0}\dot\gamma(0)-\dot\gamma(0)\right\|_p+\epsilon/4\right)\Delta t_1.\end{gathered}$$

Note that the first summand is bounded by

$$\left\|e^{s_1Z_0}\dot\gamma(0)-\dot\gamma(0)\right\|_p=\left\|\left(e^{s_1Z_0}-I\right)\dot\gamma(0)\right\|_p\leq\|\dot\gamma(0)\|_p\left\|e^{s_1Z_0}-I\right\|_p\leq M\Delta t_1,$$

where $M=\max\limits_{t\in[0,1]}\|\dot\gamma(t)\|_p$. Thus,

$$\|\pi_{x_0}(\Gamma(t_1))-\gamma(t_1)\|_p\leq(M\Delta t_1+\epsilon/4)\,\Delta t_1.$$

Now, note that $\|\pi_{x_0}(\Gamma(t_2))-\gamma(t_2)\|_p$ is less or equal than

$$\left\|e^{(t_2-t_1)Z_1}e^{t_1Z_0}x_0-e^{(t_2-t_1)Z_1}\gamma(t_1)\right\|_p+\left\|e^{(t_2-t_1)Z_1}\gamma(t_1)-\gamma(t_2)\right\|_p$$

and that,

$$\begin{gathered}\left\|e^{(t_2-t_1)Z_1}e^{t_1Z_0}x_0-e^{(t_2-t_1)Z_1}\gamma(t_1)\right\|_p=\left\|e^{(t_2-t_1)Z_1}\left(e^{t_1Z_0}x_0-\gamma(t_1)\right)\right\|_p\leq\\ \left\|e^{(t_2-t_1)Z_1}\right\|\left\|e^{t_1Z_0}x_0-\gamma(t_1)\right\|_p\leq(M\Delta t_1+\epsilon/4)\Delta t_1.\end{gathered}$$

Also, proceeding analogously as before, the second summand can be bounded by:

$$\left\|e^{(t_2-t_1)Z_1}\gamma(t_1)-\gamma(t_2)\right\|_p\leq(M\Delta t_2+\epsilon/4)\,\Delta t_2.$$

Thus,

$$\left\|\pi_{x_0}\left(e^{(t_2-t_1)Z_1}e^{t_1Z_0}\right)-\gamma(t_2)\right\|_p\leq\frac{2}{n}\left(\frac{M}{n}+\frac{\epsilon}{4}\right).$$

Inductively, we obtain

$$\|\pi_{x_0}(\Gamma(1))-\gamma(1)\|_p\leq\frac{M}{n}+\frac{\epsilon}{4}<\epsilon/2.$$

□

In the next theorem, the rectifiable distance will be characterized as the quotient distance of groups, identifying $\mathcal{O}_A\cong U_p(\mathcal{H})/G_A$. This fact will also imply the completeness of $\mathcal{O}_A$ with the rectifiable distance. In the proof of this theorem, the next lemma (c.f. [17] page 109) will be used.

Lemma 6.2. *Let H be a metrizable topological group, and G be a closed subgroup. If d is a complete distance function on H inducing the topology of H, and if d is invariant under the right translation by G, i.e., $d\,(xg, yg) = d\,(x, y)$ for any x, $y \in H$ and $g \in G$, then the left coset space $H/G = \{xG : \ x \in H\}$ is a complete metric space under the metric $\dot{d}$ given by*

$$\dot{d}\,(xG, yG) = \inf\left\{d\,(xg_1, yg_2) : \ g_1, g_2 \in G\right\}.$$

Moreover, the distance $\dot{d}$ is a metric for the quotient topology.

In our context we shall take $H = U_p\,(\mathcal{H})$, $G = G_A$ and $d = d_p$.

Theorem 6.3. *Let A be a selfadjoint operator with closed range. Let U_0 and $U_1 \in U_p\,(\mathcal{H})$ and*

$$\dot{d}_p(U_0A, U_1A) = \inf\left\{d_p(U_0, U_1U) : \ U \in G_A\right\}.$$

Then, $\dot{d}_p = d$. Where d is the rectifiable distance in $\mathcal{O}_A$, previously defined.

In particular, $(\mathcal{O}_A, d)$ is a complete metric space and d metricates the quotient topology.

Proof. It is known that $(U_p\,(\mathcal{H}), d_p)$ is a complete metric space and that G_A is d_p-closed in $U_p\,(\mathcal{H})$, then the quotient distance $\dot{d}_p$ is well defined and it can be calculated:

$$\dot{d}_p(U_0A, U_1A) = \inf\left\{d_p(U_0, U_1U) : \ U \in G_A\right\}.$$

In order to prove the equality between distances first fix $\epsilon > 0$. By lemma 6.1, there exists a curve Γ in $U_p\,(\mathcal{H})$ satisfying:

(1) $\Gamma(0) = U_0$; $\Gamma(1) = U_1U$, $U \in G_A$.
(2) $L_p(\Gamma) < d(U_0A, U_1A) + \epsilon$.

Then

$$\dot{d}_p(U_0A, U_1A) \leq d_p(U_0, U_1U) \leq L_p(\Gamma) < d(U_0A, U_1A) + \epsilon.$$

Since ϵ is arbitrary, we have proved the first inequality.

To show the reversed inequality, note that given $\epsilon > 0$, there exists $U \in G_A$ such that

$$d_p(U_0, U_1U) < \dot{d}_p(U_0A, U_1A) + \epsilon.$$

Then there exists a curve Γ in $U_p\,(\mathcal{H})$ such that $\Gamma(0) = U_0$, $\Gamma(1) = U_1U$ and

$$L_p(\Gamma) < d_p(U_0, U_1U) + \epsilon.$$

Thus, we get

$$d(U_0A, U_1A) \leq L_p(\Gamma) < d_p(U_0, U_1U) + \epsilon < \dot{d}_p(U_0A, U_1A) + 2\epsilon,$$

hence the equality holds.

The completeness of $(\mathcal{O}_A, d)$ and the fact that d metricates the quotient topology follow from lemma 6.2.

□

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